Modelling and Control of a Two-link Flexible Robot Manipulator

J. de Wit
D&C 2014.043

Master Internship

Coach: Dr. M. Reyhanoglu
Embry-Riddle Aeronautical University
Department of Engineering Physics
Systems & Control

Supervisor: Prof.dr. H. Nijmeijer
Eindhoven University of Technology
Department of Mechanical Engineering
Dynamics & Control

Daytona Beach, September, 2014
Contents

Acknowledgements v
Abstract vi

1 Introduction 1
  1.1 Background . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
  1.2 Literature review . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
  1.3 Problem definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
  1.4 Outline . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

2 Dynamical model 4
  2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
  2.2 Hybrid parameter modeling . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
  2.3 Discussion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11

3 Controller design and simulations 12
  3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
  3.2 PD controller . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
  3.3 Notch filter . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
  3.4 Control cases . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
    3.4.1 Case 1: PD control . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
    3.4.2 Case 2: PD control including filtered input \( u_1 \) . . . . . . . . . . . 15
    3.4.3 Case 3: PD control including filtered input \( u_1 \) and \( u_2 \) . . . . . . . 16
  3.5 Physical values . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  3.6 Simulations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
    3.6.1 Case 1: PD control . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
    3.6.2 Case 2: PD control including filtered input \( u_1 \) . . . . . . . . . . . . 18
    3.6.3 Case 3: PD control including filtered input \( u_1 \) and \( u_2 \) . . . . . . . 18
  3.7 Discussion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

4 Conclusions and recommendations 24
  4.1 Conclusions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
    4.1.1 Dynamic model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
    4.1.2 Control cases . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
  4.2 Recommendations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
    4.2.1 Dynamic model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
    4.2.2 Control cases . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25
CONTENTS

References 26
A Appendix 1 28
Acknowledgements

First of all I would like to thank professor Reyhanoglu for his close guidance during this project. Without his wisdom and support this thesis would have been less successful. Furthermore, I would like to give a word of appreciation to professor Nijmeijer, who gave me the opportunity for this educational and memorable journey.

My grateful thanks are also extended to Michael Hsu, Mayuri Kokate, Kyle Rivers, and Lane Wilkinson who accepted me and became close and helpful friends during my stay. Finally, I would like to thank my family for supporting me financially and emotionally and encouraging me to take this once in a lifetime chance.
Abstract

Over the last years, flexible structured manipulators have increased in popularity in aeronautical research because of their advantages compared to conventional rigid structures. Unfortunately, the flexibility of the structure has one big disadvantage; it causes undesired motions of the spacecraft. The flexibility of the structure generates structural vibrations that strongly interferes with rigid-body attitude dynamics. The dynamics of the structure increases drastically when the amount of bodies increase, and so far researchers were not capable to develop an adequately dynamical model. This study describes the derivation of a new dynamical model for a two-link flexible robot manipulator with, usually unconsidered, incorporated effects of practical parameters. Based on this theoretical framework, three different controller designs are introduced. To regulate the rigid body motions a proportional-plus-derivative (PD) control method is implemented and a notch filter is introduced to suppress the residual vibrations. The three control cases are individually designed by a combinations of these control techniques. Finally, computer simulation results are used to analyze the effects of the three different control cases.
Chapter 1

Introduction

1.1 Background

Over the last decade, research regarding the dynamic modelling and control of flexible manipulators have received increased attention. Flexible robotics are a hot topic in aerospace research because they can be applied to a large scale of structures such as large flexible satellites, space antennae, and space telescopes. An example of a satellite with flexible solar panels is represented in Figure 1.1. Besides the aerospace applications, these types of robots are used in a wide spectrum of industries. Flexible manipulators in the field of civil infrastructure, bridge/vehicle systems, and military intelligence are namely very common. Compared to the conventional rigid body manipulator, the flexible structured manipulators have several advantages: they are lighter in weight, consume less power, require less material, are more manoeuvrable and transportable, require smaller actuators, and have less cost. Next to the advantages, flexibility of the structure has one big disadvantage: it causes undesired motions of the spacecraft.
1.2 Literature review

In the literature, many mathematical models of single flexible manipulators have been established. Generally, these models can be divided into two different categories: the numerical analysis approach and the assumed mode method. The numerical analysis approach mainly uses finite element methods [2, 14] to characterise the dynamics of a single-link flexible manipulator system. The assumed mode method models the deflection of the elastic structure by a finite series of space-dependent shape functions that are multiplied by time-dependent amplitude function [3, 8, 10].

In order to regulate the angular position of the single flexible manipulator as accurate as possible, a wide spectrum of control strategies has been proposed. Paper [12] proposes a finite-time control method and gives an overview of the developed control strategies. In [6] various standard controller designs for the single-link flexible manipulator are compared. A Lyapunov-type controller based on the deflection feedback can be found in [15].

For two-link flexible manipulator the complexity of the models increases drastically. Therefore, less research regarding flexible multi-body manipulators can be found. Papers [16, 17] propose a finite element model to describe the dynamics of a planar two-link flexible robot manipulator. A dynamical model using the finite assumed mode method and incorporating hub inertia, and payload can be found in [1]. In paper [13] a comparative study on methods for model-based and model-free control of flexible-link robots is reported. Furthermore, several case studies of two-link flexible manipulators are computed. A study with a rigid first and flexible second link can be found in [5] and in [7] the first motor is replaced by a rigid clamping.

Control methods for two-link flexible manipulators are developed using energy-based nonlinear control [4, 13], neural adaptive control [13], and iterative learning control [9]. Other controller designs for two-link flexible systems can be found in references herein. The book by Junkins and Kim [8] describes a complete approach to develop the dynamics and control of infinite-dimensional flexible structures.

1.3 Problem definition

As a result of the flexibility, the link generates undesired structural vibrations that strongly interfere with rigid-body attitude dynamics. Therefore, the dynamics of the flexibility need to be taken into account while achieving high precision attitude demands. This results in an underactuated system which makes it challenging to develop a proper control algorithm.

By developing a detailed mathematical model of the motions a solid theoretical base can be realized. By means of this model a controller can be designed to regulate the motions of the system as accurate as possible. In preliminary research of these models, the effects of practical parameters such as the hubs and actuators on the dynamic characteristics of the system are not adequately discussed. Therefore, a detailed description of a complete mechanical model needs to be included in the report.

Based on the theoretical model framework, a controller design has been introduced to suppress the undesired vibrations. Because of the relatively short available time for this project, the controller design will be limited to a proportional-and-derivative control technique to control the rigid body motions. On top of this, a filter must be implemented to investigate its effect on the vibrational motion
of the flexible link.

1.4 Outline

The outline of this report will be as follows. A detailed derivation of the new dynamical model is given in chapter 2. Chapter 3 introduces the design of the controller. Next, chapter 4 will show simulation results of the applied control cases. Finally, chapter 5 gives concluding remarks and recommendations for future research.
Chapter 2

Dynamical model

2.1 Introduction

A typical spacecraft structure can be described by two principle parts: rigid bodies and flexible appendages. The rigid bodies represent the spacecraft construction which contains all the payload instrumentation and control hardware. In order to withstand the mechanical loads during the launch stage, this construction must be rigid. The flexible appendages indicate lightweight equipment, such as solar arrays and solar panels, used for making the spacecraft operational. This study is focussed on an application of two serial ordered rigid bodies (hubs) and a flexible first and rigid second link.

2.2 Hybrid parameter modeling

Figure 2.1 represents a schematic model of a two-link serial flexible robot manipulator. Both ends of link 1 are clamped in by hub 1 and hub 2. The second link is clamped to the mounting bracket of the second hub on one end and free at the other end. The links are actuated by individual motors integrated in the rigid hubs, both depicted by the circles. Frame $X_0Y_0Z_0$ denote the inertial coordinate frame and $x_iy_iz_i$ represent the rigid body frame associated with the $i$th link. Vectors $\hat{e}_0$, $\hat{e}_1$, and $\hat{e}_2$ include the unit vectors of the inertial, first, and second coordinate frames respectively. $\theta_i(t)$ is the angular position of the $i$th link with respect to the base frame. The actuators’ control torque about the z-axis are given as $\tau_1$ and $\tau_2$. $y_1(x_1, t)$ is designated as the transverse deformation of the first link at point $x_1$ with respect to the rigid link configuration. Because there is no payload attached to the end of the second link, this link will be considered as rigid. More physical parameters of this robot manipulator are shown in Table 2.1. The inertias and mass values are obtained from [11] and the remaining parameters are identified by measuring the set-up.

Because of the rigid and flexible link combination, a hybrid technique of distributed and discrete parameter modeling will be used to obtain the dynamical model [8]. In the development of the model the following assumptions are considered:

I. The robot manipulator only performs planar motion in a horizontal plane;
II. The links are considered to be uniform flexible beams;
III. Euler-Bernoulli assumptions for the long and slender links are made, i.e. negligible shear deformation and distributed rotary inertia;
IV. The deflection of each link is assumed to be small, therefore a linear theory of elasticity can be used;
V. Due to the final application of the setup in outer space, effects from atmospheric drag are ignored;
VI. Coulomb friction effects and backlash in the reduction gear are neglected;
VII. The velocity component of the flexible link in x-direction is neglected.

The formulation of the dynamic model is based on Lagrange’s equation of motion [8], which are derived from the total energy in the system. This total energy can be separated into the potential \((U)\) and kinetic energy \((T)\). For the two-link flexible system, the potential energy consists only of the elastic energy of the first link \((U_1)\) expressed as:

\[
U = U_1 \tag{2.1}
\]

where

\[
U_1 = \frac{1}{2} \int_{0}^{l_1} EI_1 (y_1''(x_1, t))^2 dx_1 \tag{2.2}
\]

The kinetic energy consists of the translational \(T_t\) and rotational \(T_r\) energy of the links and associated hubs, so

\[
T = T_{t1} + T_{t2} + T_{r1} + T_{r2} \tag{2.3}
\]
Table 2.1: Physical parameters of the robot manipulator

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>Mass density link 1</td>
<td>0.7597</td>
<td>kgm$^{-1}$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>Mass density link 2</td>
<td>0.2662</td>
<td>kgm$^{-1}$</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Length rotary axis motor to link clamping</td>
<td>0.06465</td>
<td>m</td>
</tr>
<tr>
<td>$l_1$</td>
<td>Length link 1</td>
<td>0.22</td>
<td>m</td>
</tr>
<tr>
<td>$l_2$</td>
<td>Length link 2</td>
<td>0.22</td>
<td>m</td>
</tr>
<tr>
<td>$m_{H_2}$</td>
<td>Mass of hub 2</td>
<td>0.593</td>
<td>kg</td>
</tr>
<tr>
<td>$m_{mb_2}$</td>
<td>Mass of second mounting bracket</td>
<td>0.4036</td>
<td>kg</td>
</tr>
<tr>
<td>$m_{l_2}$</td>
<td>Mass of link 2</td>
<td>0.0586</td>
<td>kg</td>
</tr>
<tr>
<td>$EI_1$</td>
<td>Flexural rigidity link 1</td>
<td>2.69</td>
<td>Nm$^2$</td>
</tr>
<tr>
<td>$J_{z_1}$</td>
<td>First motor and mounting bracket inertia</td>
<td>$7.424 \cdot 10^{-4}$</td>
<td>kgm$^2$</td>
</tr>
<tr>
<td>$J_{z_2}$</td>
<td>Second motor and mounting bracket inertia</td>
<td>$4.456 \cdot 10^{-4}$</td>
<td>kgm$^2$</td>
</tr>
</tbody>
</table>

where

\[
T_{t_1} = \frac{1}{2} \int_0^{l_1} \rho_1 \dot{r}_1^2 dx_1
\]

(2.4)

\[
T_{t_2} = \frac{1}{2} m_{H_2} \dot{r}_{H_2}^2 + \frac{1}{2} m_2 \dot{r}_2^2
\]

(2.5)

\[
T_{r_1} = \frac{1}{2} J_{z_1} \dot{\theta}_1^2
\]

(2.6)

\[
T_{r_2} = \frac{1}{2} J_{z_2} \dot{\theta}_2^2
\]

(2.7)

Herein, $\dot{r}_1^2$ and $\dot{r}_2^2$ denote the square of the velocity vector for respectively the first link and second link with associated mounting bracket. $\dot{r}_{H_2}^2$ is the square of the velocity vector of the second hub. In this case the second hub consists of the structure that connects the end of the first link to the second actuator. Position vector $\vec{r}_1$ is derived by a summation of individual position vectors. Fig 2.2 depicts the method for deriving the position vector $\vec{r}_1$ of the first link. If now a point $A$ is placed on the clamping of the link to the first rigid hub, then vector $\vec{r}_A$ describes the distance from the origin of the inertial axis to point $A$. When a second point $B$ is placed on the link at a distance $x_1$ from point $A$, vector $\vec{r}_{B/A}$ can be computed. Consequently, position vector $\vec{r}_1$ can be written as:

\[
\vec{r}_1(x_1, t) = \vec{r}_B(x_1, t) = -\vec{r}_A + \vec{r}_{B/A}(x_1, t) \quad \forall x_1 \in [0, l_1]
\]

(2.8)

where

\[
-\vec{r}_A = - \begin{bmatrix} l_0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -e_1^4 \\ e_2^4 \\ e_3^4 \end{bmatrix}
\]

(2.9)

\[
\vec{r}_{B/A} = - \begin{bmatrix} x_1 & y_1(x_1, t) & 0 \end{bmatrix} \cdot \begin{bmatrix} e_1^4 \\ e_2^4 \\ e_3^4 \end{bmatrix}
\]

(2.10)

Resulting in

\[
\vec{r}_1 = \begin{bmatrix} x_1 + l_0 & y_1(x_1, t) & 0 \end{bmatrix} \cdot \vec{e}_4
\]

(2.11)
2.2. HYBRID PARAMETER MODELING

The velocity vector of the first link is computed by taking the time derivative of (2.11) as follows:

\[
\dot{\mathbf{r}}_1 = \begin{bmatrix} \dot{x}_1 & \dot{y}_1(x, t) & 0 \end{bmatrix} \cdot \mathbf{e}_1^d + \begin{bmatrix} x_1 + l_0 & y_1(x, t) & 0 \end{bmatrix} \cdot \mathbf{e}_2^d
\] (2.12)

where

\[
\mathbf{e}_2^d = \mathbf{\omega} \times \mathbf{e}_1^d
\] (2.13)

From APPENDIX 1 follows

\[
\mathbf{\omega} = \dot{\theta}_1 \mathbf{e}_3
\] (2.14)

Equation (2.13) becomes

\[
\dot{\mathbf{e}}^d = \dot{\theta}_1 \mathbf{e}_3 \times \mathbf{e}_1^d
\] (2.15)

Substituting (2.15) in (2.12) gives

\[
\dot{\mathbf{r}}_1 = \dot{x}_1 \mathbf{e}_1^d + \dot{y}_1(x_1, t) \mathbf{e}_2^d + (x_1 + l_0) \dot{\theta}_1 \mathbf{e}_3^d - y_1(x_1, t) \dot{\theta}_1 \mathbf{e}_1^d
\] (2.16)

Based on assumption VII, (2.16) can be simplified to

\[
\dot{\mathbf{r}}_1 = \begin{bmatrix} 0 & (x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t) & 0 \end{bmatrix} \cdot \mathbf{e}_1^d
\] (2.17)

The square of the first link's velocity vector is computed via

\[
\dot{\mathbf{r}}_1^2 = \begin{bmatrix} 0 & (x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t) & 0 \end{bmatrix} \cdot \mathbf{e}_1^d \cdot \mathbf{e}_1^d^T \cdot \begin{bmatrix} (x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t) \end{bmatrix}
\] (2.18)

\[
= \begin{bmatrix} 0 & (x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t) & 0 \end{bmatrix} \cdot \begin{bmatrix} (x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t) \end{bmatrix}
\] (2.19)

\[
= ((x_1 + l_0) \dot{\theta}_1 + \dot{y}_1(x_1, t))^2
\] (2.20)
Similar to the case of the first position vector, \( \vec{r}_2 \) can be computed. Fig 2.3 depicts the method for deriving the position vector \( \vec{r}_2 \) of the second link and associated mounting bracket. When a point \( H_2 \) is placed on the rotation axis of the second actuator, the position vector of the second hub can be expressed as:

\[
\vec{r}_{H_2} = \begin{bmatrix} 2l_0 + l_1 & y_1(l_1, t) \gamma & 0 \end{bmatrix} \cdot \vec{e}_1
\] (2.21)

where

\[
\gamma = \frac{l_1 + l_0}{l_1}
\] (2.22)

The velocity vector is computed by taking the time derivative of Equation 2.21 as follows

\[
\dot{\vec{r}}_{H_2} = \begin{bmatrix} 0 & \dot{y}_1(l_1, t) \gamma & 0 \end{bmatrix} \cdot \vec{e}_1 + \begin{bmatrix} 2l_0 + l_1 & y_1(l_1, t) \gamma & 0 \end{bmatrix} \cdot \dot{\vec{e}}_1
\] (2.23)

By substitution of (2.15), Equation 2.23 can be written as

\[
\dot{\vec{r}}_{H_2} = \begin{bmatrix} -y_1(l_1, t) \gamma \dot{\theta}_1 & \dot{y}_1(l_1, t) \gamma & 0 \end{bmatrix} \cdot \vec{e}_1
\] (2.24)

The square of Equation 2.24 can now be given as

\[
\dot{\vec{r}}^2_{H_2} = \left[ -y_1(l_1, t) \gamma \dot{\theta}_1 \right]^2 + \left[ \dot{y}_1(l_1, t) \gamma \right]^2
\] (2.25)

Let point \( CG_2 \) denote the center of gravity of the second link including mounting bracket. The distance from \( H_2 \) to \( CG_2 \) can now be described by vector \( \vec{r}_{CG_2} \) as

\[
\vec{r}_{CG_2} = \begin{bmatrix} x_{CG_2} & 0 & 0 \end{bmatrix} \cdot \vec{e}_2
\] (2.26)
2.2. HYBRID PARAMETER MODELING

Herein, \( x_{CG_2} \) is the x-coordinate of the center of gravity of the second link including mounting bracket with respect to the second coordinate frame given as

\[
x_{CG_2} = \frac{(l_0 + \frac{1}{2}l_2)m_{l_2}}{m_2} + \frac{1}{2}l_0m_{mb_2} \tag{2.27}
\]

where

\[
m_2 = m_{l_2} + m_{mb_2} \tag{2.28}
\]

Since

\[
\vec{x}^2 = \Delta^{21, \alpha} = \begin{bmatrix}
  c(\theta_2 - \theta_1) & s(\theta_2 - \theta_1) & 0 \\
  -s(\theta_2 - \theta_1) & c(\theta_2 - \theta_1) & 0 \\
  0 & 0 & 1
\end{bmatrix} \vec{\xi}^4 \tag{2.29}
\]

Equation (2.26) can be written as

\[
\vec{r}_{CG_2} = \begin{bmatrix}
x_{CG_2}(\theta - \theta_1) \\
x_{CG_2}(\theta - \theta_1) \\
0
\end{bmatrix} \cdot \vec{\xi}^4 \tag{2.30}
\]

Consequently, position vector \( \vec{r}_2 \) can be computed by a summation of (2.21) and (2.30):

\[
\vec{r}_2 = \vec{r}_{H_2} + \vec{r}_{CG_2} = \begin{bmatrix}
2l_0 + l_1 + x_{CG_2}(\theta_2 - \theta_1) \\
0
\end{bmatrix} \cdot \vec{\xi}^4 \tag{2.31}
\]

The velocity vector is computed by taking the time derivative of (2.31) as follows

\[
\dot{\vec{r}}_2 = \begin{bmatrix}
-x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) \\
y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)
\end{bmatrix} \cdot \vec{\xi}^4 + \begin{bmatrix}
2l_0 + l_1 + x_{CG_2}(\theta_2 - \theta_1) \\
y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1)
\end{bmatrix} \cdot \vec{\xi}^4 \tag{2.32}
\]

Substitution of (2.15) results in

\[
\dot{\vec{r}}_2 = \begin{bmatrix}
-x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) - y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) \\
y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1)
\end{bmatrix} \cdot \vec{\xi}^4 \tag{2.33}
\]

Now the square of the velocity vector of the second link with associated mounting bracket can be expressed as

\[
\dot{\vec{r}}_2^2 = \left[ -x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) - y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) \right]^2 + \left[ y_1(l_1, t)\gamma + x_{CG_2}(\theta_2 - \theta_1) \right]^2 \tag{2.34}
\]

The bending deflection \( y_1(x_1, t) \) can be expressed using the assumed mode method:

\[
y_1(x_1, t) = \phi_j(x_1)s_j(t) \tag{2.36}
\]

where \( s_j(t) \) is the generalized coordinate corresponding to the \( j \)th vibrational mode, and \( \phi_j(x_1) \) is the \( j \)th mode shape function of, which is given [8] as

\[
\phi_j(x_1) = 1 - \cos\left(\frac{j\pi x_1}{l_1}\right) + \frac{1}{2}(-1)^{(j+1)}\left(\frac{j\pi x_1}{l_1}\right)^2 \tag{2.37}
\]

In order to avoid unnecessary complexities, the simplest model will be used. Therefore, only the first mode is considered, so \( N = j = 1 \). This leads to a strong simplification of the model and the
Moreover, the non-conservative virtual work can be evaluated as
\[ \delta W_{nc} = Q^T \delta q \] (2.54)
Here, \( Q \) is the nonconservative generalized force vector associated with \( q \) and \( \delta q \) is the associated virtual displacement. Considering the two-link flexible robot manipulator (2.54) can be written as

\[
\delta W_{nc} = (\tau_1 - \tau_2)\delta \theta_1 + \tau_2 \delta \theta_2 - \tau_2 a_h' \delta s_1
\]  

(2.55)

Therefore, the generalized forces become

\[
Q = \begin{bmatrix}
\tau_1 - \tau_2 \\
\tau_2 \\
-\tau_2 a_h'
\end{bmatrix}
\]  

(2.56)

Using (2.45) and (2.54) Hamilton’s principle can be extended resulting in Lagrange’s equations of motion for the hybrid two-link system:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = Q^T
\]  

(2.57)

Herein \( R \) is the Rayleigh dissipation function which includes the structural damping expressed as

\[
R = \frac{1}{2} c s_1^2
\]  

(2.58)

By substitution of (2.45), (2.56), (2.58) and (2.36) in (2.57) Lagrange’s equations of motion for the hybrid two-link robot manipulator can be written as

\[
\ddot{\theta}_1 = \frac{1}{(I_1 + m_{s_1}s_1^2)} \left( -m_{s_1}s_1^2 \ddot{\theta}_1 - m_{s_1} \dot{s}_1 \dot{\theta}_1 - m_{\dot{\theta}_1} \dot{s}_1 \dot{\theta}_1 + \frac{\partial m_{s_1}\dot{s}_1}{\partial \theta_1} \dot{s}_1 \dot{\theta}_1 \\
+ \frac{\partial m_{\dot{s}_1}\dot{\theta}_1}{\partial \theta_1} \dot{s}_1 \dot{\theta}_1 + \tau_1 - \tau_2 \right)
\]  

(2.59)

\[
\ddot{\theta}_2 = \frac{1}{I_2} \left( -\dot{m}_{s_1} \dot{s}_1 \dot{\theta}_1 - m_{s_1} \dot{s}_1 \dot{\theta}_1 - \dot{m}_{\dot{\theta}_1} \dot{s}_1 \dot{\theta}_1 - m_{\dot{s}_1} \dot{\theta}_1 \dot{s}_1 \dot{\theta}_1 + \frac{\partial m_{\dot{s}_1}\dot{\theta}_1}{\partial \theta_2} \dot{s}_1 \dot{\theta}_2 + \tau_2 \right)
\]  

(2.60)

\[
\ddot{s}_1 = \frac{1}{m_{s_1}} \left( -m_{s_1} \dot{s}_1 \dot{\theta}_1 - \dot{m}_{s_1} \dot{\theta}_1 \dot{s}_1 - m_{s_1} \dot{s}_1 \dot{\theta}_1 - m_{s_1} s_1 \dot{\theta}_1^2 - k \dot{s}_1 - c \dot{s}_1 + \frac{\partial m_{\dot{s}_1}\dot{\theta}_1}{\partial s_1} \dot{s}_1 \dot{\theta}_1 - a_h' \tau_2 \right)
\]  

(2.61)

\[ \]  

2.3 Discussion

In this chapter, a dynamic model is developed that describes the motions of a two-link serial flexible robot manipulator. Obviously this system is highly non-linear, and because of the fact that there are three generalized coordinates and only two inputs this dynamical system is underactuated. Both aforementioned observations makes it challenging to design a controller to regulate the angular position of the rigid bodies and suppresses the residual vibrations at the mean time. In the next chapter different controller designs will be proposed to solve this problem.
Chapter 3

Controller design and simulations

3.1 Introduction

This section includes the design of a controller in order to regulate the dynamics of the two-link flexible robot manipulator. First, the application of a proportional-plus-derivative (PD) controller will be discussed. Next, section 3.3 treats the background of a notch filter. Hereafter, in section 3.4 different combinations of controller designs are introduced and for each design a state space realization is develop for future simulations. To obtain results from these simulations, first the necessary physical values are derived in section 3.5. Finally, section 3.6 will show the results of the implementation of the different control cases obtained by simulations.

3.2 PD controller

To control the rigid body motion a PD controller is introduced. This control algorithm has to be applied to the mathematical model (2.59) - (2.61) derived in chapter 2 as follows

\begin{align*}
\ddot{\theta}_1 &= \frac{1}{(I_1 + m_s s_1^4)} \left( -m_s s_1^2 \dot{\theta}_1 - m_{s_1} \dot{\theta}_1 \dot{s}_1 - \dot{m}_{s_1} \dot{\theta}_1 \dot{s}_1 - \dot{m}_{s_1} \dot{s}_1 - \ddot{m}_{s_1} \dot{\theta}_1 + \frac{\partial m_{s_1}}{\partial \theta_1} \dot{s}_1 \dot{\theta}_2 + \frac{\partial m_{s_1}}{\partial \theta_2} \dot{s}_1 \dot{\theta}_2 \right) \\
\ddot{\theta}_2 &= \frac{1}{I_2} \left( -\dot{m}_{s_1} \dot{\theta}_2 \dot{s}_1 - m_{s_1} \dot{\theta}_2 \dot{s}_1 - \dot{m}_{s_1} \dot{\theta}_2 \dot{s}_1 - m \dot{\theta}_2 \dot{s}_1 + \frac{\partial m_{s_1}}{\partial \theta_2} \dot{s}_1 \dot{\theta}_1 + \frac{\partial m_{s_1}}{\partial \theta_1} \dot{s}_1 \dot{\theta}_2 + \ddot{\theta}_1 + \ddot{\theta}_2 \right) \\
\dot{s}_1 &= \frac{1}{m_s} \left( -m_{s_1} \dot{\theta}_1 - \dot{m}_{s_1} \dot{\theta}_1 \dot{s}_1 - m_{s_1} \dot{\theta}_1 \dot{s}_1 - m_{s_1} \dot{s}_1 \dot{\theta}_1 - k \dot{s}_1 - c \dot{s}_1 + \frac{\partial m_{s_1} \dot{\theta}_1}{\partial \theta_1} \dot{s}_1 \dot{\theta}_2 - \alpha \tau_2 \right)
\end{align*}

To design the controller these equations of motion are rewritten using partial feedback linearization

\begin{align*}
\ddot{\theta}_1 &= u_1 \\
\ddot{\theta}_2 &= u_2 \\
\dot{s}_1 &= \frac{1}{m_s} \left( -m_{s_1} \dot{\theta}_1 - \dot{m}_{s_1} \dot{\theta}_1 \dot{s}_1 - m_{s_1} \dot{\theta}_1 \dot{s}_1 - m_{s_1} \dot{s}_1 \dot{\theta}_1 - k \dot{s}_1 - c \dot{s}_1 + \frac{\partial m_{s_1} \dot{\theta}_1}{\partial \theta_1} \dot{s}_1 \dot{\theta}_2 \right) + \alpha \tau_2
\end{align*}
3.3. NOTCH FILTER

Where $u_1$ and $u_2$ are the control inputs, which can be typically expressed as

\[
\begin{align*}
  u_1 &= -K_{P1}\dot{\theta}_1 - K_{D1}\ddot{\theta}_1 \\
  u_2 &= -K_{P2}\dot{\theta}_2 - K_{D2}\ddot{\theta}_2
\end{align*}
\]  

(3.4) (3.5)

Herein $K_{P1}$, $K_{P2}$, $K_{D1}$, $K_{D2}$ denote the controller gains. From (2.60), $\tau_2$ can be given by

\[
\tau_2 = I_2 u_2 + m_{s_1} \dot{\theta}_1 \ddot{s}_1 + m_{s_1} \dot{\theta}_2 \ddot{s}_1 + m_{s_2} \dot{\theta}_1 \ddot{s}_2 + \frac{\partial m_{s_1} \dot{\theta}_2}{\partial \theta_1} \dot{s}_1 \dot{\theta}_1 - \frac{\partial m_{s_2} \dot{\theta}_1}{\partial \theta_2} \dot{s}_1 \dot{\theta}_2
\]

(3.6)

Consequently, (3.3) can be rewritten as

\[
\ddot{s}_1 + 2\zeta \omega_n \dot{s}_1 + \omega_n^2 s_1 = \alpha_1(\theta_1, \theta_2, s_1) u_1 + \alpha_2(\theta_1, \theta_2) u_2 + b_1(\theta_1, \theta_2) \ddot{s}_2 + b_2(\theta_1, \theta_2, s_1) \ddot{s}_1
\]

(3.7)

where

\[
\zeta = \frac{c}{2m_{s_1} k} \\
\omega_n = \sqrt{\frac{k}{m_{s_1}}}
\]

(3.8) (3.9)

\[
\begin{align*}
  \alpha_1 &= -m_{s_1} \dot{\theta}_1 + a'_h m_{s_1} \dot{\theta}_2 \\
  \alpha_2 &= -a'_h I_2 + m_{s_1} \dot{\theta}_2 \\
  b_1 &= -s(\theta_2 - \theta_1) \Omega \\
  b_2 &= \frac{c(\theta_2 - \theta_1) \Psi a'_h s_1 - s(\theta_2 - \theta_1) \Phi}{m_{s_1}} \\
  b_3 &= \frac{s(\theta_2 - \theta_1)(\Psi - \Omega)}{m_{s_1}} \\
  b_4 &= \frac{s(\theta_2 - \theta_1) a'_h (\Psi + \Omega)}{m_{s_1}}
\end{align*}
\]

(3.10) (3.11) (3.12) (3.13) (3.14) (3.15)

Here $\omega_n$ is the natural frequency, $\zeta$ is the damping ratio for the flexible link, and

\[
\begin{align*}
  \Omega &= m_2 x_{CG}^2 \phi h \gamma \\
  \Psi &= x_{CG}^2 \phi h \gamma \\
  \Phi &= x_{CG}^2 (2l_0 + l_1)
\end{align*}
\]

(3.16) (3.17) (3.18)

3.3 Notch filter

Notch filters are generally used to suppress the system’s input at its resonance frequency. In this case the notch filter will be applied to ensure that inputs at the vibration frequency of the flexible beam are not passed through to the system. Notch filters are specialized forms of second-order filters [6, 12]. A generalized format of these filters is given by

\[
\begin{align*}
  u &= \frac{s^2/\omega_n^2 + 2\zeta \omega_n s/\omega_n + 1}{s^2/\omega_p^2 + 2\zeta_p s/\omega_p + 1}
\end{align*}
\]

(3.19)
where \( u \) is the filter output, \( a \) is the filter input; and \( \omega_z, \omega_p, \zeta_z, \zeta_p \) are filter parameters. To create a notch filter these parameters are chosen as \( \omega_z = \omega_p = \omega_n, \zeta_z = \zeta \) and \( \zeta_p = 1 \). Substituting these parameters in (3.19) yields to the transfer function of the notch filter

\[
\frac{u}{a} = \frac{s^2 + 2\zeta\omega_n s + 1}{s^2 + 2\omega_n s + 1}
\] (3.20)

### 3.4 Control cases

To investigate the effect of the filter on the system, three different control cases will be proposed. In order to simulate the developed controller, each design will be translated into a state space realization to make it suitable for simulations. First, the feedback loop with only PD control will be obtained. Thereafter, the notch filter will be applied to as well input \( u_1 \) only as both inputs \( u_1 \) and \( u_2 \).

#### 3.4.1 Case 1: PD control

Applying PD control to the system, the feedback loop block diagram will look like Figure 3.1. Here \( d \) denotes the rigid body dynamics of the system. If now the state variables are defined as \((x_1, x_2, x_3, x_4, x_5, x_6) = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, s_1, \dot{s}_1)\), the state space description of the system can be written as

\[
\dot{x}_1 = x_2 \tag{3.21}
\]

\[
\dot{x}_2 = u_1 \tag{3.22}
\]

\[
\dot{x}_3 = x_4 \tag{3.23}
\]

\[
\dot{x}_4 = u_2 \tag{3.24}
\]

\[
\dot{x}_5 = x_6 \tag{3.25}
\]

\[
\dot{x}_6 = -2\zeta\omega_n x_6 - \omega_n^2 x_5 + \alpha_1 u_1 + \alpha_2 u_2 + b_1 x_4^2 + b_2 x_2^2 + b_3 x_2 x_4 + b_4 x_6 x_2 \tag{3.26}
\]

---

**Figure 3.1:** Feedback loop block diagram of system with PD controller

---
3.4. CONTROL CASES

3.4.2 Case 2: PD control including filtered input $u_1$

Implementing a filter to input $u_1$, the feedback loop block diagram will look like Figure 3.2. The notch filter of (3.20) can be written as

$$\ddot{u}_1 + 2\omega_n \dot{u}_1 + \omega_n^2 u_1 = \ddot{a}_1 + 2\zeta\omega_n \dot{a}_1 + \omega_n^2 a_1$$  \hspace{1cm} (3.27)

To obtain a state space description of the system an auxiliary signal $z_1$ is defined satisfying

$$\ddot{z}_1 + 2\omega_n \dot{z}_1 + \omega_n^2 z_1 = a_1$$  \hspace{1cm} (3.28)

Now the transfer function from $a_1$ to $z_1$ can be expressed in the Laplace domain as

$$\frac{z_1}{a_1} = \frac{1}{s^2 + 2\omega_n s + 1}$$  \hspace{1cm} (3.29)

Substituting (3.29) and (3.40) in (3.20) results in the filter output as

$$u_1 = (1 + \zeta)2\omega_n \dot{z}_1 + a_1$$  \hspace{1cm} (3.30)

If now the state variables are defined as $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, s_1, \dot{s}_1, z_1, \dot{z}_1)$, the state space realization of the system can be written as

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u_2 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -2\zeta\omega_n x_6 - \omega_n^2 x_5 + \alpha_1 u_1 + \alpha_2 u_2 + b_1 x_4^2 + b_2 x_2^2 + b_3 x_2 x_4 + b_4 x_6 x_2 \\
\dot{x}_7 &= x_8 \\
\dot{x}_8 &= -2\omega_n x_8 - \omega_n^2 x_7 + a_1
\end{align*}

where filter input $a_1$ can be written as

$$a_1 = -K_{P1} x_1 - K_{D1} x_2$$  \hspace{1cm} (3.39)
3.4.3 Case 3: PD control including filtered input $u_1$ and $u_2$

When the filter is applied to both inputs $u_1$ and $u_2$, the feedback loop block diagram will look as depicted in Figure 3.3. Following the methodology of the previous section an second auxiliary signal $z_2$ can be defined satisfying

$$\ddot{z}_2 + 2\omega_n \dot{z}_2 + \omega_n^2 z_2 = a_2$$

(3.40)

Consequently, the new set of state variables become

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, s_1, \dot{s}_1, z_1, \dot{z}_1, z_2, \dot{z}_2),$$

and the state space description of the system can be written as

$$\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = u_1 \\
\dot{x}_3 & = x_4 \\
\dot{x}_4 & = u_2 \\
\dot{x}_5 & = x_6 \\
\dot{x}_6 & = -2\omega_n x_6 - \omega_n^2 x_5 + \alpha_1 u_1 + \alpha_2 u_2 + b_1 x_2^2 + b_2 x_2^2 + b_3 x_2 x_4 + b_4 x_2 x_4 \\
\dot{x}_7 & = x_8 \\
\dot{x}_8 & = -2\omega_n x_8 - \omega_n^2 x_7 + a_1 \\
\dot{x}_9 & = x_{10} \\
\dot{x}_{10} & = -2\omega_n x_{10} - \omega_n^2 x_9 + a_2
\end{align*}$$

(3.41)  (3.42)  (3.43)  (3.44)  (3.45)  (3.46)  (3.47)  (3.48)  (3.49)  (3.50)

where filter input $a_2$ can be written as

$$a_2 = -K_{p2} x_3 - K_{d2} x_4$$

(3.51)
3.5 Physical values

Through integration by parts, substitution, and by using the parameters from Table 2.1, (2.46), (2.47), (2.48), (2.49), and (2.52) can be evaluated as follows:

\[ m_{s_1} = \rho_1 l_1 \left( \frac{1}{2} + \frac{\pi^2}{3} + \frac{\pi^4}{20} \right) + (m_{H_2} + m_2) a_h^2 \gamma^2 \]
\[ I_{l_1} = \rho_1 \left( \frac{1}{3} (l_1 + l_0)^3 - \frac{1}{3} l_0^3 \right) + I_{z_1} + (m_{H_2} + m_2) (2l_0 + l_1) \]
\[ = 0.1353 \text{ kgm}^2 \]
\[ I_{l_2} = m_2 x_{CG_2}^2 + I_{z_2} \]
\[ = 0.001618 \text{ kgm}^2 \]
\[ m_{s_1\theta_1} = \rho_1 l_1^2 \left( \frac{1}{2} + \frac{\pi^2}{8} + \frac{2}{\pi^2} \right) + \rho_1 l_0 l_1 (1 + \frac{\pi^2}{6}) + (m_{H_2} + m_2) a_h \gamma (2l_0 + l_1) \]
\[ = 3.407 \text{ kgm} \]
\[ k = EI_1 \frac{3\pi^4}{2l_1^4} \]
\[ = 36912 \text{ Nmrad}^{-1} \]

From (3.9) the natural frequency of the system can be calculated as

\[ \omega_n = \sqrt{\frac{k}{m_{s_1}}} = 20.687 \text{ rads}^{-1} = 3.29 \text{ Hz} \]

Because of the assumption that \( \zeta = 0.05 \), using (3.8) the dissipative constant becomes

\[ c = 2\zeta \omega_n m_{s_1} = 28.376 \text{ Nsm}^{-1} \]

Moreover, the coupling constants from (3.16) until (3.18) result in

\[ \Omega = 0.0265 \text{ kgm} \]
\[ \Psi = 0.4519 \text{ m} \]
\[ \Phi = 0.1759 \text{ Nsm}^2 \]

3.6 Simulations

The different control cases are tested by implementing the state space system of every separate case in the ode45 solver of MATLAB. For each case the controller gains are determined to be \( K_{P_1} = K_{P_2} = 1 \), and \( K_{D_1} = K_{D_2} = 2 \). Furthermore, the initial conditions are set to \((\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, s_1, \dot{s}_1) = (30^\circ \frac{\pi}{180}, 0, 30^\circ \frac{\pi}{180}, 0, 0, 0) \) and additional \((z_1, \dot{z}_1) = (0, 0)\), and \((z_2, \dot{z}_2) = (0, 0)\).

3.6.1 Case 1: PD control

Figures 3.4 until 3.6 show the responses of the generalized coordinates for the system with PD control. It can be seen the the attitudes of both links \( \theta_1 \) and \( \theta_2 \) converge to the equilibrium in
CHAPTER 3. CONTROLLER DESIGN AND SIMULATIONS

approximately 7 seconds. The residual vibrations are clearly visible in Figure 3.6 and decay over time. After 0.144 seconds the flexible link reaches the maximum deflection of approximately \(2.8 \times 10^{-4} \text{m} \).

![Figure 3.4: Response of the rigid link angle \(\theta_1\) for the system with PD control](image)

3.6.2 Case 2: PD control including filtered input \(u_1\)

Figures 3.7 until 3.9 show the responses of the generalized coordinates for the system with PD control and filtered input \(u_1\). The responses of \(\theta_1\) and \(\theta_2\) are similar to case 1, but in Figure 3.9 the effect of the controller can be obviously noticed. Compared to case 1, the response of the residual vibration is significantly decreased; the maximal lateral deflection is reduced with 52 %. Furthermore, all the residual vibrations disappear after approximately 8 seconds.

3.6.3 Case 3: PD control including filtered input \(u_1\) and \(u_2\)

Figures 3.10 until 3.12 show the responses of the generalized coordinates for the system with PD control and filtered input \(u_1\) and \(u_2\). The response of \(\theta_1\) is similar to the previous cases, but the attitude of \(\theta_2\) is clearly influenced by the filtered input. It takes the system more than 100 seconds to achieve the equilibrium state. Observing the resulting residual vibration in Figure 3.12, only little improvements can be noticed. The maximal lateral deflection of the flexible link decreases approximately 53 % compared to case 1. Furthermore, the residual vibrations show similar behaviour compared to case 2 and again all again the residual vibrations disappear after approximately 8 seconds.

3.7 Discussion

In this final section, three different control designs are proposed and tested by simulations. The designs exist out of a PD-controller in combination with a notch-filter. Simulations prove that implementing a notch filter has a significant effect on the residual vibrations, but causes an increase
3.7. DISCUSSION

Figure 3.5: Response of the rigid link angle $\theta_2$ for the system with PD control

Figure 3.6: Response of coordinate $s_1$ for the system with PD control
Figure 3.7: Response of the rigid link angle $\theta_1$ for the system with PD control and filtered input $u_1$

Figure 3.8: Response of the rigid link angle $\theta_2$ for the system with PD control and filtered input $u_1$
3.7. DISCUSSION

Figure 3.9: Response of coordinate $s_1$ for the system with PD control and filtered input $u_1$.

Figure 3.10: Response of the rigid link angle $\theta_1$ for the system with PD control and filtered input $u_1$ and $u_2$. 
CHAPTER 3. CONTROLLER DESIGN AND SIMULATIONS

Figure 3.11: Response of the rigid link angle $\theta_2$ for the system with PD control and filtered input $u_1$ and $u_2$

Figure 3.12: Response of coordinate $s_1$ for the system with PD control and filtered input $u_1$ and $u_2$
in settling time for the second input. In the next chapter the conclusion and recommendations regarding the project will follow.
Chapter 4

Conclusions and recommendations

4.1 Conclusions

4.1.1 Dynamic model

In this study a new dynamic model is developed for a two-link flexible robot manipulator, derived from the Euler-Lagrange approach. This complete dynamic model includes, usually unconsidered, actuator and hub dynamics. Furthermore, the effort of the second torque ($\tau_2$) on the flexibility of the first link is incorporated.

4.1.2 Control cases

To control the angular position of the links, a control algorithm is introduced using the proportional-plus-derivative control method. In order to suppress the residual vibrations due to the flexibility of the link, a notch filter is applied to the inputs of the actuators. Consequently, adding a filter to the input of the first actuator resulted in a significantly decrease of the residual vibrations; after approximately 8 seconds all the residual vibrations are vanished. Moreover, an improvement of 52% and 53% in maximum link deflection are obtained for respectively case 2 and case 3. However, applying the filter to both actuators’ inputs results in a considerable increase in settling time for the second input.

4.2 Recommendations

4.2.1 Dynamic model

Future research on the dynamic model could include incorporation of a payload to the end effector. By using the currently developed model as a basis, a new dynamical model can be developed including the payload and flexibility of the second link. Moreover, it is recommended to validate the assumed value of the dissipative constant $c$ experimentally. In order to achieve a more accurate dynamical model, an adaptive mode method model can be developed and integrated. Hence, the mode shape function can be adapted when systems’ eigenmode changes.
4.2 RECOMMENDATIONS

4.2.2 Control cases

To check the reliability of the simulated results, applying the control cases in an experimental setup is advised. Therefore, the actuator dynamics of the experimental setup are required first. Fine-tuning of the controller gains will result in an improvement of the controller performance. Furthermore, an energy-based Lyapunov non-linear control method can be developed which will improve the controller performance as well. To reduce the residual vibrations it is recommended to apply a notch filter to the input of the first actuator.
References


Appendix A

Appendix 1

The angular velocity matrix \( \tilde{\omega} \) is given by

\[
\tilde{\omega} = -A^{10} \cdot A^{10\mathrm{T}}
\]

Herein \( A^{10} \) is the transformation matrix of frame 2 with respect to frame 1, given by

\[
A^{10} = \begin{bmatrix}
\cos(\theta_1) & -\sin(\theta_1) & 0 \\
\sin(\theta_1) & \cos(\theta_1) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Now the angular velocity matrix can be written as

\[
\tilde{\omega} = \begin{bmatrix}
\sin(\theta_1) \dot{\theta}_1 & -\cos(\theta_1) \dot{\theta}_1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \cdot \begin{bmatrix}
\cos(\theta_1) & -\sin(\theta_1) & 0 \\
\sin(\theta_1) & \cos(\theta_1) & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & -\dot{\theta}_1 & 0 \\
-\sin(\theta_1) \cos(\theta_1) \dot{\theta}_1 - \sin(\theta_1) \sin(\theta_1) \dot{\theta}_1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -\omega_3 \\
-\omega_2 & 0 & \omega_1 \\
-\omega_3 & \omega_1 & 0
\end{bmatrix}
\]

The angular velocity vector can be computed as follows

\[
\tilde{\omega} = \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} \cdot \tilde{e}^3 = \begin{bmatrix}
0 & 0 & \dot{\theta}_1
\end{bmatrix} \cdot \begin{bmatrix}
\tilde{e}^3_1 \\
\tilde{e}^3_2 \\
\tilde{e}^3_3
\end{bmatrix} = \dot{\theta}_1 e^3_3
\] (A.1)