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# Explicit Optimal Binary Pebbling for One-Way Hash Chain Reversal

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**Abstract.** We present explicit optimal binary pebbling algorithms for reversing one-way hash chains. For a hash chain of length  $2^k$ , the number of hashes performed per output round is at most  $\lceil \frac{k}{2} \rceil$ , whereas the number of hash values stored throughout is at most  $k$ . This is optimal for binary pebbling algorithms characterized by the property that the midpoint of the hash chain is computed just once and stored until it is output, and that this property applies recursively to both halves of the hash chain.

We develop a framework for easy comparison of explicit binary pebbling algorithms, including simple speed-1 binary pebbles, Jakobsson’s binary speed-2 pebbles, and our optimal binary pebbles. Explicit schedules describe for each pebble exactly how many hashes need to be performed in each round. The optimal schedule exhibits a nice recursive structure, which allows fully optimized implementations that can readily be deployed. In particular, we develop in-place implementations with minimal storage overhead (essentially, storing only hash values), and fast implementations with minimal computational overhead.

## 1 Introduction

Originally introduced by Lamport to construct an identification scheme resisting eavesdropping attacks [Lam81], one-way hash chains have become a truly fundamental primitive in cryptography. The idea of Lamport’s identification scheme is to generate a hash chain as a sequence of successive iterates of a one-way hash function applied to a random seed value, revealing only the last element of the hash chain upon registration. Later, during successive rounds of identification, the remaining elements of the hash chain are output in reverse order, one element at a time.

Due to the one-way property of the hash function, efficient reversal of a hash chain is non-trivial for long chains. In 2002, Jakobsson introduced a simple and efficient pebbling algorithm for reversal of one-way hash chains [Jak02], building on the pebbling algorithm of [IR01] for efficient key updates in a forward-secure digital signature scheme. Pebbling algorithms for one-way hash chain reversal strike a balance between storage requirements (measured as the number of hash values stored) and computational requirements (measured as the number of hashes performed). The performance constraint is that each next element of

the reversed hash chain should be produced within a limited amount of time after producing the preceding element. For a hash chain of length  $n = 2^k$ , Jakobsson’s algorithm stores  $O(\log n)$  hash values only and the number of hashes performed in each round is  $O(\log n)$  as well.

The problem of efficient hash chain reversal was extensively studied by Copersmith and Jakobsson [CJ02]. They proved nearly optimal complexity for a binary pebbling algorithm storing at most  $k + \lceil \log_2(k + 1) \rceil$  hash values and performing at most  $\lfloor \frac{k}{2} \rfloor$  hashes per round. Later, it was observed by Yum et al. that a greedy implementation of Jakobsson’s original algorithm actually stores no more than  $k$  hash values, requiring no more than  $\lceil \frac{k}{2} \rceil$  hashes per round [YSEL09].

In this paper we consider the class of *binary* pebbling algorithms, covering the best algorithms of [Jak02,CJ02,YSEL09] among others. A binary pebbling algorithm is characterized by the property that the *midpoint* of the hash chain is computed just once and stored until it is output; moreover, this property applies recursively to both halves of the hash chain. In particular, this means that after producing the last element of the hash chain as the first output, a binary pebbling algorithm stores (at least) the  $k$  elements at distances  $2^i - 1$ , for  $1 \leq i \leq k$ , from the end of the hash chain.

We introduce a simple yet general framework for efficient binary pebbling algorithms for hash chain reversal, and we completely resolve the case of binary pebbling by constructing an *explicit* optimal algorithm. The storage required by our optimal algorithm does not exceed the storage of  $k$  hash values and the number of hashes performed in any output round does not exceed  $\lceil \frac{k}{2} \rceil$ . This matches the performance of the greedy algorithm of [YSEL09], which is an optimal binary pebbling algorithm as well. However, we give an exact schedule for all hashes performed by the algorithm (rather than performing these hashes in a greedy fashion).

Our optimal schedule is defined explicitly, both as a recursive definition and as a closed formula, specifying exactly how many hashes should be performed in a given round by each pebble. Apart from the mathematically appealing structure thus uncovered, the explicit optimal schedule enables the development of fully optimized solutions for one-way hash chain reversal. We show the first in-place (or, *in situ*) hash chain reversal algorithms which require essentially no storage beyond the hash values stored for the pebbles; at the same time, the computational overhead for each round is limited to a few basic operations only beyond the evaluation of the hash function. Finally, as another extreme type of solution, we show how to minimize the computational overhead to an almost negligible amount of work, at the expense of increased storage requirements.

Concretely, for hash chains of length  $2^{32}$  using a 128-bit one-way hash, our in-place algorithm only stores 516 bytes (32 hash values and one 32-bit counter) and performs, at most 16 hashes per round. We note that our results may be especially interesting in the context of post-quantum cryptography, as the security of one-way hash chains is not affected dramatically by the potential of quantum computers.

## 2 One-Way Hash Chains

Throughout, we use the following notation for finite sequences. We write  $A = \{a_i\}_{i=1}^n = \{a_1, \dots, a_n\}$  for a sequence  $A$  of length  $n$ ,  $n \geq 0$ , with  $\{\}$  denoting the empty sequence. We use  $|A| = n$  to denote the length of  $A$ , and  $\#A = \sum_{i=1}^n a_i$  to denote the weight of  $A$ . We write  $A \parallel B$  for the concatenation of sequences  $A$  and  $B$ , and  $A + B$  for element-wise addition of sequences  $A$  and  $B$  of equal length, where  $+$  takes precedence over  $\parallel$ . Constant sequences are denoted by  $c = c^{*n} = \{c\}_{i=1}^n$ , suppressing the length  $n$  when it is understood from context; e.g.,  $A + c$  denotes the sequence obtained by adding  $c$  to all elements of  $A$ .

Let  $f$  be a cryptographic hash function. The length- $2^k$  (one-way) hash chain  $f_k^*(x)$  for seed value  $x$  is defined as the following sequence:

$$f_k^*(x) = \{f^i(x)\}_{i=0}^{2^k-1}.$$

For authentication mechanisms based on hash chains, we need an efficient algorithm for producing the sequence  $f_k^*(x)$  *in reverse*. The problem arises from the fact that computation of  $f$  in the forward direction is easy, while it is *intractable* in the reverse direction. So, given  $x$  it is easy to compute  $y = f(x)$ , but given  $y$  it is very hard to compute any  $x$  at all such that  $y = f(x)$ . For long hash chains the straightforward solutions of either (i) storing  $f_k^*(x)$  and reading it out in reverse or (ii) computing each element of  $f_k^*(x)$  from scratch starting from  $x$  are clearly too inefficient.

## 3 Binary Pebbling

We introduce a framework that captures the essence of binary pebbling algorithms as follows. We define a pebble as an algorithm proceeding in a certain number of rounds, where the initial rounds are used to compute the hash chain in the forward direction given the seed value  $x$ , and the hash chain is output in reverse in the remaining rounds, one element at a time.

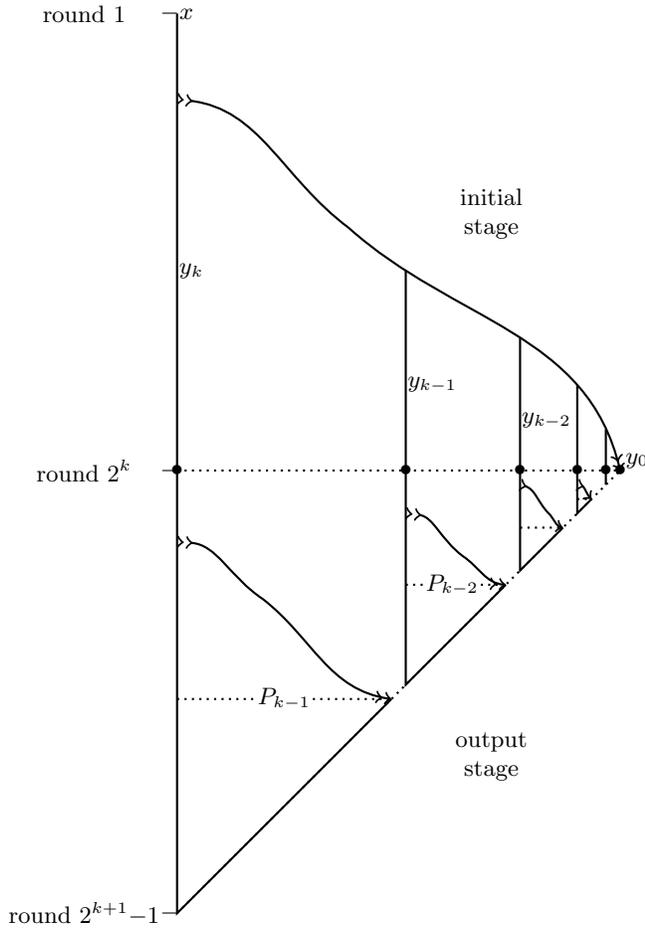
For  $k \geq 0$ , we define pebble  $P_k(x)$  below as an algorithm that runs for  $2^{k+1} - 1$  rounds in total, and outputs  $f_k^*(x)$  in reverse in its last  $2^k$  rounds. It is essential that we include the initial  $2^k - 1$  rounds in which no outputs are produced as an integral part of pebble  $P_k(x)$ , as this allows us to define and analyze binary pebbling in a fully recursive manner. In fact, in terms of a given **schedule**  $T_k = \{t_r\}_{r=1}^{2^k-1}$  with  $\#T = 2^k - 1$ , a binary pebble  $P_k(x)$  is completely specified by the following recursive definition, see also Figure 1:

**Rounds**  $[1, 2^k)$ : Compute  $\{y_i = f^{2^k-2^i}(x)\}_{i=0}^k$  using  $t_r$  hashes in round  $r$ .

**Round**  $2^k$ : Output  $y_0$ .

**Rounds**  $(2^k, 2^{k+1})$ : Run  $P_{i-1}(y_i)$  in parallel for  $i = 1, \dots, k$ .

We will refer to rounds  $[1, 2^k)$  as the **initial stage** of pebble  $P_k$  and to rounds  $[2^k, 2^{k+1})$  as its **output stage**. Running pebbles in parallel means that pebbles take turns to execute for one round each, where the order in which this happens within a round is irrelevant.



**Fig. 1.** Binary pebble  $P_k(x)$ , where  $y_i = f^{2^k - 2^i}(x)$  for  $i = k, \dots, 0$ .

The behavior of pebbles  $P_0$  and  $P_1$  is fixed since  $T_0 = \{\}$  and  $T_1 = \{1\}$ , respectively. Pebble  $P_0(x)$  runs for one round only, in which  $y_0 = x$  is output, using no hashes at all. Similarly,  $P_1(x)$  runs for three rounds, performing one hash in its first round to compute  $y_1 = x$  and  $y_0 = f(x)$ , outputting  $f(x)$  in its second round, and then running  $P_0(y_1)$  in the third round, which will output  $x$ . More generally, the following theorem shows that correct behavior follows for any pebble  $P_k$  independent of the particular schedule  $T_k$ , and furthermore that the total number of hashes performed by  $P_k$  is fixed as well.

**Theorem 1.** *Pebble  $P_k(x)$  produces  $f_k^*(x)$  in reverse in its output stage, performing  $k2^{k-1}$  hashes in total.*

**Proof** The proof is by induction on  $k$ . For  $k = 0$ , we have that  $P_0(x)$  outputs  $f_0^*(x) = x$  in its one and only round, using 0 hashes.

For  $k \geq 1$ , we see that  $P_k(x)$  first outputs  $y_0 = f^{2^k-1}(x)$  in round  $2^k$ , which is the last element of  $f_k^*(x)$ . Next, pebbles  $P_{i-1}(y_i)$  run in parallel for  $i = 1, \dots, k$ . The induction hypothesis yields that each  $P_{i-1}(y_i)$  produces  $f_{i-1}^*(f^{2^k-2^i}(x))$  in reverse in its last  $2^{i-1}$  out of  $2^i - 1$  rounds. Hence, in round  $2^k + 1$ ,  $P_0(y_1)$  outputs  $y_1 = f_0^*(f^{2^k-2}(x))$ . In the next two rounds,  $P_1(y_2)$  outputs  $f_1^*(f^{2^k-4}(x))$  in reverse. And so on until finally  $P_{k-1}(y_k)$  outputs  $f_{k-1}^*(f^{2^k-1}(x))$  in reverse in the last  $2^{k-1}$  rounds of  $P_k(x)$ . The total number of hashes performed by  $P_k$  is  $2^k - 1 + \sum_{i=1}^k (i-1)2^{i-2} = k2^{k-1}$ , using that  $P_{i-1}$  performs  $(i-1)2^{i-2}$  hashes per the induction hypothesis.  $\square$

Schedule  $T_k$  specifies the number of hashes for the initial stage of  $P_k$ . To analyze the work done by  $P_k$  in the output stage, we introduce sequence  $W_k$  of length  $2^k - 1$  to denote the number of hashes performed by  $P_k$  in each of its last  $2^k - 1$  rounds—noting that by definition no hashes are performed by  $P_k$  in round  $2^k$ . The following recurrence relation for  $W_k$  will be used throughout our analysis.

**Lemma 1.**  $W_0 = \{\}$ ,  $W_k = T_{k-1} + W_{k-1} \parallel 0 \parallel W_{k-1}$ .

**Proof** Pebble  $P_0$  runs for 1 round only, so  $W_0 = \{\}$ . For  $k \geq 1$ , we see that in the last  $2^k - 1$  rounds of pebble  $P_k$ , a pebble  $P_{k-1}$  runs in parallel to pebbles  $P_i$  for  $i = 0, \dots, k-2$ . In these rounds, pebble  $P_{k-1}$  performs  $T_{k-1} \parallel 0 \parallel W_{k-1}$  hashes, whereas pebbles  $P_i$  for  $i = 0, \dots, k-2$  perform  $W_{k-1} \parallel 0^{*2^{k-1}}$  hashes in total, as this matches the number of hashes for a pebble  $P_{k-1}$  in its last  $2^{k-1} - 1$  rounds. Hence, in total  $W_k = T_{k-1} + W_{k-1} \parallel 0 \parallel W_{k-1}$  hashes.  $\square$

We have the following lower bound for  $\max(W_k)$ , the maximal number of hashes performed by  $P_k$  in any round of the output stage. Interestingly, this lower bound holds for *any* schedule  $T_k$ . In Section 5 we will present an optimal schedule achieving the lower bound.

**Theorem 2.**  $\max(W_k) \geq \lceil k/2 \rceil$ , for  $k \geq 2$ .

**Proof** Let  $k \geq 2$  and consider the average number of hashes per round during the first half of the output stage. From Theorem 1, Lemma 1, and  $|T_{k-1}| = |W_{k-1}| = 2^{k-1} - 1$ , we have

$$\max(W_k) \geq \frac{\#T_{k-1} + \#W_{k-1}}{|T_{k-1} + W_{k-1}|} = \frac{(k-1)2^{k-2}}{2^{k-1} - 1} > \frac{k-1}{2}.$$

Hence,  $\max(W_k) \geq \lceil k/2 \rceil$ .  $\square$

To analyze the storage needed by  $P_k$  we will count the number of hash values stored by  $P_k$  at the start of each round. We introduce sequence  $S_k = \{s_r\}_{r=1}^{2^{k+1}-1}$  to denote the total storage used by  $P_k$  in each round. For instance,  $s_1 = 1$  as  $P_k$  only stores  $x$  at the start, and  $s_{2^k} = k+1$  as  $P_k$  stores  $y_0, \dots, y_k$  at the start of round  $2^k$  independent of schedule  $T_k$ .

## 4 Analysis of Speed-1 and Speed-2 Pebbles

In this section we analyze the performance of speed-1 pebbles and speed-2 pebbles. We use speed-1 pebbles to demonstrate our framework, whereas the analysis of speed-2 pebbles, which correspond to Jakobsson's original algorithm, will be used in the analysis of our optimal pebbles in the next section.

Speed-1 pebbles are defined by setting  $T_k = 1^{*2^k-1}$ , hence one hash evaluation in each initial round of  $P_k$ . To define speed-2 pebbles we set  $T_0 = \{\}$ , and  $T_k = 0^{*2^{k-1}-1} \parallel 2^{*2^{k-1}-1} \parallel 1$  for  $k \geq 1$ , hence a speed-2 pebble is idle in the first part of the initial stage and then hashes twice in each round until the end of the initial stage. As can be seen from Theorem 4 below, the storage requirements are reduced by a factor of 2 for speed-2 pebbles over speed-1 pebbles.

**Theorem 3.** *Both speed-1 and speed-2 pebbles  $P_k$  use up to  $\max(W_k) = k - 1$  hashes in any output round, for  $k \geq 1$ .*

**Proof** For a speed-1 pebble, Lemma 1 implies  $\max(W_k) = k - 1$  for  $k \geq 1$ , as all elements of  $T_{k-1}$  are equal to 1.

For a speed-2 pebble we prove by induction on  $k$  that  $\max(W_k) = k - 1$ . This clearly holds for  $k = 1, 2$ . For  $k \geq 3$ , we have, using Lemma 1,

$$\begin{aligned} T_{k-1} &= 0^{*2^{k-2}-1} \parallel 2^{*2^{k-2}-1} \parallel 1 \\ W_{k-1} &= 0 \parallel T_{k-2} + W_{k-2} \parallel 0 \parallel W_{k-2}. \end{aligned}$$

Therefore,

$$\max(W_k) = \max(T_{k-1} + W_{k-1}) = \max(W_{k-1}, 2 + W_{k-2}),$$

noting that the last element of  $W_{k-2} = 0$ . Applying the induction hypothesis twice, we conclude  $\max(W_k) = \max(k - 2, k - 1) = k - 1$ .  $\square$

**Lemma 2.**

$$\begin{aligned} S_0 &= \{1\}, \\ S_k &= (1^{*2^k} \parallel S_{k-1}) + (0 \parallel 1^{*2^{k-1}-1} \parallel S_{k-1} \parallel 0^{*2^{k-1}}), \text{ for a speed-1 } P_k, \\ S_k &= (1^{*2^k} \parallel S_{k-1}) + (0^{*2^{k-1}} \parallel S_{k-1} \parallel 0^{*2^{k-1}}), \text{ for a speed-2 } P_k. \end{aligned}$$

**Proof** Pebble  $P_0(x)$  only needs to store  $x$  during its one and only round, therefore  $S_0 = \{1\}$ . For  $k \geq 1$ , any pebble  $P_k(x)$  also needs to store  $x$  throughout all of its rounds, where pebble  $P_{k-1}(y_k) = P_{k-1}(x)$  takes over the storage of  $x$  during the output stage. This accounts for the term  $1^{*2^k} \parallel S_{k-1}$ .

In addition, a speed-1 pebble needs to store a hash value from round 2 until it reaches  $y_{k-1}$  in round  $2^{k-1}$ . From thereon, the total additional storage corresponds to running a speed-1  $P_{k-1}(y_{k-1})$  pebble. This accounts for the term  $0 \parallel 1^{*2^{k-1}-1} \parallel S_{k-1} \parallel 0^{*2^{k-1}}$ .

A speed-2 pebble needs no additional storage during its first  $2^{k-1}$  rounds. Then it needs to store an additional hash value from round  $2^{k-1} + 1$  on. By taking  $0^{*2^{k-1}} \parallel S_{k-1} \parallel 0^{*2^{k-1}}$  as additional term, we account for both the additional

hash value stored by a speed-2 pebble during rounds  $(2^{k-1}, 2^{k-1} + 2^{k-2}]$  and the storage corresponding to a speed-2  $P_{k-1}(y_{k-1})$  pebble, running from round  $2^{k-1} + 1$ .  $\square$

**Theorem 4.** *A speed-1 pebble  $P_k$  uses up to  $\max(S_k) = \max(k + 1, 2k - 2)$  storage, and a speed-2 pebble  $P_k$  uses up to  $\max(S_k) = k + 1$  storage.*

**Proof** Using that  $s_{2^k} = k + 1$ , we write  $S_k = A_k \parallel k+1 \parallel B_k$ , where  $|A_k| = |B_k| = 2^k - 1$ .

For a speed-1 pebble  $P_k$ , it can easily be checked that  $\max(S_k) = \max(k + 1, 2k - 2)$  holds for  $k = 0, 1$ . To prove this for  $k \geq 2$ , we note that it suffices to show  $\max(A_k, B_k) = 2k - 2$ , as  $\max(S_k) = \max(A_k, k + 1, B_k)$ . Lemma 2 implies

$$\begin{aligned} A_k &= 1 \parallel 2^{*2^{k-1}-1} \parallel 1 + A_{k-1} \\ B_k &= A_{k-1} + B_{k-1} \parallel k \parallel B_{k-1}, \end{aligned}$$

so we have that  $\max(A_k, B_k) = \max(A_{k-1} + B_{k-1}, k) = \max(2k - 2, k) = 2k - 2$  follows if we can show  $\max(A_k + B_k) = 2k$ , for  $k \geq 1$ . We prove the latter by induction on  $k$ . For  $k = 1$ , clearly true as  $A_1 = B_1 = \{1\}$ . For  $k \geq 2$ , we see that  $\max(A_k + B_k) = \max(2 + A_{k-1} + B_{k-1}, k + 2) = \max(2k, k + 2) = 2k$  follows from the induction hypothesis, also using that the first element of  $A_{k-1} + B_{k-1} = k$ .

For a speed-2 pebble  $P_k$ , we note that  $\max(S_k) = k + 1$  follows from the fact that  $A_k + B_k = k + 1$ , which we show by induction on  $k$ . For  $k = 0$ ,  $A_k + B_k = k + 1$  is vacuously true, as  $A_0, B_0$  are empty sequences. For  $k \geq 1$ , we see from Lemma 2 that

$$\begin{aligned} A_k &= 1^{*2^{k-1}} \parallel 1 + A_{k-1} \\ B_k &= A_{k-1} + B_{k-1} \parallel k \parallel B_{k-1}. \end{aligned}$$

Thus, from the induction hypothesis we have  $A_{k-1} + B_{k-1} = k$ , hence  $A_k + B_k = k + 1$ .  $\square$

## 5 Optimal Binary Pebbles

In this section, we will reduce the maximal number of hashes per round from  $k-1$  for a speed-2 pebble  $P_k$  to  $\lceil k/2 \rceil$  for an optimal pebble  $P_k$ , without increasing the storage requirements. We do so by letting our optimal pebbles  $P_k$  be idle for the first  $2^{k-1} - 1$  rounds, just as speed-2 pebbles do. During rounds  $[2^{k-1}, 2^k)$ , an optimal pebble will work at varying speeds, roughly as follows: the average speeds in each quarter are 2, 1, 2, and 3 hashes per round, respectively. To streamline the presentation, we will at first allow “ $\frac{1}{2}$  hashes” in the definition of our optimal schedule. At the end of this section, we will show how to round the schedule to integer values without affecting optimality.

We define optimal schedule  $T_k$  as follows:

$$T_0 = \{\}, \quad T_k = 0^{*2^{k-1}-1} \parallel U_k \parallel V_k,$$

where

$$\begin{aligned} U_1 &= \{1\}, & U_k &= \frac{1}{2} + U_{k-1} \parallel 1^{* \lfloor 2^{k-3} \rfloor}, \\ V_1 &= \{\}, & V_k &= \frac{1}{2} + U_{k-1} \parallel \frac{1}{2} + V_{k-1}. \end{aligned}$$

For example,  $T_1 = \{1\}$ ,  $T_2 = \{0, \frac{3}{2}, \frac{3}{2}\}$ , and  $T_3 = \{0, 0, 0, 2, 1, 2, 2\}$ .

Optimality is proved in the next two theorems.

**Theorem 5.** *An optimal pebble  $P_k$  uses up to  $\max(W_k) = k/2$  hashes in any output round, for  $k \geq 2$ .*

**Proof** We use Lemma 1 without explicitly referring to it.

Since  $\max(W_k) = \max(T_{k-1} + W_{k-1})$ , we obtain  $\max(W_k) = k/2$ , if we prove by induction on  $k$  that

$$T_k + W_k = T_{k-1} + W_{k-1} \parallel \frac{k+1}{2}^{*2^{k-1}}.$$

This property clearly holds for  $k = 1, 2$ . For  $k \geq 3$ , the definition of  $T_k$  implies that the property is in fact equivalent to

$$(U_k \parallel V_k) + (0 \parallel W_{k-1}) = \frac{k+1}{2}^{*2^{k-1}}.$$

From the definition of  $U_k, V_k$  and the induction hypothesis for  $T_{k-2} + W_{k-2}$  we obtain

$$\begin{aligned} U_k \parallel V_k &= \frac{1}{2} + U_{k-1} \parallel 1^{*2^{k-3}} \parallel \frac{1}{2} + (U_{k-1} \parallel V_{k-1}), \\ 0 \parallel W_{k-1} &= 0 \parallel T_{k-3} + W_{k-3} \parallel \frac{k-1}{2}^{*2^{k-3}} \parallel 0 \parallel W_{k-2}. \end{aligned}$$

Since  $0 \parallel T_{k-3} + W_{k-3}$  is equal to the first half of  $0 \parallel W_{k-2}$ , we get from the induction hypothesis that indeed all elements of  $(U_k \parallel V_k) + (0 \parallel W_{k-1})$  are equal to  $\frac{k+1}{2}$ .  $\square$

Let  $\text{len}(x) = \lceil \log_2(x+1) \rceil$  denote the bit length of nonnegative integer  $x$ . The next two lemmas give closed formulas for the optimal schedule  $T_k$  and its partial sums. Lemma 4 will be used to prove Theorem 6, but these formulas also provide the basis for our efficient in-place implementation of optimal binary pebbling.

**Lemma 3.** *For optimal schedule  $T_k = \{t_r\}_{r=1}^{2^k-1}$ , we have for  $2^{k-1} \leq r < 2^k$ :*

$$t_r = \frac{1}{2} \left( k + 1 - \text{len} \left( (2r) \bmod 2^{\text{len}(2^k-r)} \right) \right).$$

**Proof** The proof is by induction on  $k$ . For  $0 \leq k \leq 2$ , the formula is easily checked. For  $k \geq 3$ , we distinguish two cases.

**Case**  $2^{k-1} \leq r < 2^{k-1} + 2^{k-2}$ . We first note that  $(2r) \bmod 2^{\text{len}(2^k-r)} = 2r - 2^k$ . If  $r \geq 2^{k-1} + 2^{k-3}$ , we have  $t_r = 1$  by definition and we see the formula for  $t_r$  yields 1 as well as  $\text{len}(2r - 2^k) = k - 1$ . Otherwise  $r < 2^{k-1} + 2^{k-3}$ , hence we have  $t_r = t_{r+2^{k-2}}$ . So, this case reduces to the case below by noting that also  $(2(r + 2^{k-2})) \bmod 2^{\text{len}(2^k-(r+2^{k-2}))} = 2r - 2^k$ .

**Case**  $2^{k-1} + 2^{k-2} \leq r < 2^k$ . From the definition of the optimal schedule we see that in this case  $t_r = \frac{1}{2} + t'_{r-2^{k-1}}$ , where  $T_{k-1} = \{t'_z\}_{z=1}^{2^{k-1}-1}$ . From the induction hypothesis we get:

$$t'_{r-2^{k-1}} = \frac{1}{2} \left( k - \text{len}((2(r - 2^{k-1})) \bmod 2^{\text{len}(2^{k-1} - (r - 2^{k-1}))}) \right).$$

Rewriting this formula for  $t'_{r-2^{k-1}}$  we obtain

$$t_r = \frac{1}{2} + \frac{1}{2} \left( k - \text{len}((2r - 2^k) \bmod 2^{\text{len}(2^k - r)}) \right).$$

Noting that  $\text{len}(2^k - r) \leq k$ , we see that the formula holds for  $t_r$  as well.  $\square$

**Lemma 4.** For optimal schedule  $T_k = \{t_r\}_{r=1}^{2^k-1}$ , we have for  $0 \leq j \leq 2^{k-1}$ :

$$\sum_{r=2^{k-j}}^{2^k-1} t_r = \frac{1}{2} (j(k-m) + (m+3-l)2^l - 2^m) - 1,$$

where  $l = \text{len}(j)$  and  $m = \text{len}(2^l - j)$ .

**Proof** The proof is by induction on  $j$ . For  $j = 0$ , both sides are equal to 0.

For  $1 \leq j \leq 2^{k-1}$ , Lemma 3 implies that

$$t_{2^k-j} = \frac{1}{2} (k+1 - \text{len}((-2j) \bmod 2^l)).$$

Combined with the induction hypothesis for  $j-1$  we obtain

$$\sum_{r=2^{k-j}}^{2^k-1} t_r = \frac{1}{2} \left( j(k-m') + m'+1 - \text{len}((-2j) \bmod 2^l) + (m'+3-l')2^{l'} - 2^{m'} \right) - 1,$$

where  $l' = \text{len}(j-1)$  and  $m' = \text{len}(2^{l'} - j + 1)$ . We distinguish two cases.

**Case**  $l' = l - 1$ . This means that  $j = 2^{l-1}$ , and hence  $m = l$  and  $m' = 1$ . We are done as both sides are equal to  $\frac{1}{2}j(k+4-l) - 1$ .

**Case**  $l' = l$ . This means that  $2^{l-1} < j < 2^l$ , hence  $0 < 2^{l+1} - 2j < 2^l$ . This implies  $\text{len}((-2j) \bmod 2^l) = m+1$ , so we see that both sides are equal if  $m = m'$ . If  $m' = m+1$ , we see that  $2^l - j = 2^m - 1$  and that therefore both sides are equal as well.  $\square$

**Theorem 6.** An optimal pebble  $P_k$  uses up to  $\max(S_k) = k+1$  storage.

**Proof** We prove that the storage requirements of an optimal pebble do not exceed the storage requirements of a speed-2 pebble, hence that  $\max(S_k) = k+1$  for an optimal pebble as well.

Consider the rounds in which a speed-2 pebble and an optimal pebble store the values  $y_i = f^{2^k-2^i}(x)$  for  $i = k, \dots, 1$ . We claim that an optimal pebble will never store  $y_i$  before a speed-2 pebble does. Clearly, a speed-2 pebble stores  $y_i$  in round  $2^k - 2^{i-1}$  for  $i = k, \dots, 1$ . However, in round  $2^k - 2^{i-1}$  an optimal pebble

still has to compute at least as many hashes as a speed-2 pebble needs to reach  $y_0$ :

$$\sum_{r=2^k-2^{i-1}}^{2^k-1} t_r = 2^{i-2}(k+4-i) - 1 \geq 2^i - 1,$$

using Lemma 4 for  $j = 2^{i-1}$ .  $\square$

As a final step we will round the optimal schedule  $T_k$  to integer values, without affecting optimality. For example, we round  $T_2 = \{0, \frac{3}{2}, \frac{3}{2}\}$  to  $\{0, 1, 2\}$  or to  $\{0, 2, 1\}$ . In general, we make sure that if an element is rounded up then its neighbors are rounded down, and vice versa. The rounding also depends on the parity of  $k$  to alternate between rounding up and rounding down. Hence, we define the rounded optimal schedule by:

$$t_r = \left\lfloor \frac{1}{2} \left( (k+r) \bmod 2 + k + 1 - \text{len}((2r) \bmod 2^{\text{len}(2^k-r)}) \right) \right\rfloor, \quad (1)$$

for  $2^{k-1} \leq r < 2^k$ . Accordingly, we see that optimal pebble  $P_k$  will use up to  $\max(W_k) = \lceil k/2 \rceil$  hashes in any output round, matching the lower bound of Theorem 2.

## 6 Optimized Implementations

A hash chain is deployed as follows as part of an authentication mechanism like Lamport's identification scheme. Given a random seed value  $x$ , the initial stage of any type of binary pebble  $P_k(x)$  is simply performed by iterating the hash function  $f$  and storing the values  $y_i = f^{2^k-2^i}(x)$  for  $i = k, \dots, 0$ . The value of  $y_0$  is then output, e.g., as part of the registration protocol for Lamport's scheme. The other hash values  $y_1, \dots, y_k$  are stored for the remainder of the output stage, e.g., for use in later runs of Lamport's identification protocol.

The initial stage is preferably executed inside the secure device that will later use the hash chain for identification. However, for lightweight devices such as smart cards, RFID tags, sensors, etc., the initial stage will typically be run on a more powerful device, after which hash values  $y_1, \dots, y_k$  will be inserted in the lightweight device and hash value  $y_0$  can be used for registration.

To implement the output stage of a pebble  $P_k$  one needs to handle potentially many pebbles all running in parallel. The pseudocode in [Jak02,CJ02,YSEL09] suggests rather elaborate techniques for keeping track of the (state of) pebbles. On the contrary, we will show how to minimize storage and computational overhead by exploiting specific properties of Jakobsson's speed-2 pebbles and our optimal pebbles. In particular, we present in-place hash chain reversal algorithms, where the entire state of these algorithms (apart from the hash values) is represented between rounds by a single  $k$ -bit counter only.

We introduce the following terminology to describe the state of a pebble  $P_k$ . This terminology applies to both speed-2 pebbles and optimal pebbles. Pebble  $P_k$  is said to be **idle** if it is in rounds  $[1, 2^{k-1})$ , **hashing** if it is in rounds  $[2^{k-1}, 2^k]$ , and **redundant** if it is in rounds  $(2^k, 2^{k+1})$ . An idle pebble performs no hashes

at all, while a hashing pebble will perform at least one hash per round, except for round  $2^k$  in which  $P_k$  outputs its  $y_0$  value. The work for a redundant pebble  $P_k$  is taken over by its child pebbles  $P_0, \dots, P_{k-1}$  during its last  $2^k - 1$  output rounds.

The following theorem provides the basis for our in-place algorithms by showing precisely how the state of all pebbles running in parallel during the output stage of a pebble  $P_k$  can be determined from the round number. Let  $x_i \in \{0, 1\}$  denote the  $i$ -th bit of nonnegative integer  $x$ ,  $0 \leq i < \text{len}(x)$ .

**Theorem 7.** *For a speed-2 pebble or optimal pebble  $P_k$  in output round  $2^{k+1} - c$ ,  $1 \leq c \leq 2^k$ , we have for every  $i$ ,  $0 \leq i \leq k$ , exactly one non-redundant pebble  $P_i$  present if and only if bit  $c_i = 1$ , and if present,  $P_i$  is in round  $2^i - (c \bmod 2^i)$ .*

**Proof** The proof is by induction on  $c$ . For  $c = 2^k$ , only  $c_k = 1$ , which corresponds to pebble  $P_k$  being the only pebble around. Also,  $P_k$  is in its  $2^k$ th round.

For  $1 \leq c < 2^k$ , write  $c' = c + 1$  and let  $k' \geq 0$  be maximal such that  $c' \bmod 2^{k'} = 0$ . Hence  $c'_{k'} = 1$ . By the induction hypothesis for  $c'$ , pebble  $P_{k'}$  is in its first output round  $2^{k'}$ . So, in the next round  $P_{k'}$  becomes redundant, and is replaced by child pebbles  $P_{k-1}, \dots, P_0$  who will all be in their first round. As  $c = c' - 1$ , this corresponds to the fact that  $c_{k'} = 0$  and  $c_{k'-1} = \dots = c_0 = 1$ , also noting that  $2^i - (c \bmod 2^i) = 1$  for  $i = k' - 1, \dots, 0$ .

For  $i > k'$ , we have  $c_i = c'_i$ . All non-redundant pebbles in round  $2^{k+1} - c'$  remain so in round  $2^{k+1} - c$ , and for these pebbles the round number becomes  $2^i - (c' \bmod 2^i) + 1 = 2^i - (c \bmod 2^i)$ , as required.  $\square$

As a corollary, we see that a non-redundant pebble  $P_i$  is hashing precisely when  $c_{i-1} = 0$ , and  $P_i$  is idle otherwise, since for  $c_i = 1$  we have that  $c_{i-1} = 0$  if and only if  $2^i - (c \bmod 2^i) \geq 2^{i-1}$ . This holds also for  $i = 0$  if we put  $c_{-1} = 0$ .

## 6.1 In-Place Speed-2 Pebbles

We present an in-place implementation of a speed-2 pebble  $P_k$  for which the overall storage is limited to the space for  $k$  hash values and one  $k$ -bit counter  $c$ . As explained above, we will assume that hash values  $y_1, \dots, y_k$  are given as input and that  $y_0$  has been output already. Thus,  $P_k$  has exactly  $2^k - 1$  output rounds remaining. We use  $c$  to count down the output rounds.

The basis for our in-place algorithm is given by the next theorem.

**Theorem 8.** *For a speed-2 pebble  $P_k$  in output round  $2^{k+1} - c$ ,  $1 \leq c \leq 2^k$ , each non-redundant pebble  $P_i$  present stores  $t+1$  hash values, where  $t$  is maximal such that  $c_{i-1} = \dots = c_{i-t} = 0$  with  $0 \leq t \leq i$ .*

**Proof** From Theorem 7 it follows that non-redundant pebble  $P_i$  is in round  $r = 2^i - (c \bmod 2^i)$ . Since  $0 \leq t \leq i$  is maximal such that  $c_{i-1} = \dots = c_{i-t} = 0$ , we have that  $2^i - 2^{i-t} < r \leq 2^i - 2^{i-t-1}$ . This implies that  $P_i$  stores  $t+1$  hash values in round  $r$ , as Lemma 2 says that for a speed-2 pebble  $P_i$  the storage requirements throughout its first  $2^i$  rounds are given by sequence  $D_i$ , where  $D_0 = \{1\}$  and  $D_i = 1 * 2^{i-1} \parallel 1 + D_{i-1}$ .  $\square$

---

**Algorithm 1** In-place speed-2 pebble  $P_k$ , round  $r = 2^{k+1} - c$ ,  $1 \leq c < 2^k$ .

---

```

1: output  $z[0]$ 
2:  $k \leftarrow \text{pop}_0(c)$ 
3: for  $i \leftarrow 1$  to  $k$  do  $z[i-1] \leftarrow z[i]$ 
4:  $k \leftarrow k+1$ ;  $c \leftarrow \lfloor c/2 \rfloor$ 
5:  $p \leftarrow k-1$ 
6: while  $c \neq 0$  do
7:    $z[p] \leftarrow f(z[k])$ 
8:   if  $p \neq 0$  then  $z[p] \leftarrow f(z[p])$ 
9:    $k \leftarrow k + \text{pop}_0(c) + \text{pop}_1(c)$ 
10:   $p \leftarrow k$ 

```

---

Theorem 8 suggests a simple approach to store the hash values for a speed-2 pebble  $P_k$  throughout its output stage. We use a single array  $z$  of length  $k$  to store all hash values as follows. Initially,  $z[0] = y_1, \dots, z[k-1] = y_k$ , and counter  $c = 2^k - 1$ . This corresponds to pebble  $P_k$  being at the start of its output stage, where it starts to run pebbles  $P_i(y_{i+1})$  in parallel, for  $i = 0, \dots, k-1$ , each of these (non-redundant) pebbles  $P_i$  storing exactly one hash value in array  $z$ . In general, in output round  $2^{k+1} - c$  of  $P_k$ , we let each non-redundant pebble  $P_i$  store its hash values in array  $z$  in segment  $z[i..i-t]$  (corresponding to  $c_i = 1$  and  $c_{i-1} = 0, \dots, c_{i-t} = 0$ ). As a result, the non-redundant pebbles jointly occupy consecutive segments of array  $z$ , storing exactly  $\text{len}(c)$  hash values in total.

Algorithm 1 describes precisely what pebble  $P_k$  does in round  $r$ ,  $2^k < r < 2^{k+1}$ . Note that we let  $c = 2^{k+1} - r$  at the start of round  $r$ . Based on Theorem 7, we process the bits of  $c$  as follows, using operations  $\text{pop}_0(c)$  and  $\text{pop}_1(c)$  to count and remove all trailing 0s and 1s from  $c$ , respectively.

Let  $k' \geq 0$  be maximal such that  $c \bmod 2^{k'} = 0$ . Hence  $c_{k'} = 1$ . From Theorem 7, we see that pebble  $P_{k'}$  is in its first output round  $2^{k'}$ , hence  $P_{k'}$  becomes redundant in the next round, and each of its children will take over one hash value. The hash values  $y_0, \dots, y_{k'}$  computed by  $P_{k'}$  in its initial stage are stored in  $z[0], \dots, z[k']$ . So, we output  $z[0] = y_0$  for  $P_{k'}$  and move  $y_1, \dots, y_{k'}$  to entries  $z[0], \dots, z[k'-1]$ . This makes entry  $z[k']$  available. We distinguish two cases.

**Case**  $\text{len}(c-1) = \text{len}(c) - 1$ . In this case no new hash values need to be stored, and  $z[k']$  will be unused from this round on.

**Case**  $\text{len}(c-1) = \text{len}(c)$ . Let  $k'' \geq k' + 1$  be maximal such that  $c \bmod 2^{k''} = 2^{k'}$ . Hence  $c_{k''} = 1$ . We claim that speed-2 pebble  $P_{k''}$  is the unique pebble that needs to store an additional hash value. Pebble  $P_{k''}$  is in round  $2^{k''} - (c \bmod 2^{k''}) = 2^{k''} - 2^{k'}$ , so it is  $2^{k'}$  rounds from the end of its initial stage. We store its additional hash value in  $z[k']$ .

This explains Algorithm 1. In the first iteration of the loop in lines 6–10, we have that  $p = k'$  holds at line 7. Each hashing pebble performs two hashes, except when a pebble is at the end of its initial stage (corresponding to  $p = 0$ ). Essentially no processing is done for idle pebbles, due to the use of operation  $\text{pop}_1(c)$  in line 9.

---

**Algorithm 2** In-place optimal pebble  $P_k$ , round  $r = 2^{k+1} - c$ ,  $1 \leq c < 2^k$ .

---

```

1: output  $z[0]$ 
2:  $k \leftarrow \text{pop}_0(c)$ 
3: for  $i \leftarrow 1$  to  $k$  do  $z[i-1] \leftarrow z[i]$ 
4:  $k \leftarrow k+1$ ;  $c \leftarrow \lfloor c/2 \rfloor$ 
5: while  $c \neq 0$  do
6:    $p \leftarrow k$ 
7:    $k \leftarrow k + \text{pop}_0(c)$ 
8:    $j \leftarrow (-r) \bmod 2^k$ 
9:    $t \leftarrow (k+j) \bmod 2$ 
10:   $l \leftarrow \text{len}(j)$ 
11:   $m \leftarrow \text{len}(2^l - j)$ 
12:   $s \leftarrow (m+1) \bmod (l+1)$ 
13:   $h \leftarrow \lfloor (t+j(k-m) + (m+3-l)2^l - 2^m)/2 \rfloor$ 
14:   $g \leftarrow \text{len}(h) - 1$ 
15:   $p \leftarrow p + g - l$  ,
16:  for  $d \leftarrow 1$  to  $\lfloor (t+k+1-s)/2 \rfloor$  do
17:     $x \leftarrow z[p]$ 
18:    if  $h = 2^g$  then  $g \leftarrow g-1$ ;  $p \leftarrow p-1$ 
19:     $z[p] \leftarrow f(x)$ 
20:     $h \leftarrow h-1$ 
21:   $k \leftarrow k + \text{pop}_1(c)$ 

```

---

## 6.2 In-Place Optimal Pebbles

In this section we turn the algorithm for speed-2 pebbles into one for optimal pebbles by making three major changes. See Algorithm 2.

First, we make sure that the number of hashes performed by each hashing pebble  $P_k$  is in accordance with Eq. (1). The actual hashing by  $P_k$  is done in the loop in lines 16–20. To apply Eq. (1), the actual round number  $j = (2^k - r) \bmod 2^k$  for  $P_k$  is determined in line 8. Writing  $l = \text{len}(j)$  and  $m = \text{len}(2^l - j)$ , we have that the number of hashes as specified by Eq. (1) can be computed as  $\lfloor (t+k+1-s)/2 \rfloor$ , where  $t = (k+j) \bmod 2$  and  $s = (m+1) \bmod (l+1)$ , actually using that  $\text{len}((2^l - 2j) \bmod 2^l) = \text{len}(2^{l+1} - 2j) \bmod (l+1)$  holds for  $j \geq 1$ .

Second, we make sure that each hashing pebble  $P_k$  will store the correct hash values for  $y_k, \dots, y_0$ . To this end, note that Lemma 4 tells precisely how many hashes pebble  $P_k$  still needs to compute at the start of round  $j$ . Thus we set  $h$  to this value (plus one) in line 13, and test in line 18 if the current hash value must be stored (that is, whether  $h$  is an integral power of 2).

Finally, we make sure that hashing pebble  $P_k$  will store its hash values in the right entries of array  $z$ . In line 6, we let variable  $p$  point to the first entry of the segment of array  $z$  used by pebble  $P_k$ . Then in line 15, the value of  $p$  is adjusted by adding the difference between the number of hash values that a speed-2 pebble  $P_k$  would still need to store at the start of round  $j$  and the number of hash values that an optimal pebble  $P_k$  still needs to store at the start of round  $j$ .

---

**Algorithm 3** Fast optimal pebble  $P_k$ , round  $r = 2^{k+1} - c$ ,  $1 \leq c < 2^k$ .

---

```

1: output  $z[0]$ 
2:  $k \leftarrow \text{pop}_0(c)$ 
3: for  $i \leftarrow 1$  to  $k$  do  $z[i-1] \leftarrow z[i]$ 
4:  $k \leftarrow k+1$ ;  $c \leftarrow \lfloor c/2 \rfloor$ 
5: if  $c$  odd then  $a[v] \leftarrow (1, k, k, k, 2^k, 2^k)$ ;  $v \leftarrow v+1$ 
6:  $u \leftarrow v$ 
7: while  $c \neq 0$  do
8:    $k \leftarrow k + \text{pop}_0(c)$ 
9:    $u \leftarrow u-1$ ;  $(e, l, s, p, g, h) \leftarrow a[u]$ 
10:   $e \leftarrow e-1$ 
11:  if  $e \leq 0$  then
12:    if  $s = 0$  then
13:       $l \leftarrow l-1$ ; if  $l \neq 1$  then  $s \leftarrow 2$ 
14:    else
15:      if  $s = l$  then  $s \leftarrow 0$  else  $e \leftarrow 2^{s-1}$ ;  $s \leftarrow s+1$ 
16:    for  $d \leftarrow 1$  to  $\lfloor ((k+e) \bmod 2 + k + 1 - s)/2 \rfloor$  do
17:       $x \leftarrow z[p]$ 
18:      if  $h = g$  then  $g \leftarrow \lfloor g/2 \rfloor$ ;  $p \leftarrow p-1$ 
19:       $z[p] \leftarrow f(x)$ 
20:       $h \leftarrow h-1$ 
21:    if  $h \neq 1$  then  $a[u] \leftarrow (e, l, s, p, g, h)$  else  $v \leftarrow v-1$ 
22:     $k \leftarrow k + \text{pop}_1(c)$ 

```

---

This explains the design of Algorithm 2. Note that in total only three bit length computations are used per hashing pebble (cf. lines 10, 11, and 14).

### 6.3 Optimal Pebbles with Minimal Computational Overhead

Even though the computational overhead for our in-place implementation is small, it may still be relatively large if hash evaluations themselves take very little time. For instance, if the hash function is (partly) implemented in hardware. Using Intel's AES-NI instruction set one can implement a 128-bit hash function that takes a few cycles only (e.g., see [BÖS11], noting that for one-way hash chains no collision-resistance is needed such that one can use Matyas-Meyer-Oseas for which the key is fixed). Therefore, we also provide an implementation minimizing the computational overhead at the expense of some additional storage.

We will keep some state for each pebble, or rather for each *hashing* pebble only. Although an optimal pebble  $P_k$  will store up to  $k$  hash values at any time, we observe that no more than  $\lceil k/2 \rceil$  hashing pebbles will be present at any time. As in our in-situ algorithms we will thus avoid any storage (and processing) for idle pebbles, as can be seen from Algorithm 3.

A segment  $a[0..v-1]$  of an array  $a$  of length  $\lfloor k/2 \rfloor$  suffices to store the relevant hashing pebbles, where initially  $v = 0$ . In each round, at most one idle pebble  $P_k$  will become hashing, and if this happens pebble  $P_k$  is added to array  $a$ , cf. line 5.

Later, once pebble  $P_k$  is done hashing, it will be removed again from array  $a$ , cf. line 21.

For each hashing pebble we store six values called  $e, l, s, p, g, h$ , respectively. Except for counter  $e$ , these values correspond to the variables in Algorithm 2. Using  $j$  to refer to the round pebble  $P_k$  is at, as in Algorithm 2, we have  $l = \text{len}(j)$  and also  $s = (m + 1) \bmod (l + 1)$ , where  $m = \text{len}(2^l - j)$ . We use  $e$  to count down to zero starting from the appropriate powers of 2, cf. line 15.

As a result, Algorithm 3 limits the computations for each hashing pebble to a few simple operations only.

## 7 Concluding Remarks

We have completely resolved the case of binary pebbling of hash chains by constructing an explicit optimal schedule. A main advantage of our optimal schedule is that it allows for very efficient *in-place* pebbling algorithms. This compares favorably with the greedy pebbling algorithms of [YSEL09], which require a substantial amount of storage beyond the hash values themselves. The pseudocode of Algorithms 1–3 is readily translated into efficient program code, applying further optimizations depending on the target platform.<sup>1</sup>

The security of one-way hash chains for use in authentication mechanisms such as Lamport’s identification scheme does not depend on the collision resistance of the hash function. Therefore, it suffices to use 128-bit hash values—rather than 256-bit hash values, say. Using, for instance, the above mentioned Matyas-Meyer-Oseas construction one obtains a fast and simple one-way function  $f : \{0, 1\}^{128} \rightarrow \{0, 1\}^{128}$  defined as  $f(x) = \text{AES}_{\text{IV}}(x) \oplus x$ , where IV is a 128-bit string used as fixed “key” for the AES block cipher. Consequently, even for *very long* hash chains of length  $2^{32}$ , our in-place optimal pebbling algorithm will just store 516 bytes (32 hash values and one 32-bit counter) and perform at most 16 hashes per identification round. Similarly, *long* hash chains of length  $2^{16}$  would allow devices capable only of lightweight cryptography to run 65535 rounds of identification (e.g., more than twice per hour over a period of three years), requiring only 258 bytes of storage and using at most 8 hashes per round.

We leave as an open problem whether binary pebbling yields the lowest space-time product. Reversal of a length- $n$  hash chain using optimal binary pebbling requires  $\log_2 n$  storage and  $\frac{1}{2} \log_2 n$  time per round, yielding  $\frac{1}{2} \log_2^2 n$  as space-time product. Coppersmith and Jakobsson [CJ02] derived a lower bound of approx.  $\frac{1}{4} \log_2^2 n$  for the space-time product. Whether this lower bound is achievable is doubtful, because the lower bound is derived *without* taking into account that the maximum number of hashes during any round needs to be minimized. As a natural alternative, we have done a preliminary study of “Fibonacci” pebbling, considering hash chains of length  $n = F_k$ , the  $k$ th Fibonacci number. Initial results, however, suggest that the space-time product is not lower than for binary pebbling.

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<sup>1</sup> Sample code (in C, Java, Python) available at [www.win.tue.nl/~berry/pebbling/](http://www.win.tue.nl/~berry/pebbling/).

As another direction for further research we suggest to revisit the problem of efficient Merkle tree traversal studied in [Szy04], which plays a central role in hash-based signature schemes [Mer87,Mer89]; in particular, it would be interesting to see whether algorithms for generating successive authentication paths can be done in-place. More generally, research into optimal (in-place) algorithms for hash-based signatures is of major interest both in the context of lightweight cryptography (e.g., see [PCTS02,MSS13]; more references in [YSEL09]) and in the context of post-quantum cryptography (e.g., see [BDE<sup>+</sup>13]).

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