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Executable Behaviour and the $\pi$-Calculus

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Abstract. Reactive Turing machines extend Turing machines with a facility to model observable interactive behaviour. A behaviour is called executable if, and only if, it is behaviourally equivalent to the behaviour of a reactive Turing machine. In this paper, we study the relationship between executable behaviour and behaviour that can be specified in the $\pi$-calculus. We establish that all executable behaviour can be specified in the $\pi$-calculus up to divergence-preserving branching bisimilarity. For the converse, however, we need to let go of the requirement of divergence-preservation: we exhibit behaviour specifiable in the $\pi$-calculus that is not executable up to divergence-preserving behavioural equivalence, but do obtain that every behaviour definable in the $\pi$-calculus is executable up to the version of branching bisimilarity that does not preserve divergence.

1 Introduction

The Turing machine [16] is generally accepted as the machine model that captures precisely which functions are algorithmically computable. As a theoretical model of the behaviour of actual computing systems, however, it has limitations, as was already observed, e.g., by Petri [14]. Most notably, Turing machines lack facilities to adequately deal with two important ingredients of modern computing: interaction and non-termination. Concurrency theory emerged from the work of Petri and developed into an active field of research. It resulted in a plethora of calculi for the formal specification of the behaviour of reactive systems, of which the $\pi$-calculus [13,15] is probably the best-known to date.

Research in concurrency theory has focussed on defining expressive process specification formalisms, modal logics, studying suitable behavioural equivalences, etc. Expressiveness questions have also been addressed extensively in concurrency theory, especially in the context of the $\pi$-calculus (see, e.g., [10,6]), but mostly pertaining to the so-called relative expressiveness of process calculi. The absolute expressiveness of process calculi, and in particular the question as to which interactive behaviour can actually be executed by a conventional computing system, has received considerably less attention. In this paper, we consider the expressiveness of the $\pi$-calculus with respect to the model of reactive Turing machines, proposed in [3] as an orthogonal extension of conventional Turing machines with a facility to model interaction in the style of concurrency theory.

Reactive Turing machines serve to define which behaviour can be executed by a computing system. Formally, we associate with every reactive Turing machine a tran-
sition system, which mathematically represents its behaviour. Then, we say that a transition system is executable if it is behaviourally equivalent to the transition system of a reactive Turing machine. Process calculi generally also have their operational semantics defined by means of transition systems. Hence, we get a method to investigate the absolute expressiveness of a process calculus, by determining to what extent transition systems specified in the calculus are executable, and by determining to what extent executable transition systems can be specified in the calculus. Note that the behavioural equivalence is a parameter of the method: if a behaviour specified in the process calculus is not executable up to some fine notion of behavioural equivalence, it may still be executable up to some coarser notion of behavioural equivalence, and we may look for the dividing line. The entire spectrum of behavioural equivalences (see [9]) is at our disposal to draw precise conclusions.

We shall use the aforementioned method to investigate the expressiveness of the \( \pi \)-calculus. We shall prove that the \( \pi \)-calculus is, indeed, expressive: every executable behaviour can be specified in the \( \pi \)-calculus up to divergence-preserving branching bisimilarity [7,8], which is the finest behavioural equivalence discussed in van Glabbeek’s seminal paper on behavioural equivalences [9]. We also prove that the converse is not true: it is possible to specify, in the \( \pi \)-calculus, transition systems that are not executable up to divergence-preserving branching bisimilarity. The culprit turns out to be the preservation of divergence, for we prove that every \( \pi \)-calculus behaviour is executable up to the divergence-insensitive variant of branching bisimilarity.

Milner already established in [12] that the \( \pi \)-calculus Turing powerful, by exhibiting an encoding of the \( \lambda \)-calculus in the \( \pi \)-calculus by which every reduction in \( \lambda \)-calculus by a sequence of reductions in \( \pi \)-calculus. Our result that all executable behaviour can be specified in the \( \pi \)-calculus also implies that the \( \pi \)-calculus is Turing powerful, and thus subsumes Milner’s result. Note that Milner’s result confirms the computational power of the \( \pi \)-calculus, but does not make a qualitative statement about its interactive expressiveness. Similarly, in [4] several expressiveness results for variants of CCS are obtained via an encoding of Random Access Machines, and also those results only make claims about the computational expressiveness of the calculi.

In our framework, we believe that it is better to distinguish with the computable functions and executable systems when we study the expressiveness of \( \pi \)-calculus. So we investigated the expressiveness power and executability of transition systems associated with \( \pi \)-calculus processes modulo (divergence-preserving) branching bisimilarity. Like the method used in [4], we establish the powerfulness of a model by encoding it into a machine which has a certain computability/executeability. With this basic idea, we prove the reactive Turing powerfulness of \( \pi \)-calculus modulo divergence-preserving branching bisimilarity by establishing a finite specification of a reactive Turing machine in \( \pi \)-calculus. Our specification simulates the behaviour of a reactive Turing machine by encoding every functional part of the machine directly. The tape memory is modeled by making use of the link structure of \( \pi \)-calculus. On the other hand, we prove the executability of \( \pi \)-calculus modulo branching bisimilarity. We prove that it is not a divergence preserving result by defining an unboundedly branching processes in \( \pi \)-calculus. The two statements above refines the absolute expressiveness result for \( \pi \)-calculus.
The paper is organized as follows. In Section 2, the basic definitions of reactive Turing machines and divergence-preserving branching bisimilarity are recapitulated, and we also recall the operational semantics of π-calculus with replication. In Section 3, we prove the reactive Turing powerfulness of π-calculus modulo divergence-preserving branching bisimilarity: a finite specification of reactive Turing machines in the π-calculus is proposed and verified. In Section 4, we discuss the executability of the transition systems associated with π-calculus processes, establishing that every transition system specifiable in the π-calculus is executable modulo branching bisimilarity, but at the cost of introducing divergence to simulate unboundedly branching behaviour. The paper ends with a conclusion in Section 5.

Note to the reviewers: the full version of this article with proofs of the results is available as [11].

2 A Mathematical Theory of Behaviour

The transition system is the central notion in the mathematical theory of discrete-event behaviour. It is parameterised by a set 𝒜 of action symbols, denoting the observable events of a system. We shall later impose extra restrictions on 𝒜, e.g., requiring that it be finite or has a particular structure, but for now we let 𝒜 be just an arbitrary abstract set. We extend 𝒜 with a special symbol τ, which intuitively denotes unobservable internal activity of the system. We shall abbreviate 𝒜∪{τ} by 𝒜τ.

Definition 1. [Labelled Transition System] An 𝒜τ-labelled transition system T is a triple (S, −→, ↑), where,

1. S is a set of states,
2. −→ ⊆ S × 𝒜τ × S is an 𝒜τ-labelled transition relation. If (s, a, t) ∈ −→, we write s a −→ t.
3. ↑ ∈ S is an initial state.

Transition systems can be used to give semantics to programming languages and process calculi. The standard method is to first associate with every program or process expression a transition system (its operational semantics), and then consider programs and process expressions modulo one of the many behavioural equivalences on transition systems that have been studied in the literature. In this paper, we shall use the notion of (divergence-preserving) branching bisimilarity [7,8], which is the finest behavioural equivalence from van Glabbeek’s linear time - branching time spectrum [9].

In the definition of (divergence-preserving) branching bisimilarity we need the following notation: we use Reach(s) to denote the set of reachable states from s through transitions. Let −→ be an 𝒜τ-labelled transition relation on a set S, and let a ∈ 𝒜τ; we write s (a) −→ τ for “s −→ t” or “a = τ and s = t”. Furthermore, we denote the transitive closure of −→ by −→* and the reflexive-transitive closure of −→ by −→*.

Definition 2. [Branching Bisimilarity] Let T1 = (S1, −→1, ↑1) and T2 = (S2, −→2, ↑2) be transition systems. A branching bisimulation from T1 to T2 is a binary relation 𝒮 ⊆ S1 × S2 and, for all states s1 and s2, s1 ∈ 𝒮 implies
1. If \( s_1 \xrightarrow{a_1} s'_1 \), then there exist \( s'_2, s''_2 \in S_2 \), s.t. \( s_2 \xrightarrow{a_2} s'_2 \xrightarrow{(a)} s''_2 \xrightarrow{(a)} s'_2 \), \( s_1 \mathcal{R} s''_2 \) and \( s'_1 \mathcal{R} s'_2 \).

2. If \( s_2 \xrightarrow{a_2} s'_2 \), then there exist \( s'_1, s''_1 \in S_1 \), s.t. \( s_1 \xrightarrow{a_1} s'_1 \xrightarrow{(a)} s''_1 \xrightarrow{(a)} s'_1 \), \( s'_1 \mathcal{R} s_2 \) and \( s'_1 \mathcal{R} s'_2 \).

The transition systems \( T_1 \) and \( T_2 \) are branching bisimilar (notation: \( T_1 \equiv_b T_2 \)) if there exists a branching bisimulation \( \mathcal{R} \) from \( T_1 \) to \( T_2 \) s.t. \( \uparrow_1 \mathcal{R} \uparrow_2 \).

A branching bisimulation \( \mathcal{R} \) from \( T_1 \) to \( T_2 \) is divergence-preserving if, for all states \( s_1 \) and \( s_2 \), \( s_1 \mathcal{R} s_2 \) implies

3. If there exists an infinite sequence \( (s_1,i)_{i \in \mathbb{N}} \) s.t. \( s_1 = s_{1,0}, s_{1,i} \xrightarrow{\tau} s_{1,i+1} \) and \( s_{1,i} \mathcal{R} s_2 \) for all \( i \in \mathbb{N} \), then there exists a state \( s'_2 \) s.t. \( s_2 \xrightarrow{+} s'_2 \) and \( s_{1,j} \mathcal{R} s'_2 \) for some \( i \in \mathbb{N} \); and

4. If there exists an infinite sequence \( (s_2,i)_{i \in \mathbb{N}} \) s.t. \( s_2 = s_{2,0}, s_{2,i} \xrightarrow{\tau} s_{2,i+1} \) and \( s_{2,i} \mathcal{R} s_1 \) for all \( i \in \mathbb{N} \), then there exists a state \( s'_1 \) s.t. \( s_1 \xrightarrow{+} s'_1 \) and \( s_{2,j} \mathcal{R} s'_1 \) for some \( i \in \mathbb{N} \).

The transition systems \( T_1 \) and \( T_2 \) are divergence-preserving branching bisimilar (notation: \( T_1 \equiv^d_b T_2 \)) if there exists a divergence-preserving branching bisimulation \( \mathcal{R} \) from \( T_1 \) to \( T_2 \) s.t. \( \uparrow_1 \mathcal{R} \uparrow_2 \).

We call the infinite \( \tau \) transition sequence in the above definition state-preserving \( \tau \) sequence. For \( \equiv^d_b \), a divergence is an infinite sequence \( s_{1,1} \xrightarrow{\tau} s_{1,2} \xrightarrow{\tau} \ldots \), with \( s_{1,i} \equiv^d_b s_{1,j} \) for any \( i, j \in \mathbb{N} \). Sometimes we adopt the branching bisimulation over two states in the same transition system, we also denote the relation by \( \equiv^d_b \), which is actually a bisimilarity from a transition system to itself. Next we define the notion of bisimulation up to \( \equiv^d_b \), which is useful in establishing proofs of \( \equiv^d_b \).

**Definition 3.** Let \( T_1 = (S_1, \xrightarrow{1}, \uparrow_1) \) and \( T_2 = (S_2, \xrightarrow{2}, \uparrow_2) \) be two transition systems. A relation \( \mathcal{R} \subseteq S_1 \times S_2 \) is a bisimulation up to \( \equiv^d_b \) if, whenever \( (s,t) \in \mathcal{R} \), then for all \( a \in \mathcal{A}_r \):

1. If \( s \xrightarrow{a} s' \), then there exists \( t' \) such that \( t \xrightarrow{a} t' \) and \( s' \equiv^d_b \mathcal{R} t' \); and
2. If \( t \xrightarrow{a} t' \), then there exists \( s' \) such that \( s \xrightarrow{a} s' \) and \( s' \equiv^d_b \mathcal{R} t' \).

**Lemma 1.** If \( \mathcal{R} \) is a bisimulation up to \( \equiv^d_b \), then \( \mathcal{R} \subseteq \equiv^d_b \).

**Proof.** We first prove that \( \equiv^d_b \circ \mathcal{R} \) is a branching bisimulation. Suppose we have \( P \equiv^d_b P_1 \mathcal{R} Q \), then,

1. If \( P \xrightarrow{a} P' \), then according to Definition 2 we have \( P' = P_1 \xrightarrow{a} P' \), with \( P_1 \equiv^* \xrightarrow{(a)} P_1' \), and \( P_1' \equiv^d_b P, P_1' \equiv^d_b P' \). By Definition 3 we have \( Q' \equiv^* Q' \), and \( Q' \equiv^* Q' \) such that \( Q \equiv^* Q' \equiv^d_b Q', \) and for some \( P_2' \) and \( P_2'' \equiv^d_b P_2' \mathcal{R} Q' \), \( P_2' \equiv^d_b P_2' \mathcal{R} Q' \), from which it follows \( P' \equiv^d_b \mathcal{R} Q' \), and \( P \equiv^d_b \mathcal{R} Q' \).

2. If \( Q \xrightarrow{a} Q' \), we can establish a similar proof.

Therefore, a branching bisimulation up to \( \equiv^d_b \) is included in \( \equiv^d_b \).

Now we proceed to prove that \( \equiv^d_b \circ \mathcal{R} \) is divergence preserving. We still use assumption that \( P \equiv^d_b P_1 \mathcal{R} Q \), then,
1. If there is a divergence from \( P \), then since \( P \leftrightarrow P_1 \), there is also a divergence from \( P_1 \). As \( R \) is a bisimulation up to \( \leftrightarrow \), every state preserving \( \tau \) transition \( P_1 \xrightarrow{\tau} P'_1 \) is simulated by \( Q \xrightarrow{\tau} Q' \) with \( P'_1 \leftrightarrow_{\bullet} RQ' \), which is also state preserving. Therefore the divergence is preserved.

2. If there is a divergence from \( Q \), then we can establish a similar proof.

### 2.1 Executable behaviour

The notion of reactive Turing machine (RTM) was put forward in [3] to mathematically characterise which behaviour is executable by a conventional computing system. In this section, we recall the definition of RTMs and the ensued notion of executable transition system. The definition of RTMs is parameterised with the set \( A \), which we now assume to be a finite set. Furthermore, the definition is parameterised with another finite set \( D \) of data symbols. We extend \( D \) with a special symbol \( \square \) to denote a blank tape cell, and denote the set \( D \cup \{ \square \} \) of tape symbols by \( D_\square \).

**Definition 4.** [Reactive Turing Machine] A reactive Turing machine (RTM) \( M \) is a triple \( (S, \rightarrow, \uparrow) \), where

1. \( S \) is a finite set of states,
2. \( \rightarrow \subseteq S \times D_\square \times \mathcal{A}_r \times D_\square \times \{L, R\} \times S \) is a \((D_\square \times \mathcal{A}_r \times D_\square \times \{L, R\})\)-labelled transition relation (we write \( s \xrightarrow{a|d/e}M t \) for \( (s, d, a, e, M, t) \in \rightarrow \)),
3. \( \uparrow \in S \) is a distinguished initial state.

**Remark 1.** The original definition of RTMs in [3] includes an extra facility to declare a subset of the states of an RTM as being final states, and so does the associated notion of executable transition system. In this paper, however, our goal in this paper is to explore the relationship between the transition systems associated with RTMs and those that can be specified in the \( \pi \)-calculus. Since the \( \pi \)-calculus does not include the facility to specify that a state has the option to terminate, we leave it out from the definition of RTMs too.

Intuitively, the meaning of a transition \( s \xrightarrow{a|d/e}M t \) is that whenever \( M \) is in state \( s \), and \( d \) is the symbol currently read by the tape head, then it may execute the action \( a \), write symbol \( e \) on the tape (replacing \( d \)), move the read/write head one position to the left or the right on the tape (depending on whether \( M = L \) or \( M = R \)), and then end up in state \( t \).

To formalise the intuitive understanding of the operational behaviour of RTMs, we associate with every RTM \( M \) an \( \mathcal{A}_r \)-labelled transition system \( \mathcal{T}(M) \). The states of \( \mathcal{T}(M) \) are the configurations of \( M \), which consists of a state from \( S \), its tape contents, and the position of the read/write head. We define tape contents by an element of \((D_\square)^*\), replacing precisely one occurrence of a tape symbol \( d \) by a marked symbol \( \bar{d} \), indicating that the read/write head is on this symbol. We denote by \( \mathcal{D}_\square = \{ \bar{d} \mid d \in D_\square \} \) the set of marked symbols; a tape instance is a sequence \( \delta \in (D_\square \cup \mathcal{D}_\square)^* \) such that \( \delta \) contains exactly one element of \( \mathcal{D}_\square \).

We adopt a convention to concisely denote new placement of the tape head marker. Let \( \delta \) be an element of \( D_\square^{\mathcal{A}_r} \). Then by \( \delta^\leftarrow \) we denote the element of \((D_\square \cup \mathcal{D}_\square)^*\) obtained
by placing the tape head marker on the right-most symbol of \( \delta \) (if it exists), and \( \square \) otherwise. Similarly \( \check{\delta} \) is obtained by placing the tape head marker on the left-most symbol of \( \delta \) (if it exists), and \( \square \) otherwise.

**Definition 5.** Let \( M = (S, \rightarrow, \uparrow) \) be an RTM. The transition system \( T(M) \) associated with \( M \) is defined as follows:

1. its set of states \( S \) consists of the set of all configurations of \( M \);
2. its transition relation \( \rightarrow \) is the least relation satisfying, for all \( a \in \mathcal{A}_\tau \), \( d, e \in D \) and \( \delta_L, \delta_R \in D^* \):
   \[
   (s, \delta_L \check{\delta} \delta_R) \xrightarrow{a} (t, \delta_L^* e \delta_R) \text{ iff } s \xrightarrow{a[d/e]} t, \text{ and}
   \]
   \[
   (s, \delta_L \check{\delta} \delta_R) \xrightarrow{a} (t, \delta_L \check{\delta} \delta_R) \text{ iff } s \xrightarrow{a[d/e]} t, \text{ and}
   \]
3. its initial state is the configuration \((\uparrow, \square)\).

Turing introduced his machines to define the notion of *effectively computable function*. By analogy, the notion of RTM can be used to define a notion of *effectively executable behaviour*.

**Definition 6.** [Executability] A transition system is executable if it is the transition system associated with some RTM.

Usually, we shall be interested in executability up to some behavioural equivalence. In [3], a characterisation of executability up to (divergence-preserving) branching bisimilarity is given that is independent of the notion of RTM. In order to be able to recapitulate the results below, we need the following definitions, pertaining to the recursive complexity and branching degree of transition systems. Let \( T = (S, \rightarrow, \uparrow) \) be a transition system. We say that \( T \) is effective if \( \rightarrow \) is a recursively enumerable set. The mapping \( \text{out} : S \rightarrow 2^{\mathcal{A}_\tau \times S} \) associates with every state its set of outgoing transitions, i.e., for all \( s \in S \), \( \text{out}(s) = \{(a, t) \mid s \xrightarrow{a} t \} \). We say that \( T \) is computable if \( \text{out} \) is a recursive function. We call a transition system finitely branching if \( \text{out}(s) \) is finite for every state \( s \), and boundedly branching if there exists \( B \in \mathbb{N} \) such that \( |\text{out}(s)| \leq B \) for all \( s \in S \).

The following results were established in [3].

**Theorem 1.** 1. The transition system \( T(M) \) associated with an RTM \( M \) is computable and boundedly branching.
2. For every finite set \( \mathcal{A}_\tau \) and every boundedly branching computable \( \mathcal{A}_\tau \)-labelled transition system \( T \), there exists an RTM \( M \) such that \( T \bowtie_b T(M) \).
3. For every finite set \( \mathcal{A}_\tau \) and every effective \( \mathcal{A}_\tau \)-labelled transition system \( T \) there exists an RTM \( M \) such that \( T \bowtie_b T(M) \).

We also propose a criterion to decide whether a transition system is not executable. An auxiliary notion of branching degree up to a certain equivalence relation is as follows. The set of equivalence classes modulo an arbitrary equivalence relation \( \equiv \) on \( S \) is the quotient set \( S/\equiv \), and we denote the equivalence class of \( s \in S \) modulo \( \equiv \) by \( [s]_\equiv = \{s' \in S \mid s \equiv s' \} \).
Definition 7. Let \( T = (S, \rightarrow , \uparrow) \) be an LTS, and \( s \in S \).

The branching degree modulo \( \equiv \), denoted by \( \deg_\equiv(s) \) is the cardinality of the set \( \{(a, [s']_\equiv) \mid s \xrightarrow{a} s', a \in \mathcal{A}_T \} \).

The branching degree modulo \( \equiv \) of a transition system is the least upper bound of the branching degrees of all reachable states, which is defined as \( \deg_\equiv(T) = \sup(\deg_\equiv(s) \mid s \in \text{Reach}(\uparrow)) \).

We say that a transition system \( T \) is boundedly branching up to \( \equiv \) if there exists \( B \in \mathbb{N} \), such that \( \deg_\equiv(T) \leq B \), otherwise it is unboundedly branching up to \( \equiv \).

The following lemma shows that transition systems with different branching degrees cannot be divergence-preserving branching bisimilar. In other words, divergence has to be introduced to simulate a transition system with higher branching degree.

Lemma 2. Let \( T_1 \) be a transition system with \( \deg_\equiv(T_1) = b_1 \in \mathbb{N} \), and suppose that \( T_1 \) does not contain a divergence. Then there is no transition system \( T_2 \) with \( \deg_\equiv(T_2) = b_2 < b_1 \), and \( T_1 \xrightarrow{b} T_2 \).

Proof. We prove by contradiction. We suppose that \( T_1 = (S_1, \rightarrow_1, \uparrow_1) \), and \( s_1 \in \text{Reach}(\uparrow_1) \), with \( \deg_\equiv_1(s_1) = b_1 \), and a transition system \( T_2 = (S_2, \rightarrow_2, \uparrow_2) \), satisfying that \( \deg_\equiv_2(T_2) = b_2 < b_1 \), and \( T_1 \xrightarrow{b} T_2 \). Since there is no divergence in \( T_1 \), there exists a state \( s'_1 \) such that \( s_1 \xrightarrow{r} s'_1 \), \( s_1 \xrightarrow{b} s'_1 \), and there is no transition \( s'_1 \xrightarrow{r} s''_1 \) with \( s'_1 \xrightarrow{b} s''_1 \).

Since \( T_1 \xrightarrow{b} T_2 \), there is a state \( s_2 \in \text{Reach}(\uparrow_2) \), such that \( s_1 \xrightarrow{b} s_2 \). And \( T_2 \) does not contain a divergence as it preserves non-divergence with \( T_1 \). Therefore, every state-preserving \( \tau \) transition sequence in \( T_2 \) terminates. Without loss of generality, we assume that \( s'_2 \) is a state, such that \( s_2 \xrightarrow{\tau} s'_2 \), \( s'_2 \xrightarrow{b} s_2 \), and there is no transition \( s'_2 \xrightarrow{\tau} s''_2 \) with \( s'_2 \xrightarrow{b} s''_2 \).

On the other hand, \( \deg_\equiv_2(s'_2) \leq b_2 < \deg_\equiv_1(s'_1) \), and \( s'_1 \) and \( s'_2 \) do not have any state-preserving \( \tau \) transitions. We now show that \( s'_1 \xrightarrow{b} s'_2 \) only if they \( \deg_\equiv_1(s'_1) = \deg_\equiv_2(s'_2) \). Since the bisimulation requires a one-to-one correspondence between the elements from the sets \( \{(a, [s''_1]_\equiv) \mid s'_1 \xrightarrow{a} s'', a \in \mathcal{A}_T \} \) and \( \{(a, [s''_2]_\equiv) \mid s'_2 \xrightarrow{a} s'', a \in \mathcal{A}_T \} \). It follows that \( s'_2 \xrightarrow{b} s'_1 \), which leads to a contradiction.

Thus we conclude the following theorem from Theorem[11] and Lemma[2].

Theorem 2. If a transition system \( T \) has no divergence and is unboundedly branching up to \( \equiv \), then it is not executable modulo \( \equiv \).

2.2 \( \pi \)-Calculus

The \( \pi \)-calculus was proposed by Milner, Parrow and Walker in [13] as a language to specify processes with link mobility. The expressiveness of many variants of \( \pi \)-calculus has been extensively studied. In this paper, we shall consider the basic version presented in [13], excluding the match prefix. We recapitulate some definitions from [13] below and refer to the book for detailed explanations.
We presuppose a countably infinite set $\mathcal{N}$ of names; we use strings of lower case letters for elements of $\mathcal{N}$. The prefixes, processes and summations of the $\pi$-calculus are, respectively, defined by the following grammar:

$$
\pi := \overline{x}y \mid x(z) \mid \tau \quad (x, y, z \in \mathcal{N})
$$

$$
P := M \mid P \mid P \mid (z)P \mid !P
$$

$$
M := 0 \mid \pi.P \mid M + M
$$

In $x(z).P$ and $(z)P$, the displayed occurrence of the name $z$ is binding with scope $P$. An occurrence of a name in a process is bound if it is, or lies within the scope of, a binding occurrence in $P$; otherwise it is free. We use $\text{fn}(P)$ to denote the set of names that occur free in $P$, and $\text{bn}(P)$ to denote the set of names that bound in $P$.

We define the operational behaviour of a $\pi$-processes by means of the structural operational semantics in Fig. 1 in which $\alpha$ ranges over the set of $\pi$-calculus actions

$$
\mathcal{A}_\pi = \{xy, \overline{x}y, \overline{\tau}(z) \mid x, y, z \in \mathcal{N} \cup \{\tau\}
$$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Syntax</th>
<th>Meaning</th>
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<tbody>
<tr>
<td><strong>PREFIX</strong></td>
<td>$\tau.P \xrightarrow{\tau} P$</td>
<td>$x(z).P \xrightarrow{\overline{x}y} P$</td>
</tr>
<tr>
<td><strong>SUM</strong></td>
<td>$P \underline{\alpha} P'$</td>
<td>$P + Q \xrightarrow{\underline{\alpha}} P'$</td>
</tr>
<tr>
<td><strong>PAR</strong></td>
<td>$P \xrightarrow{\alpha} P'$</td>
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<td><strong>REP</strong></td>
<td>$P \xrightarrow{\alpha} P'$</td>
<td>$P</td>
</tr>
</tbody>
</table>

**Fig. 1**: Operational rules for $\pi$-calculus

The rules in Fig. 1 define on $\pi$-terms an $\mathcal{A}_\pi$-labelled transition relation $\rightarrow$. Then, we can associate with every $\pi$-term an $\mathcal{A}_\pi$-labelled transition system $\mathcal{T}(P) = (S_P, \rightarrow_P, P)$. The set of states $S_P$ of $\mathcal{T}(P)$ consists of all $\pi$-terms reachable from $P$, the transition relation $\rightarrow_P$ of $\mathcal{T}(P)$ is obtained by restricting the transition relation $\rightarrow$ defined by the structural operational rules to $S_P$ (i.e., $\rightarrow_P = \rightarrow \cap (S_P \times \mathcal{A}_\pi \times S_P)$), and the initial state of $\mathcal{T}(P)$ is the $\pi$-term $P$.

An $\alpha$-convention between $\pi$-terms is defined in [15] as a finite number of changes of bound names. In this paper, we do not distinguish with $\pi$-terms which are $\alpha$-convertible, and we write $P = Q$ if $P$ and $Q$ are $\alpha$-convertible.

For convenience, we sometimes want to abbreviate interactions which involve the transmission of no name at all, or more than one name. Instead of giving a full treatment
of the polyadic $\pi$-calculus (see [13]), we define the following abbreviations:

$$\overline{x}(y_1, \ldots, y_n).P \overset{d}{=} (w)\overline{x}w.\overline{y}_1, \ldots, \overline{y}_n.P \quad (w \notin fn(P)),$$

and

$$x(z_1, \ldots, z_n).P \overset{d}{=} x(w(z_1), \ldots, w(z_n)).P.$$

The following lemma establishes that divergence-preserving branching bisimilarity is compatible with restriction and parallel composition. This will be a useful property when establishing the correctness of our simulation of RTMs in $\pi$-calculus, in the next section.

**Lemma 3.** For all $\pi$-terms $P, P', Q,$ and $Q'$:

1. if $P \leftrightarrow^a P'$, then $(a)P \leftrightarrow^a (a)P'$;
2. if $P \leftrightarrow^d P'$ and $Q \leftrightarrow^d Q'$, then $P \mid Q \leftrightarrow^d P' \mid Q'$.

**Proof.**

1. It is straightforward to verify that the relation

$$\mathcal{R} = \{((a)P, (a)P') \mid P \text{ and } P' \text{ are } \pi\text{-terms s.t. } P \leftrightarrow^d P'\}$$

is a divergence-preserving branching bisimulation relation.

2. Define the relation $\mathcal{R}$ by

$$\{(a)(P \mid Q), (a)(P' \mid Q') \mid a \text{ is a sequence of restricted names and }$$

$$P, P', Q, \text{ and } Q' \text{ are } \pi\text{-terms s.t. } P \leftrightarrow^d P' \text{ and } Q \leftrightarrow^d Q' \} ;$$

we verify that $\mathcal{R}$ is a divergence-preserving branching bisimulation. For simplicity, we only analyze the terms with no restricted names. To this end, first suppose that $P \mid Q \overset{a}{\to} R$; we distinguish three cases according to which operational rule is applied last in the derivation of this transition:

(a) if $P \overset{a}{\to} P_1$ and $R = P_1 \mid Q$, then, since $P \leftrightarrow^d P'$, we have $P' \overset{a}{\to} P'' \overset{\alpha}{\to} P'_1$ with $P'' \leftrightarrow^d P'_1$ and $P_1 \leftrightarrow^d P'_1$. Thus $P' \mid Q' \overset{a}{\to} P'' \mid Q' = P'' \mid Q$; and, according to the definition of $\mathcal{R}$, we have $P \mid Q \mathcal{R} P'' \mid Q$ and $P_1 \mid Q \mathcal{R} P'_1 \mid Q$.

(b) If $P \overset{\pi}{\to} P_1, Q \overset{\tau}{\to} Q_1, R = P_1 \mid Q_1$, and $a = \tau$, then, since $P \leftrightarrow^d P'$ and $Q \leftrightarrow^d Q'$, we have $P' \overset{\tau}{\to} P'_1 \overset{\pi}{\to} P'_1$ with $P \leftrightarrow^d P'_1$ and $P_1 \leftrightarrow^d P'_1$, and $Q' \overset{\tau}{\to} Q'_1 \overset{\pi}{\to} Q'_1$ with $Q \leftrightarrow^d Q'_1$ and $Q_1 \leftrightarrow^d Q'_1$. Hence, it follows that $P \mid Q \overset{\tau}{\to} P'_1 \mid Q'_1 \overset{\pi}{\to} P'_1 \mid Q'_1$, with $P \mid Q \mathcal{R} P'_1 \mid Q'_1$ and $P_1 \mid Q_1 \mathcal{R} P'_1 \mid Q'_1$.

(c) If $P \overset{\pi(e)}{\to} P_1, Q \overset{\tau}{\to} Q_1, R = (e)(P_1 \mid Q_1)$, and $a = \tau$. Since $P \leftrightarrow^d P'$ and $Q \leftrightarrow^d Q'$, we have $P' \overset{\tau(e)}{\to} P'_1 \overset{\pi(e)}{\to} P'_1$ with $P \leftrightarrow^d P'_1$ and $P_1 \leftrightarrow^d P'_1$, and $Q' \overset{\tau(e)}{\to} Q'_1 \overset{\pi(e)}{\to} Q'_1$ with $Q \leftrightarrow^d Q'_1$ and $Q_1 \leftrightarrow^d Q'_1$. Hence, it follows that $P \mid Q' \overset{\tau(e)}{\to} P'' \mid Q'' \overset{\pi(e)}{\to} (e)(P'_1 \mid Q'_1)$, with $P \mid Q \mathcal{R} P'_1 \mid Q'_1$ and $(e)(P_1 \mid Q_1) \mathcal{R} (e)(P'_1 \mid Q'_1)$.

The symmetric cases can be proved analogously.
3 Specifying executable behaviour in the $\pi$-Calculus

In the previous section, we introduced $\pi$-calculus as a language to specify behaviour, and we have proposed RTMs to define a notion of executable behaviour. In this section we prove that every executable behaviour can be specified in the $\pi$-calculus up to divergence-preserving branching bisimilarity. To this end, we associate with every RTM $M$ a $\pi$-term $P$ that simulates the behaviour of $M$ up to divergence-preserving branching bisimilarity, i.e., such that $T(M) \leftrightarrow^b T(P)$. Our specification will consist of two parts: a generic finite specification of the behaviour of a tape, and a finite specification of a control process that is specific for the RTM $M$ under consideration. In the end, we establish that the RTM $M$ is simulated by the parallel composition of the two parts. The structure of our specification is illustrated in Figure 2. In this figure, each node represents a parallel component of the specification, and each labelled arrow stands for a communication channel with certain labels, and the dashed lines represent the links between cells. Moreover, the equalities on arrows and dashed lines tell the correspondence between the names defined in the linked terms.

Fig. 2. $\pi$-specification of a RTM

Tape In [1], the behaviour of the tape of a Turing machine is finitely specified in ACP, making use of finite specifications of two stacks. The specification is not easily modified to take intermediate termination into account, and therefore, in [3], an alternative solution is presented, specifying the behaviour of a tape in TCP by using a finite specification of a queue (see also [2]). In this paper, we will exploit the link passing feature of the $\pi$-calculus to give a more direct specification. In particular, we shall model the tape as a collection of cells endowed with a link structure that organises them in a linear fashion. We first give an informal description of the behaviour of a tape. The state of a tape is characterised by a tape instance $\delta_L\delta_R$, consisting of a finite (but unbounded) sequence with the current position of the tape head indicated by a marking. The tape may then exhibit the following observable actions:

1. **read**: the datum under the tape head is output along the channel read;
2. **write(e)**: a datum $e$ is written on the position of the tape head, resulting in a new tape instance $\delta_L\delta_R$; and
3. *left, right:* results in the tape head moving left or right, resulting in $\delta_L d \delta_R$ or $\delta_L d' \delta_R$, respectively.

Henceforth, we assume that tape symbols are included in the set of names, i.e., that $D_\subseteq N$.

In our $\pi$-calculus specification representing the behaviour of a tape each individual tape cell is specified as a separate component, and there is a separate component modelling the tape head. A tape cell stores a datum $d$, represented by a free name in the specification, it has pointers $l$ and $r$ to its left and right neighbour cells. Furthermore, it has two links to the component modelling the tape head: the link $u$ is used by the tape head for updating the datum, and the link $t$ serves as a general communication channel for communicating all relevant information about the cell to the tape head. The following $\pi$-term represents the behaviour of a tape cell:

$$C \defeq c(t, l, r, u, d).C(t, l, r, u, d)$$
$$C(t, l, r, u, d) \defeq u(e)C(t, l, r, u, e) \circ 7(l, r, u, d).C(t, l, r, u, d) \circ 0 .$$

Note that the behaviour of an individual tape cell $C(t, l, r, u, d)$ is very simple: either it receives along channel $u$ an update $e$ for its datum $d$, after which it recreates itself with datum $e$ in place of $d$, or it outputs all relevant information about itself to the tape head along channel $t$, after which it recreates itself. A cell is created by a synchronisation on name $c$, by which all relevant information about the cell is passed; we shall have a component $!C$ to regenerate new incarnations of existing tape cells.

At any stage during the simulation of an RTM, the number of tape cells modelled by a parallel component will be finite. To model the unbounded nature of the tape, we define a process $B$ that serves to generate new blank tape cells on either side of the tape whenever needed:

$$B \defeq b(t, r).B(t, l, r, u) + b(t, l).B(t, l, r, u)$$
$$B(t, l, r, u) \defeq 7(l, r, u, \square).\overline{C}(t, l, r, u, \square) \circ 0 . \quad B(t, l, r, u) \defeq 7(l, r, u, \square).\overline{C}(t, l, r, u, \square) \circ 0 .$$

Note that $B$ offers the choice to either create a blank tape cell at the left-hand side of the tape through $b(t, r)$, or a blank tape cell at the right-hand side of the tape through $b(t, l)$. In the first case, the cell to the right of the new blank cell already exists, and it maintains a link to itself and a link to its left neighbour; these links are passed on to the new blank cell as $r$ and $t$, respectively. At the creation of the new blank cell, two new links are generated: $u$ is the update channel of the new blank cell, and $l$ will later be used as the link to another neighbour blank cell.

Throughout the simulation of an RTM, the number of parallel components modelling individual tape cells will grow. We shall presuppose a numbering of these parallel components with consecutive integers from some interval $[m, n]$ ($m$ and $n$ are integers such that $m \leq n$), in agreement with the link structure. The numbering is reflected by a naming scheme that adds the subscript $i$ to the links $t, l, r, u$ and $d$ of the $i$th cell; we abbreviate $C(t, l, r, u, d)$ by $C_i$, $B(t, l, r, u, d)$ by $B_i$, and $B(t, l, r, u, d)$ by $B_{i,l}$ and $B_{i,r}$, and
Furthermore, we shall abbreviate
\[ H \overset{\text{def}}{=} h(t, l, r, u, d).H(t, l, r, u, d) \]

The component modelling the tape head serves as the interface between the tape cells and the RTM-specific control process. It is defined as:
\[ H(t, l, r, u, d) \overset{\text{def}}{=} \text{read} d.H(t, l, r, u, d) + \text{write}(e).\overline{e} \cdot \overline{h}(t, l, r, u, e).0 + \text{left} (l', r', u', d').H(l, l', r', u', d').0 + \text{right} (l', r', u', d').H(r, l', r', u', d').0 \]

The tape head maintains two links to the current cell (a communication channel \( t \) and an update channel \( u \)), as well as links to its left and right neighbour cells (\( l \) and \( r \), respectively). Furthermore, the tape head remembers the datum \( d \) in the current cell. The datum \( d \) may be outputted along the \( \text{read} \)-channel. Furthermore, a new datum \( e \) may be received through the \( \text{write} \)-channel, which is then forwarded through the update channel \( u \) to the current cell. Finally, the tape head may receive instructions to move left or right, which has the effect of receiving information about the left or right neighbours of the current cell through \( l \) or \( r \), respectively. In all cases, a new incarnation of the tape head is started, with a call on the \( h \)-channel.

Let \( t_{[m,n]} = t_m, t_{m+1}, \ldots, t_n \), and \( u_{[m,n]} = u_m, u_{m+1}, \ldots, u_{n-1}, u_n \) to denote restricted names. Furthermore, let \( H_l = H(t, l, r, u, d_l) \). Then, with \( d_{[m,n]} = d_m, d_{m+1}, \ldots, d_{n-1}, d_n \), we define,
\[ \text{Tape}^0_{[l,n]}(d_{[m,n]}) \overset{\text{def}}{=} (t_{[m,n]}, u_{[m,n]})((h(H_l) | !H) | Cells_{[m,n]}) \]

Furthermore, we shall abbreviate \( \text{Tape}^0_{[l,n]}(\square) \) by \( \text{Tape} \). The following lemma is about the behaviour of our tape specification:

**Lemma 4.** There are four types of transitions from \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \):

1. \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{read} d_j} \xrightarrow{\text{write} d_j} \text{Tape}^j_{[m,n]}(d_{[m,n]}) \)
2. \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{write}(e)} \xrightarrow{\text{left} d_j} \text{Tape}^{j-1}_{[m,n]}(d_{[m,n]}) \) (if \( i > m \))
3. \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{left} d_j} \text{Tape}^{j-1}_{[m,n]}(d_{[m,n]}) \square, d_{[m,n]} \) (if \( i = m \))
4. \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{right} d_j} \xrightarrow{\text{right} d_j} \text{Tape}^{j+1}_{[m,n]}(d_{[m,n]}), d_{[m,n]} \) (if \( i < n \))
5. \( \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{right} d_j} \text{Tape}^{j+1}_{[m,n+1]}(d_{[m,n]}, \square) \) (if \( i = n \))

**Proof.** Here we only verify the first case. We suppose that
\[ \text{Tape}^j_{[m,n]}(d_{[m,n]}) \xrightarrow{\text{read} d_j} (t_{[m-1,n+1]}, u_{[m,n]})((h(H_l, l, r, u, d_j) | !H) | Cells_{[m,n]}) = T'. \]

It is obvious that \( (h(H_l, l, r, u, d_j) | !H) \xrightarrow{\text{write} d_j} (h(H_l) | !H) \). According to Lemma 3, we have \( T' \xrightarrow{\text{write} d_j} \text{Tape}^j_{[m,n]}(d_{[m,n]}) \).
Finite control We associate with every RTM $M = (S_M, \rightarrow_M, \uparrow_M)$ a finite specification consisting of a tape instance and a control process. Here $m$ can be left or right.

\[
S \equiv \sum_{s \in S} \sum_{d \in D_b} d.S_{s,d}
\]

\[
S_{s,d} \equiv \sum_{(s,d,a,e,m) \in \rightarrow_M} a.\text{write } e.\overline{m}.\text{read}(f).\overline{t}.\overline{f}.0
\]

Let $s = s_1, s_2, \ldots, s_m$, where $s_1, s_2, \ldots, s_m \in S_M$, and $e = e_1, e_2, \ldots, e_n$, where $e_1, e_2, \ldots, e_n \in D_b$, then we define

\[
\text{Control}_{s,d} \equiv (s, e)(S_{s,d} | !S) .
\]

And the control process corresponding to the initial state is Control$_{1,2}$. The following lemma illustrates the behaviour of control process.

**Lemma 5.** Given an RTM $M = (S_M, \rightarrow_M, \uparrow_M)$, for every transition rule $(s, d, a, e, m, t) \in \rightarrow_M$, we have the following corresponding transition sequence:

\[
\text{Control}_{s,d} \xrightarrow{a} (s, e)(\text{write } e.\overline{m}.\text{read}(f).\overline{t}.\overline{f}.0 | !S) \xrightarrow{\text{write } e.\overline{m}.\text{read } f} \xrightarrow{\text{write } e.\overline{m}.\text{read } f} \cdots \xrightarrow{\text{write } e.\overline{m}.\text{read } f} \text{Control}_{s,d} .
\]

Finally, for a given RTM $M$, we associate with every configuration $(s, \delta_L, \delta_R)$ a $\pi$-term $M_{s,\delta_L,\delta_R}$, which is a parallel composition of the specifications of its tape instance and control process. Let $r = \text{read}, \text{write}, \text{left}, \text{write}$, and we define

\[
M_{s,\delta_L,\delta_R} = (r)(\text{Control}_{s,d} | \text{Tape}^i_{[m,n]}(d_{[m,n]})),
\]

where $d_{[m,n]} = \delta_L \delta_R$.

The following lemma concludes the behaviour of $M_{s,\delta_L,\delta_R}$.

**Lemma 6.** Given an RTM $M = (S_M, \rightarrow_M, \uparrow_M)$, for every configuration $(s, \delta_L, \delta_R)$, its transition $(s, \delta_L, \delta_R) \xrightarrow{a} (t, \delta_L', \delta_R')$ corresponds to a transition

\[
M_{s,\delta_L,\delta_R} \xrightarrow{a} \xrightarrow{\pi} M_{t,\delta_L',\delta_R'} .
\]

**Proof.** Suppose the transition is resulted from the rule $(s, d, a, e, m, t) \in \rightarrow_M$, then according to Lemma 5, we have

\[
M_{s,\delta_L,\delta_R} \xrightarrow{a} (r, s, e)(\text{write } e.\overline{m}.\text{read}(f).\overline{t}.\overline{f}.0 | !S | \text{Tape}^i_{[m,n]}(d_{[m,n]})) = M' .
\]

Then we just prove that

\[
M' \xrightarrow{\pi} (r, s, e)(\overline{m}.\text{read}(f).\overline{t}.\overline{f}.0 | !S | \text{Tape}^i_{[m,n]}(d_m, \ldots, d_{i-1}, e, d_{i+1}, \ldots, d_n)) = M'' .
\]

Using Lemma 5, we get

\[
M' \xrightarrow{\tau} (r, s, e)(\overline{m}.\text{read}(f).\overline{t}.\overline{f}.0 | !S | T') ,
\]

where $T' \xrightarrow{a} \text{Tape}^i_{[m,n]}(d_m, \ldots, d_{i-1}, e, d_{i+1}, \ldots, d_n)$. Thus $M' \xrightarrow{\pi} M''$ according to Lemma 5.

Hence, we can prove the whole statement by the above method.
Theorem 3. Given an RTM $M$, and let $M(\uparrow, \tilde{\Box})$ be the initial configuration of $M$, we have

$$\mathcal{T}(M(\uparrow, \tilde{\Box})) \equiv_b \mathcal{T}(M(\uparrow, \Box)).$$

Proof. We establish a relation $R = \{(M,s,\delta_L,\tilde{\delta}_R, M(s,\delta_L,\tilde{\delta}_R)) \mid s \in S_M, \delta_L,\tilde{\delta}_R \in D\}$. Using Lemma 6, it is enough to show that $R$ is a branching bisimulation up to $\equiv_b$ (see also Figure 3). Thus we get $\mathcal{T}(M(\uparrow, \tilde{\Box})) \equiv_b \mathcal{T}(M(\uparrow, \Box))$ by Lemma 1.

Fig. 3. Bisimulation relation between $M \in M(s,\delta_L,\tilde{\delta}_R)$ and $M(s,\delta_L,\tilde{\delta}_R)$

Thus we have the following expressiveness result for $\pi$-calculus.

Corollary 1. For every executable transition system $T$ there exists a $\pi$-term $P$, such that $T \equiv_b \mathcal{T}(P)$.

4 Executability of $\pi$-Calculus Process

We have proved that every executable behaviour can be specified in the $\pi$-calculus modulo divergence-preserving branching bisimilarity. We shall now investigate to what extent behaviour specified in the $\pi$-calculus is executable. Recall that we have defined executable behaviour as behaviour of an RTM. So, in order to prove that the behaviour specified by a $\pi$-term is executable, we need to show that the transition system associated with this $\pi$-term is behaviourally equivalent to the transition system associated with some RTM.

Note that there is an apparent mismatch between the formalisms of RTMs and $\pi$-calculus. On the one hand, the notion of RTM as we have defined it presupposes finite sets $A_\tau$ and $D_\Box$ of actions and data symbols, and also the transition relation of an RTM is finite. As a consequence, we have observed, the transition system associated with an RTM is finitely branching, and, in fact, its branching degree is bounded by a natural number. (Note that this does not mean that RTMs cannot deal with data of unbounded size, but it has to be encoded using finitely many symbols.) The $\pi$-calculus, on the other hand, presupposes an infinite set of names by which an infinite set $A_\pi$ is generated. The transition system associated to a $\pi$-term by the structural operational semantics (see
Fig. 1 may contain states with an infinite branching degree, due to the rules for input prefix and bound output prefix.

Observe, however, that the infinite branching caused by input prefix can be thought of as a technical device in the operational semantics to model the communication of an arbitrary name from one parallel component to another. The name that will be received, can either be a free name of the sending process (a value), or of a restricted name (a private channel). Since the sending parallel component will only have a finite number of free names, only finitely many values can be communicated. Although, technically speaking, according to the operational semantics, infinitely many distinct private channels may be communicated when an input prefix synchronises with a bound output prefix, the communicated private channel is not observable, and the resulting \( \pi \)-terms are all equated by \( \alpha \)-conversion, so the only observable effect of the interaction is that after the communication the sending and receiving parties share a private channel of which the name is irrelevant.

Our goal is to investigate to what extent the behaviour specified by an individual \( \pi \)-term is executable. Motivated by the above intuitive interpretation of interaction of a \( \pi \)-term with its environment, we assume that the behaviour specified by that \( \pi \)-term is executed in an environment that may offer data values from some presupposed finite set on its input channels. Furthermore, we assume that there is a facility for establishing a private channel between the \( \pi \)-term and its environment. (Such a facility could, e.g., be implemented using encryption, but we will abstract from the actual implementation of the facility.) We define a restriction on the transition systems associated with \( \pi \)-terms that is based on these assumptions.

**Definition 8.** Let \( N' \subseteq N \) be a finite set of names, let \( \mathcal{A}_x' = \mathcal{A}_x - (\{xy \mid x \in N, y \notin N'\} \cup \{xz \mid x, z \in N\}) \), and let \( P \) be \( \pi \)-term. The transition system associated with \( P \) restricted to a finite set of input names \( N' \), denoted by \( T^{N'}(P) \), is a triple \( (S_{P}, \rightarrow_{N'}^{P}, P) \), obtained from \( T(P) = (S_{P}, \rightarrow_{P}, P) \) as follows:

1. \( S_{P}^{N'} \) is the set of states reachable from \( P \) by means of transitions that are not labelled by \( xz \) (\( z \notin N' \)); and
2. \( \rightarrow_{P}^{N'} \) is the restriction of \( \rightarrow_{P} \) obtained by excluding all transitions labelled with \( xz \) (\( z \notin N' \)), and relabelling all transitions labelled with \( \overline{z} $$ (x, z \in N) $$ to \( \nu x $$, i.e.,

\[
\rightarrow_{P}^{N'} = (\rightarrow_{P} \cap (S_{P}^{N'} \times \mathcal{A}_x' \times S_{P}^{N'})) \cup \{ (s, \nu x, t) \mid s, t \in S_{P}^{N'}, s \rightarrow_{P} t \}.
\]

Using [15, Lemma 1.4.1], it is straightforward to show that for every \( \pi \)-term the set of \( \pi \)-calculus actions appearing as labels in \( T^{N'}(P) \) is finite. Furthermore, the transition system associated with \( \pi \)-term by the operational semantics, and also its restriction according to Definition 8, are clearly effective. Hence, as an immediate corollary of Theorem 1(3), we may conclude that the transition system associated with a \( \pi \)-term can be simulated by an RTM modulo (divergence-insensitive) branching bisimilarity.

**Corollary 2.** For every \( \pi \)-term \( P \) and for every finite set of input names \( N' \), there exists an RTM \( M \) such that \( T^{N'}(P) \equiv_{b} T(M) \).
The following example shows that there exist $\pi$-terms with which the structural operational semantics associates a transition system without divergence that is unboundedly branching up to $\leftrightarrow^b$. Note that by Theorem 2 such $\pi$-terms are not executable modulo divergence-preserving branching bisimilarity.

Example 1. Consider the $\pi$-process

$$P \equiv (c, i, d, \text{flip})(\overline{t} \cdot s \cdot 0 | \text{flip} \cdot 0 | !C \mid !I \mid !D)$$

with $C$, $I$ and $D$ defined as follows:

- $C \equiv c(h, t, b)(\overline{h}(t, b) \cdot 0 + \text{flip} \cdot h \cdot t \cdot 1 \cdot 0)$
- $I \equiv i(h)(\text{inc} \cdot (h' \cdot \overline{c}(h', h, 0) \cdot \overline{t} h' \cdot 0 + \text{flush} \cdot \text{flip} \cdot d h \cdot 0)$
- $D \equiv d(h)(h(t, b) \cdot b \cdot d t \cdot 0)$

Intuitively, the process $!C$ facilitates the generation of a linked list of one-bit cells with a pointer $h$ to the head of the list, a pointer $t$ to the tail of the list, and a bit $b$. Each cell may either output, along $h$, the link $t$ to the tail of the list and its bit $b$, or it may receive the instruction $\text{flip}$ after which it recreates itself with the value 1. The process $I$ serves as the interface process. It maintains a link to the head of the list. Upon receiving an $\text{inc}$-instruction, it adds another one-bit cell to the list, and upon receiving the $\text{flush}$-instruction, it flips at most one of the bits, and then calls $D$. The process $D$ then simply outputs the bits in reverse sequence.

Consider the state reached after performing $n$ $\text{inc}$-actions, followed by a $\text{flush}$-action. In this state, the list contains a string of $n$ 0s. The $\tau$-transitions that correspond to the interaction of $\text{flip}$ between $I$ and one of the $\text{flips}$ of one of the one-bit cells or $\text{flip}$ in the definition of $P$ have the effect of non-deterministically changing (at most) one of the 0s to a 1. Note that there are $n+1$ such $\tau$-transitions, and since $D$ will subsequently output the sequence in order, the states reached by these $\tau$-transitions are (pairwise) not divergence-preserving branching bisimilar. Hence, it follows that for every $n$, the transition system associated with $P$ has reachable state with a branching degree modulo $\leftrightarrow^b$ of at least $n+1$. It follows that the transition system associated with $P$ is unboundedly branching up to $\leftrightarrow^b$.

We combine the positive result in Corollary and the observations in the preceding example in a general conclusion about the executability of $\pi$-calculus.

Corollary 3. Every $\pi$-term is executable modulo branching bisimilarity, but there exist $\pi$-terms that specify transition systems that are not executable modulo divergence-preserving branching bisimilarity.

5 Conclusions

We have investigated the expressiveness of the $\pi$-calculus in relation to the theory of executability provided by reactive Turing machines. We have established that, up to divergence-preserving branching bisimilarity, every executable transition system can be specified by a $\pi$-calculus term, showing that the $\pi$-calculus is reactively Turing powerful. On the other hand, we have proved that every $\pi$-calculus process can be simulated
by a reactive Turing machine modulo branching bisimilarity, at the cost of introducing
divergence in the simulation. Divergence is then used to simulate behaviour that is un-
boundedly branching. It has been claimed (e.g., in [5]) that $\pi$-calculus provides a model
of computation that is behaviourally more expressive than Turing machines. Our results
provide further justification for this claim, and formally characterises the difference.
It should be noted that the difference in expressive power is at the level of interaction
(allowing interaction between an unbounded number of components), rather than at the
level of computation.

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