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A two-echelon production-inventory model for deteriorating items with multiple buyers

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ABSTRACT

In a production-inventory system, the manufacturer produces the items at a rate, e.g. R , dispatches the order quantities to the customers in specific intervals and stores the excess inventory for subsequent deliveries. Therefore each inventory cycle of the manufacturer can be divided into two phases, first is the period of production, the second is when the manufacturer does not do any production and utilises the inventory that is in stock. One of the challenges in these models is how to obtain the inventory level of the supplier when there is deterioration. The existing literature that considers multi-echelon systems (including models with single-buyer or multi-buyer), analyses the deterioration/inventory cost of these echelons with the assumption of having huge surplus in production capacity. Then it seems acceptable to drop part of the production period which is for producing the first batch(s) for buyer(s) at the beginning of each production period. In this paper we develop a single-manufacturer, multi-buyer model for a deteriorating item with finite production rate. We also relax the assumption on the production capacity and find the average inventory of the supplier. It is shown that in case the production rate is not high, the existing models may not be sufficiently accurate. It is also illustrated that these models are more applicable to inventory systems (and not production-inventory) as they result in fairly accurate solutions when the manufacturer has much higher production capacity compared to the demand rate. Also a sensitivity analysis is conducted to show how the model reacts to changes in parameters.

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1. Introduction

Inventory management literature is largely based on the assumption that an item in stock suffers no loss in quality. This, however, cannot hold in practice for many products. In order to address this fact, a subset of the literature studies *deteriorating items*. The deteriorating item inventory models have increasingly drawn attention in recent years. The main stream of this literature considers that a percentage of on-hand inventory goes bad.

This literature mainly includes single-echelon inventory models. Recent research papers on deteriorating item inventory strive to go beyond a single echelon and address a supply chain model in order to make it more applicable to real-world problems. The number of studies on multi-echelon supply chain however is few.

One of the challenges in modelling a multi-echelon supply chain of a deteriorating item is to evaluate the inventory level at

the supplier, specially when there are multiple buyers that can have multiple replenishments. Ghiami et al. (2013) obtain the exact *inventory level* of the supplier of a two-echelon distribution system (a single-buyer, single-supplier model). In this model however the production rate is infinite which means that the supplier receives the batches in lots. Also having multiple buyers adds to the complexity of the inventory level at the supplier.

Single-manufacturer multi-buyer models are few in the literature. Assuming finite production rate (production-inventory model) with multiple buyers makes the inventory level at the supplier complex to find. In order to analyse this complexity, the few research papers on production-inventory multi-echelon supply chain models for deteriorating items use approximations which give sufficiently accurate results under the assumption of huge surplus in production capacity. These assumptions however make the model less practical for some cases.

In this paper a multi-echelon supply chain with finite production rate is developed and analysed. Moreover it is shown that the existing models can find fairly accurate solution under specific assumptions. By using a numerical example, it is also illustrated that if those circumstances do not hold, the existing models fail to

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find accurate solutions whereas the current model can do so. The significance of these errors is also depicted.

2. Literature review

One of the first studies which analyse a multi-echelon supply chain for a deteriorating item with finite production rate is done by Yang and Wee (2000). For an overview of the deteriorating item's literature see Nahmias (1982), Raafat (1991), Goyal and Giri (2001), Li et al. (2010), and Bakker et al. (2012). Yang and Wee (2000) consider single-buyer single-vendor where the vendor produces the items with a finite production rate. The objective is to minimise the total cost function of the supply chain. Later Yang and Wee (2002) extend the work done by Yang and Wee (2000) by considering multiple buyers and aim to minimise the total cost of the system. In their model, Yang and Wee (2002) assume that there is a huge surplus in production capacity therefore they neglect the production time that is needed to produce the items for the buyers before the next production-inventory cycle starts. Rau et al. (2003) develop a single-supplier, single-manufacturer, single buyer model. Similar to Yang and Wee (2002), Rau et al. (2003) implicitly assume that the production rate is significantly larger than the demand, therefore they drop part of the manufacturer's production period. Law and Wee (2006) consider a single-buyer, single-vendor supply chain which produces and delivers a product of which the raw material is livestock. The manufacturer buys young livestock and grows them, the mature livestock is then used to make food. This food is delivered to the buyer in batches. In their research, Law and Wee (2006) consider the time value of money by discounting the cost with a specific rate, ultimately minimising the total cost of the system. Taking a discounted cash flow approach, Lo et al. (2007) model a production-inventory system which consists of one buyer and one manufacturer. Similar to the previous research works, Lo et al. (2007) aim to minimise the total cost of the system.

Yan et al. (2011) examine an integrated single-buyer, single-supplier model with finite production rate using an approximated method in order to simplify the problem. In order to keep the error of this approximation small, Yan et al. (2011) assume the deterioration rate to be very small. Yu et al. (2012) study a supply chain in which the buyers are assumed to have the same inventory period and the supplier has a huge surplus in production capacity. The model is developed and analysed based on the assumption that the manufacturer has a large production capacity compared to the total demand. Also the production set-up cost is considered to be small so the supplier starts and stops the production process very frequently. These assumptions do not allow any stock accumulation at the supplier as he starts producing the items just some time before batches are delivered. Yu et al. (2013) develop a similar model in which there is only one buyer. The deterioration rate is considered to be based on a Weibull distribution. Yu (2013) develops a single-buyer, single-supplier model and suggests a collaborative strategy for setting the inventory policies. Lee and Kim (2014) analyse a single-vendor, single-buyer supply chain for a deteriorating item considering a probability for items to be defective as a failure in the production process. In their research, Lee and Kim (2014) consider the inventory level of the supplier as in Yang and Wee (2000, 2002).

The literature shows that there are few studies on single-manufacturer multi-buyers models. In this paper a similar model to Yang and Wee (2002) and Yu et al. (2012) is developed. In order to make a more generic and practical model, however

we assume that the buyers can have different inventory periods and also the assumption of having a huge surplus in production capacity is relaxed. Thereafter the result of the current model is compared with the existing literature.

3. Model

In this paper a supply chain including one manufacturer and N buyers is considered which delivers an item with constant deteriorating rate of the on-hand inventory. The manufacturer starts the production and stores the items sometime (T_3) before the first batch is sent to all the buyers at $t=0$. He continues the production till $t=T_1$, while storing the excess inventory for next deliveries. At time $t=T_1$, the manufacturer starts a non-production period which lasts till $t=T_1+T_2$ which is followed by a production period with length of T_3 during which the manufacturer produces items for the next inventory cycle. This production-inventory process repeats itself infinitely. Shortages are not allowed neither at the supplier nor the buyers. There is no replacement for the deteriorated items. The production rate is assumed to be greater than the sum of the demand at all the buyers. The notations used in this model are as follow:

θ	the deterioration rate
N	number of the buyers
d_i	the demand rate at buyer i , $i = 1, 2, \dots, N$
p	the production rate
T	the production-inventory cycle at the manufacturer, where $T = T_1 + T_2 + T_3$
T_1	the production period after the first batch sent to all the buyers
T_2	the non-production period during T
T_3	the production period to produce enough inventory to send to all the buyers at the end of T
$\tilde{I}_{v1}(t_1)$	the echelon stock at the supplier between $t_1 = 0$ and $t_1 = T_1$
$\tilde{I}_{v2}(t_2)$	the echelon stock at the supplier between $t_2 = 0$ and $t_2 = T_2$
$\tilde{I}_{v3}(t_3)$	the echelon stock at the supplier between $t_3 = 0$ and $t_3 = T_3$
$\tilde{I}_{bi}(t)$	the echelon stock at retailer i between $t=0$ and $t=T/n_i$
n_i	number of replenishments of buyer i during T
$I_{v1}(t_1)$	the physical inventory at the supplier between $t_1 = 0$ and $t_1 = T_1$
$I_{v2}(t_2)$	the physical inventory at the supplier between $t_2 = 0$ and $t_2 = T_2$
$I_{v3}(t_3)$	the physical inventory at the supplier between $t_3 = 0$ and $t_3 = T_3$
$I_{bi}(t)$	the physical inventory at retailer i between $t=0$ and $t=T/n_i$, $I_{bi}(t) = \tilde{I}_{bi}(t)$
I_{mi}	the maximum inventory level (order quantity) of buyer i
p_v	the unit production cost at the supplier
p_b	the unit purchasing price for the buyers
F_v	the holding cost per unit of currency per time unit at the supplier
F_b	the holding cost per unit of currency per time unit at each buyer
C_{sv}	the set-up production cost for the supplier
C_{sb}	the fixed ordering cost for each buyer
TP	the average profit of the supply chain
TP_v	the average profit of the supplier
TR_v	the average revenue of the supplier
TC_v	the total average cost of the supplier
HC_v	the average holding cost of the supplier
DC_v	the average deterioration cost of the supplier
SC_v	the average ordering cost of the supplier

TP_b	the average profit of all buyers
TR_b	the average revenue of all buyers
TC_b	the total average cost of all buyers
HC_b	the sum of average holding cost of all buyers
DC_b	the sum of average deterioration cost of all buyers
SC_b	the sum of average ordering fixed cost of all buyers

In order to optimise an inventory system in an economical context, researchers aim to either maximise the total profit or minimise the total cost. However in some cases these two objective functions are not equivalent and result in different optimal solutions (see Ghiami, 2014). Therefore the decision of which objective function to choose should be made carefully. Only in situations when the revenue is not a function of decision variables can profit maximisation be replaced by cost minimisation. Let TP_v and TP_b to be the total profit function of the supplier and the sum of the profit functions of all buyers respectively. These values can be obtained from relevant revenue and cost functions which are TR_v and TC_v for the supplier, and TR_b and TC_b for the group of buyers. The total profit of the supply chain (TP) is the sum of the total profit functions:

$$TP = TP_v + TP_b$$

$$= TR_v - TC_v + TR_b - TC_b. \quad (1)$$

In case of constant demand with no shortages (or with complete backlogging), where total sale (in terms of quantity) of the buyer is equal to the total demand and hence, independent of the decision variables (T and n_i) as in this model, minimising the total cost is equivalent to maximising the total profit as the derivatives of TR_b and TR_v with respect to the decision variables are zero (relaxing the integrality assumption on n_i), therefore

$$\frac{\partial TP}{\partial T} = -\frac{\partial(TC_v + TC_b)}{\partial T}, \quad (2)$$

and

$$\frac{\partial TP}{\partial n_i} = -\frac{\partial(TC_v + TC_b)}{\partial n_i}. \quad (3)$$

Here also it is implicitly assumed that the opportunity cost of capital is negligible, therefore, taking the classic approach does not result in errors in the optimal solution. This will no longer be true in general when using a Net Present Value (NPV) approach, as it values costs and revenues based on the time they take place, hence captures the opportunity cost of capital accurately.

The inventory level at buyer i is as shown in Fig. 1. The manufacturer's physical inventory dt units of time before dispatching the order quantities to the buyers is equal to the sum of the order quantities ($\sum_{i=1}^N I_{mi}$). This level goes to zero after dispatching the order quantities to the buyers. The echelon stock however does not drop to zero (as the batches are now at the buyers' inventory) but gradually decreases due to the demand and the deterioration. Echelon stock of an entity in a supply chain refers to the sum of the physical inventory in that business unit and all

Fig. 1. Inventory level of buyer i (Yang and Wee, 2002).

the downstream firms. In order to calculate the average inventory of the supplier, Joglekar (1988) considers the difference between the echelon stock level of the supplier and the echelon stock level of the retailer. In the current model, multiple buyers with multiple replenishments during one inventory cycle at the supplier, make the supplier's inventory level complex to calculate. Therefore we use the same technique as Joglekar (1988) to find the average inventory and not the inventory level.

It should be noted that in this paper with deterministic demand and zero lead-time, the physical stock level of the supplier and the buyers are considered. This approach however should be taken carefully as Axsäter and Rosling (1993) show that in case of lead-time or stochastic demand, formulating a multi-echelon inventory problem based on physical stock policies and echelon stock policies may result in two different solutions (see also Axsäter and Juntti, 1996, 1997).

In Fig. 2 the physical inventory (solid line) and the echelon stock (dashed line) of the supplier are shown. As can be seen, finding the physical inventory level of the supplier is complicated while finding the echelon stock is trivial. The following differential equations represent the change of the echelon stock of the supplier over T :

$$\frac{d\tilde{I}_{v1}(t_1)}{dt_1} = p - \sum_{i=1}^N d_i - \theta \tilde{I}_{v1}(t_1), \quad 0 \leq t_1 \leq T_1, \quad (4)$$

$$\frac{d\tilde{I}_{v2}(t_2)}{dt_2} = - \sum_{i=1}^N d_i - \theta \tilde{I}_{v2}(t_2), \quad 0 \leq t_2 \leq T_2, \quad (5)$$

and

$$\frac{d\tilde{I}_{v3}(t_3)}{dt_3} = p - \sum_{i=1}^N d_i - \theta \tilde{I}_{v3}(t_3), \quad 0 \leq t_3 \leq T_3. \quad (6)$$

In a similar way the following differential equation holds for the inventory level at buyer i :

$$\frac{dI_{bi}(t)}{dt} = -d_i - \theta I_{bi}(t), \quad 0 \leq t \leq \frac{T}{n_i}, \quad i = 1, 2, \dots, N. \quad (7)$$

The inventory level at buyer i reaches zero at T/n_i . Considering this boundary condition together with (7), the inventory level of buyer i is as follows:

$$I_{bi}(t) = \frac{d_i}{\theta} \left[\frac{\exp\left(\frac{\theta T}{n_i}\right) - \exp(\theta t)}{\exp(\theta t)} \right]$$

$$\approx d_i \left(\frac{T}{n_i} - t \right) \left[1 + \frac{\theta \left(\frac{T}{n_i} - t \right)}{2} \right], \quad 0 \leq t \leq \frac{T}{n_i}, \quad i = 1, 2, \dots, N. \quad (8)$$

By setting t equal to zero, the maximum inventory level of buyer i (optimal order quantity) is obtained

$$I_{mi} = \frac{d_i}{\theta} \left[\exp\left(\frac{\theta T}{n_i}\right) - 1 \right], \quad i = 1, 2, \dots, N. \quad (9)$$

Using Taylor's expansion ($e^x \approx 1 + x + x^2/2$), maximum inventory level of the buyer is

$$I_{mi} \approx \frac{d_i T}{n_i} \left(1 + \frac{\theta T}{2n_i} \right), \quad i = 1, 2, \dots, N. \quad (10)$$

In order to find the echelon stock level of the supplier, the relevant boundary conditions should be considered; $\tilde{I}_{v1}(0) = \sum_{i=1}^N I_{mi}$, $\tilde{I}_{v2}(T - T_1) = 0$ (this is due to the fact that in case production does not start at $t_2 = T_2$, $\tilde{I}_{v2}$ will reach zero at $t_2 = T - T_1$) and $\tilde{I}_{v3}(T - T_1 - T_2) = \sum_{i=1}^N I_{mi}$. The results drawn from

Fig. 2. Inventory level and echelon stock of the supplier.

these boundary conditions are

$$\tilde{I}_{v1}(t_1) = \frac{p - \sum_{i=1}^N d_i}{\theta} (1 - \exp(-\theta t_1)) + \exp(-\theta t_1) \sum_{i=1}^N I_{mi}, \quad (11)$$

$$\tilde{I}_{v2}(t_2) = \frac{\sum_{i=1}^N d_i}{\theta} (\exp(\theta(T - T_1 - t_2)) - 1), \quad (12)$$

and

$$\tilde{I}_{v3}(t_3) = \frac{p - \sum_{i=1}^N d_i}{\theta} + \left(\sum_{i=1}^N I_{mi} - \frac{p - \sum_{i=1}^N d_i}{\theta} \right) \exp(\theta(T - T_1 - T_2 - t_3)). \quad (13)$$

By using Taylor's expansion and knowing that $\tilde{I}_{v1}(T_1) = \tilde{I}_{v2}(0)$, $\tilde{I}_{v2}(T_2) = \tilde{I}_{v3}(0)$ and $T = T_1 + T_2 + T_3$ the value of T_1 , T_2 and T_3 are obtained as functions of T :

$$T_1 \approx \frac{T}{p} \left(\sum_{i=1}^N d_i \left(1 - \frac{1}{n_i} \right) + \frac{\theta T}{2} \sum_{i=1}^N d_i \left(1 - \frac{1}{n_i^2} \right) \right), \quad (14)$$

$$T_2 \approx T - \frac{T}{p} \left(1 + \frac{\theta T}{2} \right) \sum_{i=1}^N d_i, \quad (15)$$

and

$$T_3 \approx \frac{T}{p} \sum_{i=1}^N \frac{d_i}{n_i} \left(1 + \frac{\theta T}{2n_i} \right). \quad (16)$$

The average inventory holding cost for all the buyers and the supplier are as shown in (17) and (18). In order to simplify the calculations, Taylor's expansion is used:

$$HC_b = \frac{p_b F_b}{T} \sum_{i=1}^N n_i \int_0^{T/n_i} I_{bi}(t) dt \approx \frac{p_b F_b T}{2} \sum_{i=1}^N \frac{d_i}{n_i} \left(1 + \frac{\theta T}{3n_i} \right), \quad (17)$$

and

$$\begin{aligned} HC_v &= \frac{p_v F_v}{T} \left[\int_0^{T_1} \tilde{I}_{v1}(t_1) dt_1 + \int_0^{T_2} \tilde{I}_{v2}(t_2) dt_2 + \int_0^{T_3} \tilde{I}_{v3}(t_3) dt_3 - \sum_{i=1}^N n_i \int_0^{T/n_i} I_{bi}(t) dt \right] \\ &\approx \frac{p_v F_v}{T} \left[\frac{p - \sum_{i=1}^N d_i}{2} T_1^2 + \left(1 - \frac{\theta T_1}{2} \right) T_1 \sum_{i=1}^N I_{mi} + T_2 \left(T_3 + \frac{T_2}{2} \right) \sum_{i=1}^N d_i \right. \\ &\quad + \frac{\theta T_2 T_3 (T - T_1)}{2} \sum_{i=1}^N d_i + \left(1 + \frac{\theta T_3}{2} \right) T_3 \sum_{i=1}^N I_{mi} - \frac{T_3^2}{2} \left(1 + \frac{\theta T_3}{3} \right) \left(p - \sum_{i=1}^N d_i \right) \\ &\quad \left. - \frac{T^2}{2} \sum_{i=1}^N \frac{d_i}{n_i} \left(1 + \frac{\theta T}{3n_i} \right) \right]. \quad (18) \end{aligned}$$

It should be noted that the term in brackets on the first line of (18) calculates the difference between the echelon stock of the supplier and the sum of inventory at all the buyers during T to obtain the total physical inventory of the supplier during this inventory period. This value is divided by T to give the average inventory of the supplier which is used to calculate the holding cost of this echelon.

Using (10), the average deterioration cost for all the buyers and the supplier are as follow:

$$DC_b = \sum_{i=1}^N \frac{n_i p_b}{T} \left(I_{mi} - \frac{T d_i}{n_i} \right) \approx \frac{p_b \theta T}{2} \sum_{i=1}^N \frac{d_i}{n_i}, \quad (19)$$

and

$$DC_v = \frac{p_v}{T} \left(p(T - T_2) - \sum_{i=1}^N n_i I_{mi} \right) \approx \frac{p_v \theta T}{2} \sum_{i=1}^N d_i \left(1 - \frac{1}{n_i} \right). \quad (20)$$

In a cost minimisation model it is necessary to consider these costs, however in a profit maximisation model these costs are captured as the deteriorated items incur purchasing cost while not producing any revenues.

Buyer i places n_i orders during one inventory period of the supplier (T), which means a total cost of $n_i C_{sb}$ is incurred by this buyer. Therefore the sum of ordering cost of all the buyers during T is given by $C_{sb} \sum_{i=1}^N n_i$. The average ordering cost for the group of buyers and the supplier are as follow:

$$SC_b = \frac{C_{sb}}{T} \sum_{i=1}^N n_i, \quad (21)$$

and

$$SC_v = \frac{C_{sv}}{T}. \quad (22)$$

The average cost functions of all the buyers, the supplier and the whole supply chain are

$$TC_b = HC_b + DC_b + SC_b, \quad (23)$$

$$TC_v = HC_v + DC_v + SC_v, \quad (24)$$

and

$$TC = TC_b + TC_v, \quad (25)$$

respectively.

In order to optimise this supply chain the following non-linear program should be solved:

$$\text{Min } TC(T, n_i)$$

$$\text{Subject to } n_i \in \{1, 2, 3, \dots\} \text{ for } i \in \{1, 2, \dots, N\}. \quad (26)$$

As the number of inventory periods of the buyers within one supplier's inventory period cannot be a large number, enumeration is suggested. In case the number of retailers, N , is large, enumeration may take a long time to find a reasonable solution. With this regard a heuristic similar to Yang and Wee (2002) is developed which can solve the problem when N is relatively large:

Step 1. Assume $n_i = n$ for $i = 1, 2, \dots, N$. For a range of values assigned to n (enumeration), find the optimal value for T . As shown in Appendix A, the second derivative of the total cost function with respect to T is positive. Therefore by assuming n to be constant, there is one global optimum for T ;

Step 2. In the range of values assigned to n , find the optimal value which results in the lowest cost, and denote this value as n^* ;

Step 3. Set all n_i values equal to n^* (n_i values are candidates for the optimal solution);

Step 4. For retailer j , having fixed n_i ($i \neq j$), find the corrected value of n_j , $j = 1, 2, \dots, N$ which satisfies the following inequalities:

$$TC(n_j, n_i) \leq TC(n_j - 1, n_i) \text{ \& } TC(n_j, n_i) \leq TC(n_j^{\xi} + 1, n_i^{\xi});$$

Step 5. If in Steps 4 all n_i values remain unchanged, then $n_i^* = n_i$ and go to Step 6, otherwise repeat Steps 4;

Step 6. Stop.

4. Comparison with the literature

In the literature, a similar model has been analysed by Yang and Wee (2002). In their research in order to calculate the average holding cost of the supplier, Yang and Wee (2002) assume that the inventory level of the manufacturer is as shown in Fig. 3. The inventory cycle at the supplier is $T = T_1 + T_2$, where T_1 is the production period and T_2 is the non-production interval.

Considering the inventory level at the buyers and the supplier, depicted in Figs. 1 and 3, Yang and Wee (2002) calculate the

Fig. 3. Inventory level of the supplier (Yang and Wee, 2002).

supplier's inventory holding cost as follows:

$$HC_v^{YW} = \frac{p_v F_v}{T} \left[\int_0^{T_1} I_{v1}(t_1) dt_1 + \int_0^{T_2} I_{v2}(t_2) dt_2 - \sum_{i=1}^N n_i \int_0^{T/n_i} I_{bi}(t) dt \right]. \quad (27)$$

It should be noted that this equation calculates the average holding cost of the supplier.

The echelon stock of the supplier, $\tilde{I}_v(t)$, is the sum of the physical inventory of the supplier, $I_v(t)$, and the total echelon stock of the downstream buyers, $\tilde{I}_b(t)$. As buyer i is delivering the product to the end customer, his echelon stock, $\tilde{I}_{bi}(t)$, is the same as his physical inventory, $I_{bi}(t)$, hence $\tilde{I}_{bi}(t) = I_{bi}(t)$. Therefore the inventory level at the supplier is

$$I_v(t) = \tilde{I}_v(t) - \tilde{I}_b(t), \quad \forall t \geq 0, \quad (28)$$

where

$$\begin{aligned} \tilde{I}_b(t) &= \sum_{i=1}^N \tilde{I}_{bi}(t) \\ &= \sum_{i=1}^N I_{bi}(t), \quad \forall t \geq 0. \end{aligned} \quad (29)$$

In (27), Yang and Wee (2002) aim to use the idea presented in (28) to obtain the average inventory at the supplier. The researchers however assume that the production rate is significantly higher than the sum of the demand rates seen by the buyers. If this assumption holds then T_3 can be considered to be very small. This assumption however changes the model from a production-inventory to an inventory model. This means that the model developed by Yang and Wee (2002) will not result in a fairly accurate solution in case that the production rate is not much larger than the total demand rates, since it falls short in obtaining the average holding cost accurately. The numerical example developed by Yang and Wee (2002) is analysed further in Section 5.

5. Numerical example and analysis

Example 1. In this part the same example as in Yang and Wee (2002) is considered in which $N=2$, $p=20 \times 10^5$, $d_1=4 \times 10^4$, $d_2=8 \times 10^4$, $F_v=0.15$, $F_b=0.17$, $C_{sb}=200$, $C_{sv}=5000$, $p_v=10$, $p_b=12$, and $\theta=0.1$. The results of the example are presented in Table 1. It should be noted that $T_1 + T_3$ represents the production period in the supplier's inventory cycle.

The optimal production-inventory policy for this supply chain is as shown in Table 1. There is a small difference between the results of Yang and Wee (2002) and the ones obtained in this research. This difference is due to the underestimation of the supplier's average inventory in Yang and Wee (2002) and also the Taylor's expansion used in both studies. As mentioned earlier, Yang and Wee (2002) drop part of the production period which in case of huge surplus in production capacity their model results in fairly accurate solutions. For the optimal solution obtained in this research (0.1900, 2.3) this underestimation (the difference between (18) and (27)) results in an error of -7.03% ($\text{Error} = 100 \times (HC_v^{YW} - HC_v) / HC_v$) in the supplier's average holding cost. As expected, ignoring T_3 in Yang and Wee (2002) results in a lower

Table 1
Optimal solution of numerical Example 1.

Model used for the analysis	n_1	n_2	$T_1 + T_3$	T_2	T	TC_b	TC_v	TC
Yang and Wee (2002)	2	2	0.0115	0.1784	0.1899	22,732	38,252	60,983
Current model	2	3	0.0116	0.1784	0.1900	19,650	43,497	63,148

average inventory at the supplier. In this example there is a huge surplus in production capacity at the supplier (due to the deterioration the excess capacity is less than $20 \times 10^5 - (4+8) \times 10^4$) which is not applicable to some cases as manufacturers aim to exploit their capacity. By setting the supplier's capacity to 20×10^4 (which leaves a small excess capacity) this error goes to -58.86% . To show the error caused by this miscalculation, the optimal total cost has been calculated for different production rates using the model developed in Section 3 and the one in Yang and Wee (2002). The results are presented in Fig. 4.

Table 2 presents the optimal solution using the current model and the model developed by Yang and Wee (2002) for two different cases; when the production rate is 20×10^4 and when the production rate extends infinitely. As can be seen the error of Yang and Wee (2002) is negligible when there is an inventory system (infinite production rate). This shows that in case of huge surplus in production capacity or infinite production rate, the model of Yang and Wee (2002) results in sufficiently accurate solutions.

Example 2. In order to see how the current model reacts to the changes in parameters, a sensitivity analysis is conducted. For this purpose the same numerical example is considered with the production rate of 1.4×10^5 . The results of this numerical example are presented in Table 3.

Furthermore the parameters of the model are decreased and increased by 5% and 10% to see the changes in T , TC , HC_v/p_vF_v (to see how the average inventory level at the vendor changes), I_{m1} , I_{m2} , and $(T_1+T_3)/T$ (the ratio of the production period to the whole cycle). Note that the changes in HC_v (average inventory holding cost) and HC_v/p_vF_v (average inventory) are exactly the same as long as p_v and F_v are fixed. The new optimal value for any of these measures (X^{new}) is compared with the corresponding value obtained from the original problem (X^*) using $\delta = 100(X^{new} - X^*)/X^*$.

In Tables 4–9 we report the parameters that the model's measures (one or some of them) show sensitivity to. The rest of the parameters are not presented here as the model shows sensitivity as little as one percent or less to any changes made to them.

The model shows the highest sensitivity to the production rate. As shown in Table 4, for the lower production rates, the model

Table 3
Optimal solution for numerical Example 2.

n_1	n_2	T_1+T_3	T_2	T	TC_b	TC_v	TC
5	7	0.2308	0.0349	0.2657	17,403	38,324	55,727

Fig. 4. Optimal TC when p changes, current model versus Yang and Wee (2002).

Table 2
Optimal solution.

$p = 20 \times 10^4$	n_1	n_2	T_1	T_2	T_3	T	TC_b	TC_v	TC
Yang and Wee (2002)	3	4	0.2004	0.1327	0	0.3331	22,225	16,244	38,468
Current model $p \rightarrow \infty$	4	5	0.1094	0.0902	0.0299	0.2295	17,520	41,729	59,248
Yang and Wee (2002)	2	2	ϵ^a	0.1829	0	0.1829	22,187	41,127	63,314
Current model	2	2	ϵ^a	0.1835	ϵ^a	0.1835	22,230	40,985	63,215

^a ϵ is a very small positive value.

Table 4The optimum and the results of the sensitivity analysis when p changes (%).

$p(10^5)$	Optimal solution		Changes in model results (%)						
	n_1	n_2	T	TC	HC_v/p_vF_v	I_{m1}	I_{m2}	$(T_1+T_3)/T$	
1.26	5	7	2.94	-2.83	-23.09	2.95	2.95	11.15	
1.33	5	7	1.36	-1.33	-10.78	1.36	1.36	5.28	
1.4	5	7	0	0	0	0	0	0	
1.47	5	7	-1.18	1.19	9.52	-1.19	-1.19	-4.78	
1.54	4	6	-7.42	2.15	18.31	15.77	8.03	-9.18	

Table 5The optimum and the results of the sensitivity analysis when d_1 changes (%).

$d_1(10^4)$	Optimal solution		Changes in model results (%)						
	n_1	n_2	T	TC	HC_V/p_VF_V	I_{m1}	I_{m2}	$(T_1+T_3)/T$	
3.6	5	7	0.98	-0.98	4.69	-9.11	0.99	-3.32	
3.8	5	7	0.48	-0.48	2.38	-4.54	0.48	-1.66	
4	5	7	0	0	0	0	0	0	
4.2	5	7	-0.46	0.47	-2.45	4.52	-0.46	1.66	
4.4	5	7	-0.91	0.92	-4.98	9.00	-0.91	3.32	

Table 6The optimum and the results of the sensitivity analysis when d_2 changes (%).

$d_2(10^4)$	Optimal solution		Changes in model results (%)						
	n_1	n_2	T	TC	HC_v/p_vF_v	I_{m1}	I_{m2}	$(T_1+T_3)/T$	
7.2	5	6	-1.00	-1.73	10.00	-1.01	3.98	-6.68	
7.6	5	7	0.80	-0.80	4.99	0.81	-4.24	-3.32	
8	5	7	0	0	0	0	0	0	
8.4	5	7	-0.73	0.74	-5.38	-0.73	4.23	3.32	
8.8	5	7	-1.38	1.42	-11.15	-1.38	8.48	6.65	

Table 7The optimum and the results of the sensitivity analysis when C_{sv} changes (%).

$C_{sv}(10^3)$	Optimal solution		Changes in model results (%)						
	n_1	n_2	T	TC	HC_v/p_vF_v	I_{m1}	I_{m2}	$(T_1+T_3)/T$	
4.5	5	7	-3.44	-3.44	-3.42	-3.45	-3.45	-0.05	
4.75	5	7	-1.71	-1.70	-1.70	-1.71	-1.71	-0.02	
5	5	7	0	0	0	0	0	0	
5.25	5	7	1.67	1.67	1.66	1.68	1.68	0.02	
5.5	5	7	3.32	3.32	3.30	3.33	3.33	0.04	

Table 8The optimum and the results of the sensitivity analysis when θ changes (%).

θ	Optimal solution		Changes in model results (%)					
	n_1	n_2	T	TC	HC_v/p_vF_v	I_{m1}	I_{m2}	$(T_1+T_3)/T$
0.09	5	7	3.09	-2.99	3.13	3.07	3.07	-0.09
0.095	5	7	1.51	-1.49	1.53	1.50	1.50	-0.05
0.1	5	7	0	0	0	0	0	0
0.105	5	7	-1.44	1.46	-1.46	-1.43	-1.43	0.05
0.11	5	7	-2.82	2.91	-2.86	-2.80	-2.81	0.09

Table 9The optimum and the results of the sensitivity analysis when p_v changes (%).

p_v	Optimal solution		Changes in model results (%)						
	n_1	n_2	T	TC	HC_v/p_vF_v	I_{m1}	I_{m2}	$(T_1+T_3)/T$	
9	5	7	3.70	-3.56	3.67	3.71	3.70	0.05	
9.5	5	7	1.80	-1.77	1.79	1.80	1.80	0.02	
10	5	7	0	0	0	0	0	0	
10.5	5	7	-1.71	1.74	-1.70	-1.71	-1.71	-0.02	
11	5	7	-3.32	3.44	-3.30	-3.33	-3.33	-0.04	

suggests higher ratio for production period to the whole cycle. This is intuitive that the lower production rate prevents from the accumulation of the items (due to the lower difference between the production and the demand) at the vendor, hence lower average inventory, a 10% decrease in the production rate results in a 23% decrease in the vendor's average inventory. Increasing the production rate, suggests accumulating the items at the vendor (and not at the retailers due to their higher deterioration rate and holding cost), and of course reducing the production cycle ratio, as longer production time results in more on-hand inventory, hence higher deterioration and holding cost. This pattern of change persists while the combination of T , n_1 and n_2 creates a balance between different cost functions and results in a minimum for the total cost function.

More increase in the production rate (for instance 1.54×10^5) will bring too much of deterioration and holding cost at the vendor, therefore the model suggests much lower inventory period as well as lower frequencies for the retailers to avoid huge increase in the total cost function. Lower replenishment frequency at the buyers shows an increase in the order quantities as the result of the trade-off between ordering cost, deterioration and holding cost. This indicates that in a flexible production environment how the manufacturer can decrease the total cost of the supply chain.

An increase/decrease in the demand at buyer i , increases/decrease the order quantity of that buyer as long as the frequency of replenishment remains unchanged (see Tables 5 and 6). This will result in a change in the average inventory at the vendor in the opposite direction. A change in the demand at any of the buyers moves the production cycle ratio in the same direction to adjust the system accordingly. Note that the effect of the changes in d_2 on HC_v/p_vF_v and $(T_1+T_3)/T$ is larger compared to the effect of changes in d_2 . Table 6 shows that an increase of 10% in d_2 increases the total cost of the system by 1.42%, however it should be reminded that the revenue obtained by buyer 2 is increased linearly (10%) that all in all results in a better performance of the system.

Table 7 shows that any change in C_{sv} , moves all the measures studied in this sensitivity analysis, in the same direction by the same amount except for the production cycle which remains unchanged. Taking into account the trade-off between the ordering cost and the deterioration and holding cost, the model suggests a shorter inventory cycle in case of a decrease in C_{sv} . In practice setting the value of C_{sv} depends on the vendor and its upstream suppliers and how they formulate the collaboration in between or how they improve the technologies (if applicable) used.

Any change in θ or p_v moves the measures in the opposite direction except for TC that changes in the same direction and the production cycle ratio that remains unchanged (see Tables 8 and 9). Higher deterioration rates motivate the model to reduce the inventory cycle to avoid deterioration at the echelons. Intuitively this results in lower order quantities. Note that the range presented for θ keeps the values for n_i unchanged. As Table 8 shows, the trade-off between the average ordering cost and holding cost, when there

is an increase in the deterioration rate, moves the optimal solution to a new optimum with a higher total cost.

6. Conclusions

In this paper a production-inventory model is developed in which a manufacturer (vendor) is delivering a deteriorating product to N retailers. Very few similar models have been studied in the literature. The existence of multiple buyers that can have different inventory periods makes the supplier's inventory level hard to obtain. In order to avoid complexity in modelling, these research works have used some approximation that restrict the application of them and reduce their accuracy.

In this study the exact average inventory level at the vendor is obtained as the difference between the echelon stock of the supplier and the physical inventory of the downstream buyers. Yang and Wee (2002) adopt the same approach and drop a part of the inventory cycle under the assumption of having a huge surplus in production capacity. In this paper it is shown that in case the production capacity is much larger than the total demand, the model developed by Yang and Wee (2002) works with high accuracy. The level of accuracy in Yang and Wee (2002), however does not stay high, in the case that the production rate is just large enough to cover the demand. In the current paper, it is shown how large the error in the vendor's holding cost can be in some cases when using the model developed in Yang and Wee (2000, 2002). It is also shown that such models are more accurate when the system is an inventory model or in case of a finite production rate, that the rate is large relative to the total demand rate.

In the sensitivity analysis conducted in this paper, it is shown which parameters can have a greater effect on the output of the model. It also has been discussed that how these parameters make changes in the results. This sensitivity analysis highlights the most important parameters that should be taken into account when making an investment or improvement in the system in order to optimise the output of the system.

Implementation of supply chain models is a huge challenge and is highly dependent on the relations within the supply chains and the level of the collaboration between the partners in the supply chain. This model can be extended in different ways in order to get closer to real-world problems. It is suggested that the time value of money should be considered in the model as it can have significant impact on optimality, especially when the opportunity cost of capital is high. Also studying the effects of backlogging on the optimal solutions may be interesting in case of high purchasing price the model aims to consider the trade-off between the high capital cost and the shortage cost. Such model gives helpful insights into the systems in which backlogging can be considered.

where $a = \sum_{i=1}^N d_i/n_i$, $b = \sum_{i=1}^N d_i/n_i^2$ and $d = \sum_{i=1}^N d_i$. The second derivative of the total cost function, presented in Eq. (30), is positive for all the values of T .

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Appendix A. Derivatives of the total cost function

The second derivative of the total cost function presented in Eq. (25) with respect to T is as follows:

$$\begin{aligned} \frac{\partial^2 TC}{\partial T^2} = & \frac{1}{3} p_b F_b (d - b) + \frac{p_v F_v \theta}{p^2} (d^2 (p - d) + abp + ad(d - a) + a^3) \\ & + \frac{2p_v F_v \theta d^2}{p} + \frac{\theta b}{3} (p_b F_b - p_v F_v) + \frac{2C_{sb} \sum_{i=1}^N n_i}{T^3} + \frac{2C_{sv}}{T^3} \end{aligned} \quad (30)$$