Degenerate two-phase flow model in porous media including dynamics effects in the capillary pressure: existence of a weak solution

Citation for published version (APA):

Document status and date:
Published: 01/01/2015

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Degenerate two-phase flow model in porous media including dynamics effects in the capillary pressure: existence of a weak solution

by

X. Cao, I.S. Pop
Degenerate two-phase flow model in porous media including dynamic effects in the capillary pressure: existence of a weak solution

X. Cao\textsuperscript{a,}\textsuperscript{*}, I.S. Pop\textsuperscript{a,b}
\textsuperscript{a}CASA, Eindhoven University of Technology, P. O. Box 513, 5600 MB Eindhoven, The Netherlands
\textsuperscript{b}Institute of Mathematics, Johannes Bruns GT. 12, University of Bergen, Bergen, Norway

Abstract
In this paper, we analyze a degenerate elliptic-parabolic system which describes the flow of two incompressible, immiscible fluids in porous media including dynamic effects in the phase-pressure difference. First, for a regularized diffusion coefficient, the existence and uniqueness of the weak solution in the non-degenerate case is obtained. Then we let the regularization parameter go to zero to show the existence of weak solutions under degenerate case.  

\textit{Keywords:} Degeneracy, dynamic capillary pressure, two-phase flow, weak solution, uniqueness, pseudo-parabolic system

1. Introduction
In this paper, we analyze the existence and, where appropriate, uniqueness of a weak solution to the elliptic-parabolic system:

\[ \partial_t u + \nabla \cdot (k(u) \nabla p) - \Delta \theta(u) = 0, \]  
(1.1)

\[ \nabla \cdot (k(u) \nabla p) + \nabla \cdot (k_v(u) \nabla (\tau(u) \partial_t u)) = 0, \]  
(1.2)

complemented with initial and boundary conditions. The equations hold in \( Q := (0, T_M] \times \Omega \). Here \( \Omega \) is a bounded domain in \( \mathbb{R}^d \), having Lipschitz continuous boundary, and \( T_M > 0 \) is a given maximal time. The unknowns are \( u \) and \( p \). The work is motivated by two-phase flow in porous media (e.g. oil and water).

1.1. Two phase flow model in porous media under non-equilibrium condition

The system (1.1) - (1.2) models two phase flow in porous media, with dynamic effects in the phase pressure difference. It is obtained by including Darcy’s law for both phases in the mass conservation laws. With \( w, n \) being indices for the wetting, respectively, non-wetting phase, the mass conservation equations are (see [4, 21]):

\[ \phi \frac{\partial s_\alpha}{\partial t} + \nabla \cdot q_\alpha = 0 \quad (\alpha = w, n). \]  
(1.3)

The coefficient \( \phi \) represents the porosity of the porous medium, while \( s_\alpha \) and \( q_\alpha \) denote the saturation and the volumetric velocity of the \( \alpha \) phase. The volumetric velocity \( q_\alpha \) is deduced from the Darcy law as

\[ q_\alpha = -\frac{k}{\mu_\alpha} k_v(s_\alpha) \nabla p_\alpha \quad (\alpha = w, n), \]  
(1.4)

\*Corresponding author

Email addresses: x.cao@tue.nl (X. Cao), i.pop@tue.nl (I.S. Pop)

URL: http://www.win.tue.nl/casa/people/tempstaff/168.html (X. Cao)
where \( \bar{k} \) is the absolute permeability of the porous medium, \( p_\alpha \) the pressure, \( \mu_\alpha \) the viscosity and \( k_{r\alpha} \) the relative permeability of the \( \alpha \) phase. The specific function of \( k_{r\alpha} \) is assumed known. Substituting (1.4) in (1.3) gives

\[
\phi \frac{\partial s_\alpha}{\partial t} - \nabla \cdot \left( \frac{\bar{k}k_{r\alpha}}{\mu_\alpha} \nabla p_\alpha \right) = 0 \quad (\alpha = w, n).
\]

(1.5)

We assume that only two phases are present,

\[
s_w + s_n = 1.
\]

(1.6)

To complete the model, one commonly assumes a relationship between the phase pressure difference, and \( s_w \). Under equilibrium assumption, this is

\[
p_n - p_w = p_c(s_w),
\]

with a given function \( p_c \). Experimental results [15, 20] have, however, proved the limitation of this assumption. Alternatively, in [20] the following relation is proposed:

\[
p_n - p_w = p_c(s_w) - \tau(s_w) \frac{\partial s_w}{\partial t},
\]

(1.7)

The damping function \( \tau \) as well as the function \( p_c \), which represents the capillary pressure under equilibrium condition, are assumed to be known. Summing the two equations from (1.5) and making use of (1.6) give:

\[
\nabla \cdot F = 0,
\]

(1.8)

where

\[ F = \frac{\bar{k}k_{r\alpha}}{\mu_\alpha} \nabla p_\alpha + \frac{\bar{k}k_{r\alpha}}{\mu_\alpha} \nabla p_\alpha \]

denotes the total flow.

Introducing the normalized relative permeabilities

\[
k_\alpha := \frac{k_{r\alpha}}{\mu_\alpha} \quad (\alpha = w, n),
\]

and with \( k = k_w + k_n \), we also define the fractional flow function

\[
f_w(s_w) := \frac{k_w}{k}.
\]

Then, we follow [2, 13], and define the global pressure \( p \)

\[
p = p_n - \int_{C_0}^{s_w} f_w(z)p'_c(z)dz,
\]

(1.9)

which leads to the following expression of the water pressure

\[
p_w = p + \int_{C_0}^{s_w} f_w(z)p'_c(z)dz - p_c(s_w) + \tau(s_w)\partial_t s_w.
\]

(1.10)

Here \( C_0 \in (0, 1) \) is a constant that will be used as the boundary value of water saturation. Furthermore, define the complementary pressure \( \theta \) as the integral (Kirchhoff) transformation

\[
\theta(s_w) = -\int_{C_0}^{s_w} \frac{k_w}{k} f_w(z)p'_c(z)dz.
\]

(1.11)

Then, from (1.5) for the wetting phase, by using (1.6) and (1.9) gives

\[
\phi \partial_t s_w + \nabla \cdot (\bar{k}k_{r\alpha} \nabla p) - \nabla \cdot (\bar{k} \nabla \theta(s_w)) = 0.
\]

(1.12)

Finally, (1.8) becomes

\[
\nabla \cdot (\bar{k}k_{r\alpha} \nabla p) + \nabla \cdot (\bar{k} \nabla (\tau \partial_t s_w)) = 0.
\]

(1.13)
The system (1.12)-(1.13) is in dimensional form. Taking $L_r$, $T_r$, and $P_r$ as characteristic values for the length, time, global pressure, respectively. Scaling the space variable $x$ with $L_r$, the time $t$ with $T_r$ and the pressure $p$ with $P_r$, and assuming

$$\frac{T_r}{L_r^2} = \frac{\phi}{kP_r},$$

we obtain the following system:

$$\partial_t u + \nabla \cdot \left( k_n \nabla p \right) - \Delta \theta(u) = 0,$$  
(1.15)

$$\nabla \cdot \left( k_n \nabla p \right) + \nabla \cdot \left( k_w \nabla \left( \tau \partial_t u \right) \right) = 0,$$  
(1.16)

with $u = s_w$.

1.2. Assumptions and known results

For the non-linearities appearing in the model, we refer to [4, 21]. The special assumptions are given below. Here we mention the typical choices of permeability in the literature ([7, 8])

$$k_w(u) = u^\alpha, \quad k_n(u) = (1 - u)^\beta, \quad \text{with} \quad \alpha, \beta > 1,$$

and

$$-p'_c(u) = u^{-\lambda}, \quad \lambda > 1.$$

These are defined for $u \in [0, 1]$ (the physical range). The model (1.15) - (1.16) is completed by the initial condition

$$u(0, \cdot) = u^0 \quad \text{in} \quad \Omega,$$  
(1.17)

and the boundary conditions

$$u = C_D, \quad p = 0 \quad \text{at} \quad \partial \Omega, \quad \text{for all} \quad t > 0,$$  
(1.18)

where $u^0$ is a given function and the constant $C_D$ satisfies $0 < C_D < 1$.

Remark 1. To avoid an excess of technicalities, the pressure boundary condition in (1.18) is assumed constant. Other types of boundary conditions, as Naumann can be considered. Similarly, the boundary values of $u$ may be non-constant, but should be bounded away from 0 and 1.

In this paper we assume

**A1**: The functions $k_w, k_n [0, 1] \rightarrow [0, 1]$ are $C^1$, $k_w$ is an increasing function, and $k_n$ is decreasing. $p_c \in C^1((0, 1], \mathbb{R}^+)\rightarrow \mathbb{R}^+$ is a decreasing function. $\tau \in C^1((0, 1], \mathbb{R}^+)$ is an increasing function satisfying $\tau \geq \tau_0$, for some $\tau_0 > 0$, and for all $u \in [0, 1]$. Furthermore, we assume $-p'_c k_w \leq C < +\infty$.

These functions (see also [4]) are defined on $[0, 1]$, which is the physically relevant range. For the analysis, we extend them to $\mathbb{R}$.

a: $k_w(u) = k_w(0)$, if $u \leq 0$, $k_w(u) = k_w(1)$, if $u \geq 1$.

b: $k_n(u) = k_n(0)$, if $u \leq 0$, $k_n(u) = k_n(1)$, if $u \geq 1$.

c: $\frac{1}{p_c(u)} = \frac{1}{p_c(0)}$, if $u \leq 0$, $\frac{1}{p_c(u)} = \frac{1}{p_c(1)}$, if $u \geq 1$.

d: $\frac{1}{\tau(u)} = \frac{1}{\tau(0)}$, if $u \leq 0$, $\frac{1}{\tau(u)} = \frac{1}{\tau(1)}$, if $u \geq 1$.

Remark 2. A1 allows $p'_c$, respectively $\tau$ to become $\infty$. Then, an extension is not meaningful. However, this extension makes sense for the reciprocals $\frac{1}{p'_c}$, $\frac{1}{\tau}$. Also, the monotonicity of $\tau$, as well as the assumption on a strictly positive lower bound can he compensated by additional technicalities. We omit this here.
In the proofs below, we will use the functions (see [10]) $G, \Gamma : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$

$$G(u) = \int_{C_D}^{u} \frac{\tau k}{k_w k} (z) dz, \quad \text{and} \quad \Gamma(u) = \int_{C_D}^{u} G(z) dz,$$  

(1.19)

and the function $T(u)$:

$$T(u) = \int_{0}^{u} \tau(z) dz + C_T,$$

(1.20)

here $C_T$ is a constant specified below. Clearly, $\Gamma$ is a convex function satisfying $\Gamma(0) = G(0) = 0$, implying that

$$\Gamma(u) \geq 0, \quad \text{for all} \quad u \in \mathbb{R}.$$  

(1.21)

**A2:** Second, we assume that the initial condition $u^0$ satisfies $u^0 \in W^{1,2}_0(\Omega) + C_D$, $\int\Omega \Gamma(u^0) < +\infty$, $T(u^0) \in W^{1,2}(\Omega)$.

The problem considered here is an extension of scalar, two-phase flow models studied in [10], while only one pseudo-parabolic equation is considered in the unknown $u$. The model in [10] can be derived from the present one assuming that the total flow is known (see [2, 18]). After non-dimensionlization and some algebraic manipulation one gets

$$\partial_t u + \nabla \cdot \mathbf{F} - \Delta \theta(u) - \tau \nabla \cdot \left( \frac{k_w}{k} \nabla \partial_t u \right) = 0.$$  

(1.22)

The existence and uniqueness of the weak solution has been shown in [18] in the linear case of the third order derivative term. In [27], the author has considered one phase flow model, and regularization is employed in to prove existence of a weak solution for degenerate case. Here we build on similar ideas to prove existence for the two phase flow model. For uniqueness but in the non-degenerate case, we refer to [5, 11].

Two-phase flow models are mainly analyzed under equilibrium assumptions, i.e. for $\tau = 0$. In particular, we refer to [3, 14, 24]. For non-equilibrium, two-phase flow models are analyzed in [23], where existence is obtained for non-degenerate cases, but including hysteresis effects, and in [12], where uniqueness is proved again, in non-degenerate cases.

In what follows, we analyze first the regularized model, and prove the existence and uniqueness of a solution, as well as a-priori energy estimates. This is achieved by Rothe’s method [22]. Finally, we pass the regularization parameter to 0 to obtain the existence of weak solutions in the degenerate case. Note that uniqueness remains open.

### 2. Existence and uniqueness of weak solutions in the regularized case

#### 2.1. The weak solution concept

We use the standard spaces $L^2(\Omega), W^{1,2}(\Omega)$ and $W^{1,2}_0(\Omega)$ in the theory of partial differential equation, and $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and the corresponding norm in $L^2(\Omega)$, or where needed, in $(L^2(\Omega))^d$. Furthermore, $L^2(0, T_M; X)$ denotes the Bochner space of $X$-valued functions.

Let us now define the set

$$V := W^{1,2}_0(\Omega) + C_D.$$  

Inspired by [27], we define a weak solution to the model $(1.15)-(1.18)$ solving

**Problem P:** Find $u \in L^2(0, T_M; V)$, such that $\partial_t u \in L^2(Q)$, and $p \in L^2(0, T_M; W^{1,r}_0(\Omega))$ (for some particular chosen $r^* \in (1, 2)$), such that $u(0, \cdot) = u^0$, $\sqrt{n} \nabla p \in L^2(0, T_M; L^2(\Omega)^d)$, $\sqrt{n} \nabla (\nabla p + \nabla \partial_t T(u)) \in L^2(0, T_M; L^2(\Omega)^d)$, and

$$\begin{align*}
(\partial_t u, \phi) - (k_w \nabla p, \nabla \phi) + (\nabla \theta(u), \nabla \phi) &= 0, \quad (1.21) \\
(k \nabla p, \nabla \psi) + (k_w \nabla \partial_t T(u), \nabla \psi) &= 0, \quad (1.22)
\end{align*}$$

for any $\phi, \psi \in L^2(0, T_M; W^{1,2}_0(\Omega))$.

Before dealing with the degenerate case, we consider first the regularized case. To this aim, we define:

**a:**\quad $k_{ud}(u) = k_w(u + \delta)$, if $0 \leq u \leq 1 - \delta$, $k_{ud}(u) = k_w(\delta)$, if $u < 0$, $k_{ud}(u) = k_w(1)$, if $u > 1 - \delta$.

**b:**\quad $k_{ud}(u) = k_w(u - \delta)$, if $\delta \leq u \leq 1$, $k_{ud}(u) = k_w(0)$, if $u < \delta$, $k_{ud}(u) = k_w(1 - \delta)$, if $u > 1$.  

4
Below we analyze the following:

\[ p_{\delta}(u) = -p_{\varepsilon}(u + \delta), \text{ if } 0 \leq u \leq 1 - \delta, -p_{\delta}(u) = -p_{\varepsilon}'(\delta), \text{ if } u < 0, -p_{\delta}(u) = -p_{\varepsilon}'(1), \text{ if } u > 1 - \delta. \]

d': \( \tau(u) = \tau(u - \delta), \text{ if } 0 \leq \delta \leq 1, \tau(u) = \tau(0), \text{ if } u < \delta, \tau(u) = \tau(1 - \delta), \text{ if } u > 1. \)

This immediately follows:

\[ G_\delta(u) = \int_{\partial D} \frac{\tau_\delta k_\delta}{k_\delta - k_\delta}(z) dz, \quad T_\delta(u) = \int_{\partial D} G_\delta(z) dz, \quad \Gamma_\delta(u) = \int_{\partial D} G_\delta(z) dz, \]

\[ k_\delta(u) = k_{\delta 0}(u) + k_{\delta 0}(u), \quad f_\delta = \frac{k_{\delta 0}}{k_\delta}, \quad \theta_\delta(u) = -\int_{\partial D} \frac{k_{\delta 0}k_{\delta 0}}{k_\delta}(z)p_{\delta}'(z) dz, \]

\[ T_\delta(u) = \int_{\partial D} \tau_\delta(z) dz + C_T. \]

We now investigate the regularized model

\[ \partial_t u_\delta + \nabla \cdot (k_{\delta 0} \nabla p_\delta) - \Delta \theta_\delta(u_\delta) = 0, \]

\[ \nabla \cdot (k_\delta \nabla p_\delta) + \nabla \cdot (k_{\delta 0} \nabla T_\delta(u_\delta)) = 0, \]

with the following initial and boundary conditions

\[ u_\delta(0, \cdot) = u^0, \quad \text{in } \Omega, \]

\[ u_\delta = C_D, \quad p_\delta = 0, \quad \text{at } \partial \Omega, \quad \text{for all } t > 0. \]

Below we analyze the following:

**Problem P_δ:** Find \( u_\delta \in L^2(0, T_M; V), \) such that \( \partial_t u_\delta \in L^2(Q), \) and \( p_\delta \in L^2(0, T_M; W_0^{1,2}(\Omega)), \) such that \( u_\delta(0, \cdot) = u^0, \)

\[ \nabla T_\delta(u_\delta) \in L^2(0, T_M; L^2(\Omega)^d) \text{ and} \]

\[ \int_0^{T_M} (\partial_t u_\delta, \phi) dt - \int_0^{T_M} (k_{\delta 0} \nabla p_\delta, \nabla \phi) dt + \int_0^{T_M} (\nabla \theta_\delta(u_\delta), \nabla \phi) dt = 0, \]

\[ \int_0^{T_M} (k_\delta \nabla p_\delta, \nabla \psi) dt + \int_0^{T_M} (k_{\delta 0} \nabla T_\delta(u_\delta), \nabla \psi) dt = 0, \]

for any \( \phi, \psi \in L^2(0, T_M; W_0^{1,2}(\Omega)). \)

We show the existence of a solution to Problem P_δ by the method of Rothe [22]. In doing so, we use the elementary inequality:

\[ ab \leq \frac{1}{2\sigma} a^2 + \frac{\sigma}{2} b^2, \quad \text{for any } a, b \in \mathbb{R}, \sigma > 0. \]

2.2. The time discretization

With \( N \in \mathbb{N}, \) let \( h = T_M/N \) and consider the Euler implicit discretization of (2.10)-(2.11).

**Problem P^n:** Given \( u^{n-1}_\delta \in V (n = 1, 2, \ldots N), \) find \( u^n_\delta \in V \) and \( p^n_\delta \in W_0^{1,2}(\Omega), \) such that

\[ (u^n_\delta - u^{n-1}_\delta, \phi) - (k_{\delta 0}(u^n_\delta) \nabla p^n_\delta, \nabla \phi) + (\nabla \theta_\delta(u^n_\delta), \nabla \phi) = 0, \]

\[ (k_\delta(u^n_\delta) \nabla p^n_\delta, \nabla \psi) + (k_{\delta 0}(u^n_\delta) \nabla T_\delta(u^n_\delta) - T_\delta(u^{n-1}_\delta), h) \nabla \psi = 0, \]

for any \( \phi, \psi \in W_0^{1,2}(\Omega). \) Here \( (\cdot, \cdot) \text{ means } L^2 \text{ inner product.} \)

**Lemma 2.1.** Problem P^n has a solution.
Proof. We start with a finite dimension approximation (Garlerin), for which we prove the existence of a solution to Problem $P^D_0$. To this aim, we use Lemma 1.4 (pp. 164 in [30]). Then we pass to the limit for the discrete solution, and use compactness to show the existence of a solution for Problem $P^D_0$.

Let $\{v_m\}_{m=1}^{\infty}$ be the countable basis of the separable space $W_0^1(\Omega)$. Set $W_m = \text{Span}\{v_m\}(k = 1, \ldots , m)$. Then, given $\alpha_1, \ldots , \alpha_m, \beta_1, \ldots , \beta_m \in \mathbb{R}$, define $v = \sum_{i=1}^m \alpha_i v_i \in W_m$, $w = \sum_{i=1}^m \beta_i v_i \in W_m$, and set $\z = v + C_D$.

A finite dimensional solution $(\z, w) \in (W_m + C_D) \times W_m$ of (2.13)-(2.14) satisfies

\[
\begin{align*}
\frac{(\z - u_0^{n-1})}{h}, & \phi - (k_{\alpha \delta}(\z) \nabla w, \nabla \phi) + (\nabla \theta_\delta(\z), \nabla \phi) = 0, \quad (2.15) \\
(k_\delta(\z) \nabla w, \nabla \psi) + (k_{\alpha \delta}(\z) \frac{T_\delta(\z) - T_\delta(u_0^{n-1})}{h}, \nabla \psi) = 0, \quad (2.16)
\end{align*}
\]

for any $\phi, \psi \in W_m$.

To avoid an excess of notations in the first part, we do not use any different indices for the solution pair $(\z, w)$ in the finite dimensional case.

For $i = 1, \ldots , m$, define

\[
\tilde{\beta}_i = (k_\delta(\z) \nabla w, \nabla v_i) + \frac{1}{h} (k_{\alpha \delta}(\z) \nabla (T_\delta(\z) - T_\delta(u_0^{n-1})), \nabla v_i), \quad (i = 1, 2, \ldots , m). \quad (2.17)
\]

Note that if $(\z, w)$ is a solution pair to (2.15)-(2.16), one has $\tilde{\beta}_i = 0$ for all $i$.

Since $w = \sum_{i=1}^m \beta_i v_i$, one clearly has

\[
\sum_{i=1}^m \beta_i \tilde{\beta}_i = (k_\delta(\z) \nabla w, \nabla w) + \frac{1}{h} (k_{\alpha \delta}(\z) \nabla (T_\delta(\z) - T_\delta(u_0^{n-1})), \nabla w)
\]

\[
= \left| \sqrt{k_\delta(\z) \nabla w} \right|^2 + \frac{1}{h} (\tau_\delta(\z) k_{\alpha \delta}(\z) \nabla \z, \nabla w) - \frac{1}{h} (\tau_\delta(\z) k_{\alpha \delta}(\z) \nabla u_0^{n-1}, \nabla w). \quad (2.18)
\]

Further, define $g : \mathbb{R} \to \mathbb{R}$ by

\[
g(\z) = \int_0^\z \frac{\tau_\delta k_{\alpha \delta}}{k_{\alpha \delta}}(z)dz. \quad (2.19)
\]

By A1, there exist $0 < M_0 \leq M_1 < \infty$ possibly depending on $\delta$, such that

\[
g(\z) \leq M_0 \z^2, \quad |g(\z)| \leq M_1 |\z|. \quad (2.20)
\]

As above, for $i = 1, \ldots , m$, we define

\[
l_i = \frac{(g(\z), v_i)}{|v_i|^2} \quad (i = 1, 2, \ldots , m), \quad (2.21)
\]

which immediately implies

\[
g(\z) = \sum_{i=1}^m l_i v_i, \quad (2.22)
\]

Define

\[
\tilde{l}_i = \left( \z - u_0^{n-1}, \frac{v_i}{h} \right) - \frac{1}{h} (k_{\alpha \delta}(\z) \nabla w, \nabla v_i) + \frac{1}{h} (\nabla \theta_\delta(\z), \nabla v_i), \quad (2.23)
\]

as in above, if $(\z, w)$ is a solution pair, then $\tilde{l}_i = 0$ for all $i$.

Since $\nabla g(\z) = \frac{\tau_\delta k_{\alpha \delta}}{k_{\alpha \delta}}(\z) \nabla \z$, we get

\[
\sum_{i=1}^m l_i \tilde{l}_i = \frac{1}{h^2} (\z - u_0^{n-1}, g(\z)) - \frac{1}{h} (k_{\alpha \delta}(\z) \nabla w, \frac{\tau_\delta k_{\alpha \delta}}{k_{\alpha \delta}}(\z) \nabla \z) + \frac{1}{h} (\nabla \theta_\delta(\z), \frac{\tau_\delta k_{\alpha \delta}}{k_{\alpha \delta}}(\z) \nabla \z). \quad (2.24)
\]
Adding (2.18) and (2.24) yields
\[
\sum_{i=1}^{m} \beta_i \tilde{\beta} + \sum_{i=1}^{m} l_i \tilde{l}_i = \left\| \sqrt{k_\beta(\varsigma)} \nabla w \right\|^2 + \frac{1}{h^2} (\varsigma - u_0^{-1}, g(\varsigma)) \\
+ \frac{1}{h} (\nabla \theta_\delta(\varsigma), \frac{\tau_\delta k_{\delta}(\varsigma)}{k_{\delta}} \nabla \varsigma) \right\| - \frac{1}{h} (\tau_\delta(u_0^{-1}k_{\delta}(\varsigma)) \nabla u_0^{-1}, \nabla w). 
\] (2.25)

The existence of a solution is provided if the sum on the left in (2.25) becomes positive for \((l_1, \ldots, l_m)\) and \((\beta_1, \ldots, \beta_m)\) sufficiently large.

To prove this, denote the terms on the right by \(T_1, T_2, T_3, T_4\). Note that \(T_1\) is positive.

For \(T_2\), we use (2.20) to obtain,
\[
\frac{1}{h^2} (\varsigma - u_0^{-1}, g(\varsigma)) = \frac{1}{h^2} (\varsigma, g(\varsigma)) - \frac{1}{h^2} (u_0^{-1}, g(\varsigma)) \\
\geq \frac{M_0}{h^2} \|\varsigma\|^2 - \frac{1}{2h^2} \frac{M^2_1}{M_0} \|u_0^{-1}\|^2 - \frac{M_0}{2h^2} \|\varsigma\|^2 \\
= \frac{M_0}{2h^2} \|\varsigma\|^2 - \frac{1}{2h^2} \frac{M^2_1}{M_0} \|u_0^{-1}\|^2. 
\] (2.26)

Recalling (2.4), \(T_3\) becomes
\[
\frac{1}{h} (\nabla \theta_\delta(\varsigma), \frac{\tau_\delta k_{\delta}(\varsigma)}{k_{\delta}} \nabla \varsigma) = \frac{1}{h} \left( -\frac{k_{\delta}}{k_{\delta}} \frac{\nabla \theta_\delta(\varsigma)}{k_{\delta}} \nabla \varsigma, \frac{\tau_\delta k_{\delta}(\varsigma)}{k_{\delta}} \nabla \varsigma \right) = \frac{1}{h} \left\| \frac{k_{\delta}}{\sqrt{k_{\delta}}} \sqrt{-\tau_\delta \frac{\nabla \theta_\delta(\varsigma)}{k_{\delta}} \nabla g(\varsigma)} \right\|^2. 
\] (2.27)

For \(T_4\), one gets
\[
\frac{1}{h} (\tau_\delta(u_0^{-1}k_{\delta}(\varsigma)) \nabla u_0^{-1}, \nabla w) \geq \frac{1}{2h^2} \left\| \tau_\delta(u_0^{-1}k_{\delta}(\varsigma)) \nabla u_0^{-1} \right\|^2 - \frac{1}{2} \left\| \sqrt{k_{\delta}(\varsigma)} \nabla w \right\|^2. 
\] (2.28)

Using (2.26)-(2.28), the Poincaré and Cauchy-Schwarz inequalities lead to
\[
\sum_{i=1}^{m} \beta_i \tilde{\beta} + \sum_{i=1}^{m} l_i \tilde{l}_i \geq \frac{M^2_2}{2} \|\nabla w\|^2 + \frac{M_0}{2h^2} \|\varsigma\|^2 + \frac{M_3}{h} \|\nabla \varsigma\|^2 - \frac{1}{2h^2} \left\| \tau_\delta(u_0^{-1}k_{\delta}(\varsigma)) \nabla u_0^{-1} \right\|^2 - \frac{1}{2h^2} \frac{M^2_1}{M_0} \|u_0^{-1}\|^2 \\
\geq C(\Omega) \frac{M^2_2}{2} \|\nabla w\|^2 + \frac{M_0}{2h^2} \|\varsigma\|^2 + \frac{M_3}{h} \|\nabla \varsigma\|^2 - \frac{1}{2h^2} \left\| \tau_\delta(u_0^{-1}k_{\delta}(\varsigma)) \nabla u_0^{-1} \right\|^2 - \frac{1}{2h^2} \frac{M^2_1}{M_0} \|u_0^{-1}\|^2. 
\] (2.29)

Note that \(M_2\) is independent of \(\delta\), and \(M_3\) may depend on \(\delta\), but they all do not depend on the dimension of the space \(W_m\).

We know that (2.17) and (2.23) define a mapping \(\zeta_m : \mathbb{R}^{2m} \to \mathbb{R}^{2m}\) by \(\zeta_m \left( \begin{array}{c} 1 \\ \beta \end{array} \right) = \left( \begin{array}{c} 1 \\ \beta \end{array} \right)\), which is continuous by (A1). As \(|\|\| + \|\beta\|\) (e.g. in the standard Euclidean norm) becomes large enough, \(\zeta_m\) is strictly positive, and Lemma 1.4 in [30] (pp. 164) guarantees that \(\zeta_m\) has a zero, that means there exists a solution to the discrete system.

We now pass to the limit as \(m \to \infty\). Denoting by \((u_m, p_m) \in [W_m + C_\Omega] \times [W_m]\) the finite-dimensional solution \((\varsigma, w)\) obtained above, from (2.29), we get
\[
C(\Omega) \frac{M^2_2}{2} \|\nabla p_m\|^2 + \frac{M_0}{2h^2} \|u_m\|^2 + \frac{M_3}{h} \|\nabla u_m\|^2 \leq \frac{1}{2h^2} \left\| \tau_\delta(u_0^{-1}k_{\delta}(u_m)) \nabla u_0^{-1} \right\|^2 + \frac{1}{2h^2} \frac{M^2_1}{M_0} \|u_0^{-1}\|^2 \leq C. 
\] (3.30)

here \(C\) may depend on \(\delta\), but not on the dimension \(m\).

This means that \(u_m\) is uniformly bounded in \(V\), \(p_m\) is uniformly bounded in \(W_0^{1,2}(\Omega)\), so we can find \(\tilde{u} \in V, \tilde{p} \in W_0^{1,2}(\Omega)\), such that, \(u_m \to \tilde{u}\) weakly in \(V\) and \(p_m \to \tilde{p}\) weakly in \(W_0^{1,2}(\Omega)\). The compact embedding of \(W^{1,2}(\Omega)\) into \(L^2(\Omega)\) gives
Therefore, for any \( \phi \) the sequence \( k_{n_0}(u_m) \nabla p_m \) is bounded uniformly in \( L^2(\Omega) \) (w.r.t. \( m \)), so it has a weak limit \( \chi \). We identify this limit as \( k_{n_0}(\bar{u}) \nabla \bar{p} \) implying that
\[
 (k_{n_0}(u_m) \nabla p_m, \nabla \phi) \rightarrow (k_{n_0}(\bar{u}) \nabla \bar{p}, \nabla \phi), \quad \text{for any } \phi \in W^{1,2}_0(\Omega).
\]
To do so, we choose the test function \( \phi \in C_0^\infty(\Omega) \), clearly, one yields
\[
 (k_{n_0}(u_m) \nabla p_m, \nabla \phi) = (\nabla p_m, k_{n_0}(\bar{u}) \nabla \phi) + (\nabla p_m, (k_{n_0}(u_m) - k_{n_0}(\bar{u})) \nabla \phi),
\]
and since \( k_{n_0}(\bar{u}) \in L^\infty(\Omega) \), one has
\[
 (\nabla p_m, k_{n_0}(\bar{u}) \nabla \phi) \rightarrow (\nabla \bar{p}, k_{n_0}(\bar{u}) \nabla \phi).
\]
We now show that the limit of \( (\nabla p_m, (k_{n_0}(u_m) - k_{n_0}(\bar{u})) \nabla \phi) \) is 0.
Since \( \nabla p_m \) is bounded uniformly in \( (L^2(\Omega))^d \), one has
\[
 |(\nabla p_m, (k_{n_0}(u_m) - k_{n_0}(\bar{u})) \nabla \phi)| \leq ||\nabla p_m|| \cdot ||(k_{n_0}(u_m) - k_{n_0}(\bar{u})) \nabla \phi|| \leq C \left( \int_{\Omega} (k_{n_0}(u_m) - k_{n_0}(\bar{u}))^2 |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}}.
\]
Because \( k_{n_0} \) is bounded, we get
\[
 |(k_{n_0}(u_m) - k_{n_0}(\bar{u})) \nabla \phi| \leq C |\nabla \phi| \quad \text{in } \Omega.
\]
Further, since \( u_m \rightarrow \bar{u} \) strongly in \( L^2(\Omega) \), in the view of the continuity of \( k_{n_0} \), we have
\[
 k_{n_0}(u_m) \rightarrow k_{n_0}(\bar{u}) \quad a.e.
\]
Then, by the Dominated Convergence Theorem
\[
 \left( \int_{\Omega} (k_{n_0}(u_m) - k_{n_0}(\bar{u}))^2 |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}} \rightarrow 0.
\]
Therefore, for any \( \phi \in C_0^\infty(\Omega) \), since \( k_{n_0}(u_m) \nabla p_m \rightarrow \chi \) (weakly in \( W^{1,2}(\Omega) \)), due to the density of \( C_0^\infty(\Omega) \) in \( W^{1,2}(\Omega) \), this allows identifying \( \chi = k_{n_0}(\bar{u}) \nabla \bar{p} \) for any \( \phi \in W^{1,2}_0(\Omega) \).
Similarly, we can also prove that
\[
 (k_{n_0}(u_m) \nabla p_m, \nabla \phi) \rightarrow (k_{n_0}(\bar{u}) \nabla \bar{p}, \nabla \phi),
\]
for any \( \phi \in W^{1,2}_0(\Omega) \).
Therefore a solution pair to Problem \( P^0_a \) from now on is denoted by \( (u^0_n, p^0_n) \).

### 2.3. A priori estimates

Having established the existence for the time discrete problems, we now prove the existence of a solution to Problem \( P_a \). To achieve this, we use the elementary result.

**Lemma 2.2.** Let \( k \in 1, \ldots, N, m \geq 1 \). Given two sets of real vectors \( a^k, b^k \in \mathbb{R}^m \) \((k = 1, \ldots, N)\), one has
\[
 \sum_{k=1}^N (a^k - a^{k-1}, a^k) = \frac{1}{2} \left( |a^0|^2 - |a^1|^2 + \sum_{k=1}^N |a^k - a^{k-1}|^2 \right).
\]
We have the following:
Lemma 2.3. A $C > 0$, not depending on $h$ exists such that, for any $N^* \in 1, 2, ..., N$, one has

$$\int_{\Omega} \Gamma_\delta(u_{N}^{N^*}) dx + \frac{1}{2} \left\| \nabla T_\delta(u_{N}^{N^*}) \right\|^2 + \sum_{n=1}^{N^*} \left\| \nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})) \right\|^2 + h \sum_{n=1}^{N^*} \left\| \sqrt{-p'_c \tau_\delta(u_{n}^{N^*})} \nabla u_{n}^{N^*} \right\|^2 \leq C, \quad (2.35)$$

$$\left\| u_{N}^{N^*} \right\|^2_{L^2(\Omega)} + \frac{1}{2} \left\| \nabla u_{N}^{N^*} \right\|^2 + h \sum_{n=1}^{N^*} \left\| \nabla u_{n}^{N^*} \right\|^2 \leq C. \quad (2.36)$$

Proof. Taking $\phi = \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) d\zeta$ in (2.13) and $\psi = \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) dz$ in (2.14) (both being in $W_0^{1,2}(\Omega)$) gives

$$\left( \frac{u_{n}^{N^*} - u_{n}^{N^*-1}}{h}, \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) d\zeta \right) - (k_\delta(u_{n}^{N^*}) \nabla p_1^{N^*}, \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*}) + (\nabla \theta_\delta(u_{n}^{N^*}), \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*} = 0, \quad (2.37)$$

and

$$(k_\delta(u_{n}^{N^*}) \nabla p_1^{N^*}, \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*}) + (k_\delta(u_{n}^{N^*}) \nabla T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})/h, 1/k_{\omega_\delta} \nabla T_\delta(u_{n}^{N^*}) = 0. \quad (2.38)$$

With $G_\delta$ defined in (2.3), in view of its monotonicity, we have

$$(u_{n}^{N^*} - u_{n}^{N^*-1}) \cdot G_\delta(u_{n}^{N^*}) \geq \Gamma_\delta(u_{n}^{N^*}) - \Gamma_\delta(u_{n}^{N^*-1}). \quad (2.39)$$

Adding (2.37) and (2.38) and using (2.39) lead to

$$\frac{1}{h} \int_{\Omega} \left( \Gamma_\delta(u_{n}^{N^*}) - \Gamma_\delta(u_{n}^{N^*-1}) \right) dx + (\nabla \theta_\delta(u_{n}^{N^*}), \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*}) + \frac{1}{h} (\nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})), \nabla T_\delta(u_{n}^{N^*})) \leq 0. \quad (2.40)$$

Summing up (2.40) for $n = 1$ to $N^*$ gives

$$\int_{\Omega} \left( \Gamma_\delta(u_{N}^{N^*}) - \Gamma_\delta(u_{1}^{0}) \right) dx + h \sum_{n=1}^{N^*} (\nabla \theta_\delta(u_{n}^{N^*}), \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*}) + \sum_{n=1}^{N^*} (\nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})), \nabla T_\delta(u_{n}^{N^*})) \leq 0. \quad (2.41)$$

Since $\theta_\delta(u_{n}^{N^*}) = (\int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) d\zeta \cdot p'_c(u_{n}^{N^*}) / k_{\omega_\delta} k_{\kappa}$), one has

$$h \sum_{n=1}^{N^*} (\nabla \theta_\delta(u_{n}^{N^*}), \int_{C_\rho}^{\zeta} \tau_\delta k_\delta \kappa(\zeta) \nabla u_{n}^{N^*}) = h \sum_{n=1}^{N^*} \left\| \sqrt{-p'_c \tau_\delta(u_{n}^{N^*})} \nabla u_{n}^{N^*} \right\|^2. \quad (2.42)$$

According to Lemma 2.2, we get

$$\sum_{n=1}^{N^*} (\nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})), \nabla T_\delta(u_{n}^{N^*})) = \frac{1}{2} \sum_{n=1}^{N^*} \left\| \nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})) \right\|^2 \leq \frac{1}{2} \left\| \nabla T_\delta(u_{n}^{N^*}) \right\|^2. \quad (2.43)$$

leading to

$$\int_{\Omega} \left( \Gamma_\delta(u_{N}^{N^*}) + h \sum_{n=1}^{N^*} \left\| \sqrt{-p'_c \tau_\delta(u_{n}^{N^*})} \nabla u_{n}^{N^*} \right\|^2 + \frac{1}{2} \sum_{n=1}^{N^*} \left\| \nabla (T_\delta(u_{n}^{N^*}) - T_\delta(u_{n}^{N^*-1})) \right\|^2 \right. \leq \Gamma_\delta(u_{1}^{0}) + \frac{1}{2} \left\| \nabla T_\delta(u_{1}^{0}) \right\|^2. \quad (2.44)$$
Recalling (1.21), (A1) and (A2), uniformly w.r. t. \( \delta \), it holds that
\[
\int_{\Omega} \Gamma_\delta(u^0) \, dx = \int_{\Omega} \int_{\Omega} \int_{\Omega} \tau_\delta k_\delta \, dz \, dx
\]
\[
= \int_{\Omega} \int_{\Omega} \int_{\Omega} \tau_\delta (\nabla u^0) \, dz \, dx + \int_{\Omega} \int_{\Omega} \int_{\Omega} \tau_\delta (\nabla u^0) \, dz \, dx
\]
\[
\leq \int_{\Omega} \int_{\Omega} \int_{\Omega} \tau_\delta (\nabla u^0) \, dz \, dx + \int_{\Omega} \int_{\Omega} \int_{\Omega} \tau_\delta (\nabla u^0) \, dz \, dx
\]
\[
= \int_{\Omega} \Gamma(u^0) \, dx
\]
\[
< \infty.
\] (2.45)

Furthermore, we also have
\[
\int_{\Omega} |\nabla T_\delta(u^0)|^2 \, dx = \int_{\Omega} \tau_\delta |\nabla u^0|^2 \, dx \leq \int_{\Omega} \tau_\delta |\nabla u^0|^2 \, dx = \int_{\Omega} \nabla |T(u^0)|^2 \, dx \leq C.
\] (2.46)

Therefore, we obtain
\[
\int_{\Omega} \Gamma_\delta(u^N) \, dx + \frac{1}{2} \|\nabla T_\delta(u^N)\|^2 + \frac{1}{2} \sum_{n=1}^{N} \|\nabla (T_\delta(u^N) - T_\delta(u^{N-1}))\|^2 + h \sum_{n=1}^{N} \left\| \frac{1}{\delta} p_n \cdot \nabla (u^N - u^{N-1}) \right\|^2 \leq C.
\] (2.47)

By the definition of \( \tau_\delta \), this immediately gives
\[
\frac{1}{2} \|\nabla u^N\|^2 + h \sum_{n=1}^{N} \|\nabla u^N\|^2 \leq C.
\] (2.48)

To complete the proof, we use the Poincaré’s inequality
\[
\|u^N\|_{L^2(\Omega)} \leq \|u^N\|_{L^2(\Omega)} + \|C_D\|_{L^2(\Omega)} \leq C(\|\nabla u^N\|_{L^2(\Omega)}) \leq C.
\] (2.49)

**Lemma 2.4.** For any \( N^* \in 1, 2, ..., N \), we have
\[
\sum_{n=1}^{N^*} \|T_\delta(u^N) - T_\delta(u^{N-1})\|^2 + \sum_{n=1}^{N^*} \|u^N_n - u^{N-1}_n\|^2 \leq Ch,
\] (2.50)
\[
\sum_{n=1}^{N^*} \|T_\delta(u^N) - T_\delta(u^{N-1})\|^2 + \sum_{n=1}^{N^*} \|u^N_n - u^{N-1}_n\|^2 \leq Ch,
\] (2.51)
\[
h \sum_{n=1}^{N^*} \|\nabla p^N_n\|^2 \leq C,
\] (2.52)

where \( C \) is independent on \( h \).

**Proof.** Testing in both (2.13) and (2.14) with \( h(T_\delta(u^N) - T_\delta(u^{N-1})) \), adding the resulting gives
\[
(u^N_n - u^{N-1}_n, T_\delta(u^N) - T_\delta(u^{N-1})) + \sum_{n=1}^{N^*} h k_n (u^N_n, T_\delta(u^N) - T_\delta(u^{N-1})) + h \sum_{n=1}^{N^*} \|\nabla p^N_n\|^2 \leq h \sum_{n=1}^{N^*} \|\nabla (u^N_n, T_\delta(u^N) - T_\delta(u^{N-1}))\|^2 = 0.
\] (2.53)
With \( \psi = h p_0^n \) in (2.14), we have

\[
h \left\| \sqrt{k_0(u_0^n)} \nabla p_0^n \right\|^2 = -(k_{\text{wcb}}(u_0^n) \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})), \nabla p_0^n) \leq \left\| \frac{k_{\text{wcb}}(u_0^n)}{\sqrt{k_0(u_0^n)}} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\| \cdot \left\| \sqrt{k_0(u_0^n)} \nabla p_0^n \right\|,
\]

(2.54)
giving

\[
-h(k_{\text{wcb}}(u_0^n) \nabla p_0^n, \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1}))) \leq \left\| \frac{k_{\text{wcb}}(u_0^n)}{\sqrt{k_0(u_0^n)}} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2. \tag{2.55}
\]

Then, (2.53) becomes

\[
(u_0^n - u_0^{n-1}, T_\delta(u_0^n) - T_\delta(u_0^{n-1})) + \left\| \frac{k_{\text{wcb}}(u_0^n)}{\sqrt{k_0(u_0^n)}} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 + h(\nabla \theta_0(u_0^n), \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1}))) \leq \left\| \frac{k_{\text{wcb}}(u_0^n)}{\sqrt{k_0(u_0^n)}} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2. \tag{2.56}
\]

In view of the definition \( k_\delta \), one has

\[
\left\| \frac{k_{\text{wcb}}(u_0^n)}{\sqrt{k_0(u_0^n)}} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 = \left\| \frac{k_{\text{wcb}}(u_0^n)}{k_\delta} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2. \tag{2.57}
\]

Further, since

\[
h \left\| \nabla \theta_0(u_0^n), \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\| \leq \frac{1}{2} \left\| \sqrt{k_{\text{wcb}}}(u_0^n) \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 + \frac{h^2}{2} \left\| \frac{k_{\text{wcb}}(u_0^n)}{k_\delta} \nabla(u_0^n) \right\|^2, \tag{2.58}
\]

(2.56) yields

\[
(u_0^n - u_0^{n-1}, T_\delta(u_0^n) - T_\delta(u_0^{n-1}))+ \frac{1}{2} \left\| \frac{k_{\text{wcb}}(u_0^n)}{k_\delta} \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 \leq \frac{h^2}{2} \left\| \frac{k_{\text{wcb}}(u_0^n)}{k_\delta} \nabla(u_0^n) \right\|^2 \leq \frac{h^2}{2} \left\| \frac{k_{\text{wcb}}(u_0^n)}{k_\delta} \nabla(u_0^n) \right\|^2. \tag{2.59}
\]

By (2.36) and (A1) and (A2), we have

\[
(u_0^n - u_0^{n-1}, T_\delta(u_0^n) - T_\delta(u_0^{n-1})) + \left\| \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 \leq C h^2. \tag{2.60}
\]

This leads to

\[
\sum_{n=1}^{N} (u_0^n - u_0^{n-1}, T_\delta(u_0^n) - T_\delta(u_0^{n-1})) + \sum_{n=1}^{N} \left\| \nabla(T_\delta(u_0^n) - T_\delta(u_0^{n-1})) \right\|^2 \leq C h. \tag{2.61}
\]

By the definition of \( T \), a \( \xi \) exists, such that

\[
(u_0^n - u_0^{n-1}, T_\delta(u_0^n) - T_\delta(u_0^{n-1})) = \left\| \frac{\sqrt{r_\delta}(\xi)}{\sqrt{r_\delta}(u_0^n - u_0^{n-1})} \right\|^2 \leq C h^2. \tag{2.62}
\]
Similarly, a $\tilde{\xi}$ exists, such that
\[ (u^n_\delta - u^{n-1}_\delta, T_\delta(u^n_\delta) - T_\delta(u^{n-1}_\delta)) = \left\| \frac{1}{\sqrt{T_\delta(\xi)}} (T_\delta(u^n_\delta) - T_\delta(u^{n-1}_\delta)) \right\|^2 \leq Ch^2. \] (2.63)
Then we obtain
\[ \sum_{n=1}^{N^n} \left\| T_\delta(u^n_\delta) - T_\delta(u^{n-1}_\delta) \right\|^2 + \sum_{n=1}^{N^n} \left\| u^n_\delta - u^{n-1}_\delta \right\|^2 \leq Ch. \] (2.64)
Furthermore, according to (A1), (2.36) and (2.60), one also has
\[ \sum_{n=1}^{N^n} \left\| \nabla(u^n_\delta - u^{n-1}_\delta) \right\|^2 \leq Ch, \] (2.65)
Finally, (2.52) follows from (2.54) and (2.60).

2.4. Existence of weak solutions to the regularized problem

To show the existence of a solution to Problem $P_\delta$, we consider linear interpolation in time:
\[ T_{\delta N}(t) = T_\delta(u^{n-1}_\delta) + \frac{t - \tau^{n-1}}{h}(T_\delta(u^n_\delta) - T_\delta(u^{n-1}_\delta)), \] (2.66)
\[ U_{\delta N}(t) = u^{n-1}_\delta + \frac{t - \tau^{n-1}}{h}(u^n_\delta - u^{n-1}_\delta), \] (2.67)
and piecewise constant functions in time
\[ \bar{T}_{\delta N}(t) = T_\delta(u^n_\delta), \quad \bar{U}_{\delta N}(t) = u^n_\delta, \quad \bar{P}_{\delta N}(t) = p^n_\delta, \] (2.68)
for $t \in (\tau^{n-1}, \tau^n], n = 1, 2, \ldots, N$. Clearly, $\bar{T}_{\delta N} \in L^2(Q), \bar{U}_{\delta N} \in L^2(0, T_\delta; V), \bar{P}_{\delta N}(t) \in L^2(0, T_\delta; W^1_0(Q))$.

We have the following result:

**Theorem 2.1.** Problem $P_\delta$ has a solution $(u_\delta, p_\delta)$.

**Proof.** According to the priori estimates in Lemma 2.3 and Lemma 2.4, we have
\[ \int_0^{T_\delta} \left\| T_{\delta N}(t) \right\|^2_{L^2(\Omega)} dt \leq C, \] (2.69)
\[ \int_0^{T_\delta} \left\| \nabla T_{\delta N}(t) \right\|^2_{L^2(\Omega)} dt \leq C, \] (2.70)
\[ \int_0^{T_\delta} \left\| \partial_t U_{\delta N}(t) \right\|^2_{L^2(\Omega)} dt \leq C, \] (2.71)
\[ \int_0^{T_\delta} \left\| \partial_t T_{\delta N}(t) \right\|^2_{L^2(\Omega)} dt \leq C, \] (2.72)
and
\[
\int_0^T \| \partial_t \nabla T_{\delta N}(t) \|^2_{L^2(\Omega)} \, dt = \sum_{n=1}^N \int_0^T \left\| \frac{1}{h} \nabla (T_\delta (u^n_\delta) - T_\delta (u^{n-1}_\delta)) \right\|^2_{L^2(\Omega)} \, dt \\
= \frac{1}{h} \sum_{n=1}^N \left\| \nabla (T_\delta (u^n_\delta) - T_\delta (u^{n-1}_\delta)) \right\|^2_{L^2(\Omega)} \\
\leq C. \tag{2.73}
\]

In the same way, one obtains similar estimates
\[
\int_0^T \| U_{\delta N}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^T \| \nabla U_{\delta N}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^T \| \partial_t U_{\delta N}(t) \|^2_{L^2(\Omega)} \, dt + \int_0^T \| \partial_t \nabla U_{\delta N}(t) \|^2_{L^2(\Omega)} \, dt \leq C. \tag{2.74}
\]

Therefore, the sequences \( \{ T_{\delta N} \}_{N \in \mathbb{N}} \) and \( \{ U_{\delta N} \}_{N \in \mathbb{N}} \) are uniformly bounded in \( W^{1,2}(0,T_M;W^{1,2}(\Omega)) \), so there exist two sub-sequences (still denoted by \( \{ T_{\delta N} \} \) and \( \{ U_{\delta N} \} \) ) which converge weakly to some \( T_\delta^* \in W^{1,2}(0,T_M;W^{1,2}(\Omega)) \) and \( u_\delta \in W^{1,2}(0,T_M;V) \).

For any \( \phi, \psi \in L^2(0,T_M;W_0^{1,2}(\Omega)) \), (2.13)-(2.14) give
\[
\int_0^T \int_\Omega \partial_t U_{\delta N} \phi dx dt - \int_0^T \int_\Omega k_{\delta N}(\bar{U}_{\delta N}) \nabla \bar{P}_{\delta N} \nabla \phi dx dt + \int_0^T \int_\Omega \nabla \partial_t (\bar{U}_{\delta N}) \nabla \phi dx dt = 0, \tag{2.75}
\]
\[
\int_0^T \int_\Omega \int_\Omega k_{\delta N}(\bar{U}_{\delta N}) \nabla \bar{P}_{\delta N} \nabla \psi dx dt + \int_0^T \int_\Omega k_{\delta N}(\bar{U}_{\delta N}) \nabla \partial_t \bar{U}_{\delta N} \nabla \psi dx dt = 0. \tag{2.76}
\]

Clearly, \( T_{\delta N}(t) \rightarrow T_\delta^* \) strongly in \( L^2(Q) \). By Lemma 3.2 in [26], it follows that \( T_{\delta N}(t) \rightarrow T_\delta^* \) strongly in \( L^2(Q) \) as well and a similar conclusion can be drawn for \( \bar{U}_{\delta N} \) and \( u_\delta \). By the continuity of \( k_\delta, k_{\delta N}, k_{\partial N} \), we also have \( k_{\partial N}(\bar{U}_{\delta N}) \rightarrow k_{\partial N}(u_\delta) \), \( k_{\delta}(\bar{U}_{\delta N}) \rightarrow k_{\delta}(u_\delta) \), and \( k_{\delta N}(\bar{U}_{\delta N}) \rightarrow k_{\delta N}(u_\delta) \). Now we show that \( \partial_t \nabla T_\delta^* = \partial_t \nabla T_\delta(u_\delta) \). Since \( T_{\delta N}(t) = T_\delta(u_\delta) \) converges to \( T_\delta^* \), by the definition of \( T_\delta \), we also find that \( T_\delta(\bar{U}_{\delta N}) \rightarrow T_\delta(u_\delta) \), then we have \( T_\delta^* = T_\delta(u_\delta) \).

Similarly, for \( \bar{P}_{\delta N} \), we have a \( p_\delta \) such that
\[
\nabla \bar{P}_{\delta N} \rightarrow \nabla p_\delta \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d).
\]

As in the proof in Lemma 2.1, we get
\[
k_{\partial N}(\bar{U}_{\delta N}) \nabla \bar{P}_{\delta N} \rightarrow k_{\partial N}(u_\delta) \nabla p_\delta \quad \text{weakly in } L^2(0,T_M;L^2(\Omega)^d),
\]
\[
k_{\delta N}(\bar{U}_{\delta N}) \nabla \bar{P}_{\delta N} \rightarrow k_{\delta N}(u_\delta) \nabla p_\delta \quad \text{weakly in } L^2(0,T_M;L^2(\Omega)^d),
\]
\[
k_{\partial N}(\bar{U}_{\delta N}) \nabla \partial_t \bar{U}_{\delta N} \rightarrow k_{\partial N}(u_\delta) \nabla \partial_t T_\delta(u_\delta) \quad \text{weakly in } L^2(0,T_M;L^2(\Omega)^d),
\]
\[
k_{\delta N}(\bar{U}_{\delta N}) \nabla \partial_t \bar{U}_{\delta N} \rightarrow k_{\delta N}(u_\delta) \nabla \partial_t T_\delta(u_\delta) \quad \text{weakly in } L^2(0,T_M;L^2(\Omega)^d).
\]

Combining the results, we obtain that \( (u_\delta, p_\delta) \) is the solution pair of Problem \( P_\delta \). \( \square \)

2.5. Uniqueness of Problem \( P_\delta \)

After having obtained the existence of a weak solution, we show its uniqueness. To do so, we consider the following system:
\[
\partial_t s - \nabla \cdot (k_{\delta N}(s) \nabla p_\delta) = 0, \tag{2.77}
\]
\[
-\partial_t s - \nabla \cdot (k_{\delta N}(s) \nabla p_\delta) = 0, \tag{2.78}
\]
\[
p_\delta - p_\delta = p_{\delta N} - p_{\delta N}(C_D) - \partial_t T_\delta(s). \tag{2.79}
\]

Formally, this is equivalent to (2.6) - (2.7), in the sense that \( u_\delta = s \) and the global pressure \( p \) is given by (1.9) or (1.10), often finding \( p_\delta, p_\delta \) in (2.77) - (2.79).

Clearly, the initial and boundary conditions should be compatible with the original ones:
\[
s(0, \cdot) = u^0 \quad \text{in } \Omega, \tag{2.80}
\]
Proof. The proof follows the ideas in [19], see [29] for the underlying ideas.

A3: $\Omega$ is a $C^{0,\epsilon}$ domain for some $0<\epsilon \leq 1$.

A4: $u^0 \in C^{0,\epsilon}(\bar{\Omega})$.

A weak solution of (2.77)-(2.79) solves

Problem P: Given $s(0,\cdot) = u^0$, find $p_w \in L^2(0, T_M; W_0^{1,2}(\Omega))$, $p_n \in L^2(0, T_M; W_0^{1,2}(\Omega))$ and $s \in L^2(0, T_M; L^2(\Omega))$, such that

$$p_w = p_n = 0, \quad \text{at } \partial \Omega, \text{ for all } t > 0.$$  \hspace{1cm} (2.81)

Furthermore, for the uniqueness, we also assume

Problem P: Given $s(0,\cdot) = u^0$, find $p_w \in L^2(0, T_M; W_0^{1,2}(\Omega))$, $p_n \in L^2(0, T_M; W_0^{1,2}(\Omega))$ and $s \in L^2(0, T_M; L^2(\Omega))$, such that

$$p_w = p_n = 0, \quad \text{at } \partial \Omega, \text{ for all } t > 0.$$  \hspace{1cm} (2.81)

The equivalence above can be made more precisely.

Lemma 2.5. Problem $P_e$ and Problem $P_\delta$ are equivalent. Specifically, $(u_s, p_\delta)$ is a solution to Problem $P_\delta$ if and only if $(s, p_w, p_n)$ solves Problem $P_e$, with $s = u_s$, $p_w = p_s + \int_{S_h} f_w(s) p_w(s)dz - p_c(u_s) + p_0(C_D) + \tau(u_s)\partial_t u_s$, $p_n = p_s + \int_{S_h} f_w(s) p_w(s)dz$.

Proof. The proof follows the ideas in [19], see [29] for the underlying ideas.

Theorem 2.2. There exists at most one solution $(u_s, p_\delta)$ for Problem $P_\delta$.

Proof. Uniqueness for Problem $P_e$ is proved in [12]. Then, by equivalence, Problem $P_\delta$ has at most one solution.

3. Existence of Problem P

Below we extend the existence result to a class of degenerate equations. Specifically, we assume the following wing asymptotic behaviors for the nonlinear functions $k_u, k_n, p_c$ and $\tau$.

A5: There exist $\alpha > 0$, $\beta > 0$, $\lambda > 0$, and for different dimensions, we assume the followings

$d = 3$: $\alpha \geq \lambda > \alpha/3 + 10/3$, $\omega > 5/2$, $e := \omega + \beta > 5$,

$d = 2$ or $d = 1$: $\alpha \geq \lambda > 4$, $\omega > 2$, $e > 4$,

and the followings hold

$$\lim_{u \searrow 0} k_u(u) - u^\alpha = C_1,$$  \hspace{1cm} (3.1)

$$\lim_{u \searrow 0} p_c'(u) - u^\beta = C_2,$$  \hspace{1cm} (3.2)

$$\lim_{u \searrow 0} k_u(u) - (1 - u)^\beta = C_3,$$  \hspace{1cm} (3.3)

$$\lim_{u \searrow 0} \tau(u) - (1 - u)^\omega = C_4.$$  \hspace{1cm} (3.4)

for some constants $C_1, C_2, C_3, C_4 \in (0, +\infty)$.

To avoid unnecessary technical complications, we restrict the proof to the cases $k_u = u^\alpha$, $k_n = (1 - u)^\beta$, $p_c'(u) = -u^\gamma$, and $\tau(u) = (1 - u)^{-\omega}$. We note that these nonlinearities are commonly encountered in the porous media literature [4, 7]. Less standard is the function $\tau$. We refer to [6, 9], where this type of behavior is proposed.

Since in (A1) we assume that the product $k_u \cdot p_c$ is uniformly bounded in $\mathbb{R}$, this implies $\alpha \geq \lambda$. According to the definition of $\tau$ and its regularization, one has $T(u)$ and $T_\delta(u)$ as followings:

$$T(u) = \begin{cases} 
\frac{u}{u^{-\omega}}, & \text{if } u < 0, \\
\frac{1}{u^{-\omega}}(1 - u)^{1-\omega}, & \text{if } 0 \leq u < 1, \\
+\infty, & \text{if } u > 1,
\end{cases} \hspace{1cm} (3.5)$$
\[ T_\delta(u) = \begin{cases} u + \frac{1}{\omega - 1} - \delta, & \text{if } u < \delta, \\ \frac{1}{\omega - 1} (1 + \delta - u)^{1-\omega}, & \text{if } \delta \leq u \leq 1, \\ \frac{1}{\omega - 1} \frac{1}{\omega - 1} u, & \text{if } u > 1. \end{cases} \] (3.6)

**Remark 3.** Choosing \( C_T \) in (1.20) \( 1/(\omega - 1) \) gives a simple exposition for \( T \), \( T(u) = \frac{1}{\omega - 1} (1 - u)^{1-\omega} \) (\( u \in [0, 1] \)). However, the choice of \( C_T \) has no importance for the proof.

Here we define the characteristic function

\[ \chi_{(a,b)} = \begin{cases} 1, & \text{if } v \in (a, b), \\ 0, & \text{if } v \notin (a, b). \end{cases} \] (3.7)

Then we have the following results

**Lemma 3.1.** Let suppose the hypotheses (A1), (A2) and (A5). Then there exists a constant \( C > 0 \), independent on \( \delta \), such that for the first component of the weak solution pair \((u_\delta, p_\delta)\) to Problem \( P_\delta \), one has

\[ ||(u_\delta + \delta)^{2-\alpha}||_{L^2(0,T;w^{1+\alpha}(\Omega))} \leq C, \] (3.8)

\[ ||(1 - u_\delta + \delta)^{2-\alpha}||_{L^2(0,T;w^{1+\alpha}(\Omega))} \leq C, \] (3.9)

\[ ||\nabla T_\delta(u_\delta)||_{L^2(0,T;w^{1+\alpha}(\Omega))} + ||\nabla u_\delta||_{L^2(0,T;w^{1+\alpha}(\Omega))} \leq C, \] (3.10)

\[ \left\| \sqrt{-p_\delta \tau_\delta \nabla u_\delta} \right\|_{L^2(0,T;w^{1+\alpha}(\Omega))} \leq C. \] (3.11)

**Proof.** With \( t \in (0, T_M) \), taking the test function \( \phi = \chi_{(0,\alpha)} \int_{C_0}^{u_\delta} \frac{\tau_\delta k_\delta}{k_\delta k_\delta} (z)dz \) in (2.10) and \( \psi = \chi_{(0,\alpha)} \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz \) in (2.11) gives

\[ \int_0^t (\partial_t u_\delta, \int_{C_0}^{u_\delta} \frac{\tau_\delta k_\delta}{k_\delta k_\delta} (z)dz)dt + \int_0^t \left\| \sqrt{-p_\delta \tau_\delta \nabla u_\delta} \right\|^2 dt = 0, \] (3.12)

\[ \int_0^t (\nabla p_\delta, \frac{\tau_\delta k_\delta}{k_\delta k_\delta} \nabla u_\delta)dt + \int_0^t (\nabla \partial_t T_\delta(u_\delta), \nabla T_\delta(u_\delta))dt = 0. \] (3.13)

Adding (3.12) and (3.13) gives

\[ \int_0^t \int_\Omega \partial_t u_\delta \int_{C_0}^{u_\delta} \frac{\tau_\delta k_\delta}{k_\delta k_\delta} (z)dzdxdt + \int_0^t \left\| \sqrt{-p_\delta \tau_\delta \nabla u_\delta} \right\|^2 dt + \frac{1}{2} \int_0^t \int_\Omega d ||\nabla T_\delta(u_\delta)||^2 dt = 0. \] (3.14)

By the definition of \( k_\delta \), one has

\[ \int_\Omega \partial_t u_\delta \int_{C_0}^{u_\delta} \frac{\tau_\delta k_\delta}{k_\delta k_\delta} (z)dzdx = \int_\Omega \partial_t u_\delta \int_{C_0}^{u_\delta} \tau_\delta (z)dzdx + \int_\Omega \partial_t u_\delta \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dzdx. \] (3.15)

The following identities hold a.e.

\[ \partial_t u \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz = \partial_t \left( u \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz - \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz \right), \] (3.16)

\[ \partial_t u_\delta \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz = \partial_t \left( (u_\delta - 1) \int_{C_0}^{u_\delta} \frac{\tau_\delta}{k_\delta k_\delta} (z)dz - \int_{1}^{u_\delta} (z - 1) \frac{\tau_\delta}{k_\delta k_\delta} (z)dz \right). \] (3.17)
As in [27], we define the functions $\tilde{E}_{\alpha\delta}, \tilde{E}_{\alpha\delta} : \mathbb{R} \to \mathbb{R}$
\[
\tilde{E}_{\alpha\delta}(y) = \int_0^y \int_{K_0} \frac{\tau_\delta}{k_0} dz dv + \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{\alpha - 1} C_D \frac{\tau_\delta^\alpha}{k_0} dz
\] (3.18)

\[
= y \int_0^y \int_{K_0} \frac{\tau_\delta}{k_0} dz - \int_0^y \int_{K_0} \frac{\tau_\delta^\alpha}{k_0} dz + \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{\alpha - 1} C_D, \quad (3.19)
\]

and
\[
\tilde{E}_{\alpha\delta}(y) = \int_0^y \int_{C_D} \frac{1}{k_0} dz dv + \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{\alpha - 1} C_D. \quad (3.20)
\]

The choice of these terms is justified by the following calculations.

Recalling (A5), we have
\[
\int_0^{u_\delta} \int_{C_D} \frac{1}{k_0} dz dv = \frac{(u_\delta + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} + \frac{(C_D + \delta)^{1-\alpha}}{\alpha - 1} u_\delta
\]
\[
- \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{(1-\alpha)(2-\alpha)} C_D. \quad (3.21)
\]

Similarly, for $u_\delta < 0$, one has
\[
\int_0^{u_\delta} \int_{C_D} \frac{1}{k_0} dz dv = \frac{\delta^{-\alpha}}{2} u_\delta^2 + \frac{(C_D + \delta)^{1-\alpha} - \delta^{1-\alpha} u_\delta + \delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D
\]
\[
- \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{(1-\alpha)(2-\alpha)} C_D, \quad (3.22)
\]

and for $u_\delta > 1 - \delta$, we get
\[
\int_0^{u_\delta} \int_{C_D} \frac{1}{k_0} dz dv = \frac{(u_\delta - (1-\delta))^2}{2} + \frac{(C_D + \delta)^{1-\alpha} - \delta^{1-\alpha} u_\delta + \delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D
\]
\[
+ \frac{1 - \delta}{\alpha - 1} \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{(1-\alpha)(2-\alpha)} C_D. \quad (3.23)
\]

Note that the calculations above hold for the choice $k_0(a) = a^{-\alpha}$. If, instead, $k_0$ behaves like in (3.1), there the expressions on the right in the above are dominating terms, the reminders being regular w.r.t $\delta$.

In this way $\tilde{E}_{\alpha\delta}(u_\delta)$ becomes
\[
\tilde{E}_{\alpha\delta}(u_\delta) = \begin{cases}
\frac{\alpha^{-\alpha}}{2} u_\delta^2 + \frac{(C_D + \delta)^{1-\alpha} - \delta^{1-\alpha} u_\delta + \delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D, & \text{for } u_\delta < 0, \\
\frac{(u_\delta - (1-\delta))^2}{2} + \frac{(C_D + \delta)^{1-\alpha} - \delta^{1-\alpha} u_\delta + \delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D + \frac{1 - \delta}{\alpha - 1} \frac{(C_D + \delta)^{2-\alpha} - (C_D + \delta)^{1-\alpha}}{(1-\alpha)(2-\alpha)} C_D, & \text{for } 0 \leq u_\delta \leq 1 - \delta, \\
\frac{\alpha^{-\alpha}}{2} u_\delta^2 + \frac{(u_\delta + 1)^{1-\alpha} - \delta^{1-\alpha} u_\delta + \delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D, & \text{for } u_\delta > 1 - \delta.
\end{cases} \quad (3.24)
\]

We note that
\[
\tilde{E}_{\alpha\delta}(u_\delta) \geq \tilde{E}_{\alpha\delta}^0(u_\delta) := \begin{cases}
\frac{\alpha^{-\alpha}}{2} u_\delta^2 + \frac{\delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D, & \text{for } u_\delta < 0, \\
\frac{(u_\delta - (1-\delta))^2}{2} + \frac{1}{(1-\alpha)(2-\alpha)} C_D, & \text{for } 0 \leq u_\delta \leq 1 - \delta, \\
\frac{\alpha^{-\alpha}}{2} u_\delta^2 + \frac{\delta^{2-\alpha}}{(1-\alpha)(2-\alpha)} C_D, & \text{for } u_\delta > 1 - \delta.
\end{cases} \quad (3.25)
\]

whereas $\tilde{E}_{\alpha\delta}(u_\delta) \geq C(u_\delta + \delta)^2$ ($C > 0$ independent on $\delta$).

In the same way, for $\tilde{E}_{\alpha\delta}$, since $\tau_\delta \cdot k_0 = (1 - z + \delta)^\alpha$ (see (A5)), for $\delta \leq u_\delta \leq 1$, one has
\[
\int_1^{u_\delta} \int_{C_D} \frac{\tau_\delta}{k_0} dz dv = \frac{(1 - u_\delta + \delta)^{2-\alpha}}{(1-\alpha)(2-\alpha)} + \frac{1 - C_D + \delta^{-\alpha}}{\alpha - 1} - \frac{(1 - u_\delta)}{(1-\alpha)} + \frac{\delta^{2-\alpha}}{(1-\alpha)(2-\alpha)}. \quad (3.26)
\]
Further, for \( u_\delta < \delta \), one gets
\[
\int_1^u \int_{C_D} \tau_\delta dzdv = \frac{(u_\delta - \delta)^2}{2} + \frac{1}{(1 - e)(2 - e)} + \frac{1 - \delta}{e - 1} + (1 + \delta - C_D)^{1 - e} - 1 \left(1 - u_\delta\right) - \frac{\delta^2 - e}{(1 - e)(2 - e)}. \tag{3.27}
\]
and for \( u_\delta > 1 \), we have
\[
\int_1^u \int_{C_D} \tau_\delta dzdv = \frac{\delta^2}{2} (u_\delta - 1)^2 + \left(\frac{\delta^1 - e}{e - 1} - (1 + \delta - C_D)^{1 - e}\right)(u_\delta - 1). \tag{3.28}
\]
Then \( E_{ab}(v) \) rewrites
\[
E_{ab}(u_\delta) = \begin{cases} 
\frac{(u_\delta - \delta)^2}{2} + \frac{1}{(1 - e)(2 - e)} + \frac{1 - \delta}{e - 1} + (1 + \delta - C_D)^{1 - e} - 1 \left(1 - u_\delta\right), & \text{for } u_\delta < \delta, \\
\frac{\delta^2}{2} (u_\delta - 1)^2 + \left(\frac{\delta^1 - e}{e - 1} - (1 + \delta - C_D)^{1 - e}\right)(u_\delta - 1) + \frac{\delta^2 - e}{(1 - e)(2 - e)}, & \text{for } u_\delta > 1,
\end{cases} \tag{3.29}
\]
so
\[
E_{ab}(u_\delta) \geq E^{0}_{ab}(u_\delta) := \begin{cases} 
\frac{(u_\delta - \delta)^2}{2} + \frac{1}{(1 - e)(2 - e)} + \frac{1 - \delta}{e - 1}, & \text{for } u_\delta < \delta, \\
\frac{\delta^2}{2} (u_\delta - 1)^2 + \left(\frac{\delta^1 - e}{e - 1} - (1 + \delta - C_D)^{1 - e}\right)(u_\delta - 1), & \text{for } u_\delta > 1,
\end{cases} \tag{3.30}
\]
and
\[
E^{0}_{ab}(u_\delta) \geq C \left[1 - u_\delta\right] + \delta^2 - e, \tag{3.31}
\]
with \( C > 0 \) independent on \( \delta \).

Substitute \( E_{ab} + E_{ab} \) into (3.14) instead of \( \int \partial_t u_\delta \int_{C_D} \tau_\delta dzdv \), we have
\[
\int \Omega \nabla T(u_\delta) = \int \Omega \nabla T(u_\delta) + \int \Omega \nabla T(u_\delta) \leq \int \Omega \nabla T(u_\delta) \leq C, \tag{3.33}
\]
and
\[
\int \Omega \nabla T(u_\delta) = \int \Omega \nabla T(u_\delta) \leq \int \Omega \nabla T(u_\delta) \leq \int \Omega \nabla T(u_\delta) \leq C. \tag{3.34}
\]
These lead to
\[
\int \Omega E_{ab}(u_\delta) dx = \int \Omega \int_{C_D} \tau_\delta dzdv + \int \Omega \frac{(C_D + \delta)^{1 - e} - 1}{(1 - e)(2 - e)} - \frac{(C_D + \delta)^{1 - e} C_D}{(1 - e)(2 - e)} dx \leq C. \tag{3.35}
\]
Similarly, we also have
\[
\int \Omega E_{ab}(u_\delta) dx = \int \Omega \int_{C_D} \tau_\delta dzdv + \int \Omega \frac{\delta^2 - e}{(1 - e)(2 - e)} dx = \int \Omega \int_{C_D} \tau_\delta dzdv + \int \Omega \frac{(1 - C_D + \delta)^{1 - e} - 1}{(1 - e)(2 - e)} dx \leq C. \tag{3.36}
\]
Further, observe that $\mathcal{E}_{n\delta}$ is convex, allowing a minimum at $u_\delta = C_D$, then we have $\mathcal{E}_{n\delta} \geq \frac{(C_{D\delta})^\alpha}{\alpha-1} + \frac{(C_{n\delta})^\alpha}{\alpha-1} C_D > 0$. And $\mathcal{E}_{n\delta}$ is also positive obtained from (3.30) and (3.31). These give the estimates from (3.32):

$$\int_{\Omega} \mathcal{E}_{n\delta}(u_\delta) + \int_{\Omega} \mathcal{E}_{n\delta}(u_\delta) \leq C.$$  \hspace{1cm} (3.37)

Since $\tau_\delta$ is far away from 0, then one has

$$\int_{\Omega} \tilde{\mathcal{E}}_{n\delta}(u_\delta) \leq C.$$  \hspace{1cm} (3.38)

These lead to

$$\int_0^{\bar{T}} \parallel \sqrt{-p_\delta \tau_\delta \nabla u_\delta} \parallel^2 dt \leq C,$$  \hspace{1cm} (3.39)

$$\int_{\Omega} (\|u_\delta\| + \delta)^{2-\alpha} dx \leq C,$$  \hspace{1cm} (3.40)

$$\int_{\Omega} (\|1 - u_\delta\| + \delta)^{2-\alpha} dx \leq C,$$  \hspace{1cm} (3.41)

$$\|\nabla T_\delta(u_\delta)\|^2 + \|\nabla u_\delta\|^2 \leq C.$$  \hspace{1cm} (3.42)

**Lemma 3.2.** Under the assumptions (A1), (A2) and (A5), there exists a constant $C > 0$, independent on $\delta$, such that the weak solution pair $(u_\delta, p_\delta)$ of Problem $P_\delta$ satisfies:

$$\int_0^{\bar{T}} \|\partial_t u_\delta\|^2 dt + \int_0^{\bar{T}} \|1 - \frac{1}{\sqrt{\tau_\delta(u_\delta)}} \partial_t T_\delta(u_\delta)\|^2 dt + \int_0^{\bar{T}} \| \frac{k_{n\delta} k_{D\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta)\|^2 dt \leq C,$$  \hspace{1cm} (3.43)

$$\int_0^{\bar{T}} \| \sqrt{k_{n\delta}} \nabla p_\delta \|^2 + \int_0^{\bar{T}} \| \sqrt{k_{n\delta}} \nabla (p_\delta + \partial_t T_\delta(u_\delta)) \|^2 \leq C.$$  \hspace{1cm} (3.44)

**Proof.** Testing by $\partial_t T_\delta(u_\delta)$ both in (2.10) and (2.11), adding the resulting gives

$$\int_0^{\bar{T}} (\partial_t u_\delta, \partial_t T_\delta(u_\delta)) dt + \int_0^{\bar{T}} (k_{n\delta} \nabla p_\delta, \nabla \partial_t T_\delta(u_\delta)) dt + \int_0^{\bar{T}} (\nabla \theta_\delta(u_\delta), \nabla \partial_t T_\delta(u_\delta)) dt + \int_0^{\bar{T}} \| \sqrt{k_{n\delta}} \nabla \partial_t T_\delta(u_\delta) \|^2 dt = 0.$$  \hspace{1cm} (3.45)

Further, taking $\psi = p_\delta$ in (2.11) gives

$$\parallel \sqrt{k_\delta} \nabla p_\delta \parallel^2 = -(k_{n\delta} \nabla \partial_t T_\delta(u_\delta), \nabla p_\delta) \leq \parallel \frac{k_{n\delta}}{\sqrt{k_\delta}} \nabla \partial_t T_\delta(u_\delta) \parallel \parallel \sqrt{k_\delta} \nabla p_\delta \parallel,$$  \hspace{1cm} (3.46)

implying

$$-(k_{n\delta} \nabla p_\delta, \nabla \partial_t T_\delta(u_\delta)) \leq \parallel \frac{k_{n\delta}}{\sqrt{k_\delta}} \nabla \partial_t T_\delta(u_\delta) \parallel^2.$$  \hspace{1cm} (3.47)

Then (3.45) becomes

$$\int_0^{\bar{T}} (\partial_t u_\delta, \partial_t T_\delta(u_\delta)) dt + \int_0^{\bar{T}} \| \sqrt{k_{n\delta}} \nabla \partial_t T_\delta(u_\delta) \|^2 dt + \int_0^{\bar{T}} (\nabla \theta_\delta(u_\delta), \nabla \partial_t T_\delta(u_\delta)) dt \leq \int_0^{\bar{T}} \| \frac{k_{n\delta}}{\sqrt{k_\delta}} \nabla \partial_t T_\delta(u_\delta) \|^2 dt.$$  \hspace{1cm} (3.48)
Further, one has
\[
\left\| \sqrt{k_{\alpha\delta}} \nabla \partial_t T_\delta(u_\delta) \right\|^2 - \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2 = \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2,
\]
and
\[
\left( \nabla \vartheta(u_\delta), \nabla \partial_t T_\delta(u_\delta) \right) \leq \frac{1}{2} \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2 + \frac{1}{2} \left\| \frac{k_{\alpha\delta}}{k_\delta} (-p'_{\alpha\delta}) \nabla u_\delta \right\|^2.
\]
Then (3.48) leads to
\[
\int_0^{T_\delta} (\partial_t u_\delta, \partial_t T_\delta(u_\delta))dt + \frac{1}{2} \int_0^{T_\delta} \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2 dt \leq \frac{1}{2} \int_0^{T_\delta} \left\| \frac{k_{\alpha\delta}}{k_\delta} p'_{\alpha\delta} \nabla u_\delta \right\|^2 dt \leq \frac{1}{2} \int_0^{T_\delta} \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2 dt. \tag{3.51}
\]
By using \( \left\| p_{\alpha\delta} \nabla \partial_t u_\delta \right\|^2 \leq C, \left| k_{\alpha\delta} p'_{\alpha\delta} \right| \leq C \), and since by (A1), \( \frac{1}{\tau_{\delta}} \) is bounded, we have
\[
\int_0^{T_\delta} (\partial_t u_\delta, \partial_t T_\delta(u_\delta))dt + \frac{1}{2} \int_0^{T_\delta} \left\| \frac{k_{\alpha\delta}}{k_\delta} \nabla \partial_t T_\delta(u_\delta) \right\|^2 dt \leq C. \tag{3.52}
\]
Then, by (3.50), this particularly implies
\[
\int_0^{T_\delta} \left| (\nabla \vartheta(u_\delta), \nabla \partial_t T_\delta(u_\delta)) \right| \leq C. \tag{3.53}
\]
Clearly,
\[
\int_0^{T_\delta} \left\| \sqrt{k_{\alpha\delta}} \partial_t u_\delta \right\|^2 dt = \int_0^{T_\delta} \left\| \frac{1}{\sqrt{k_{\alpha\delta}}} \partial_t T_\delta(u_\delta) \right\|^2 dt = \int_0^{T_\delta} (\partial_t u_\delta, \partial_t T_\delta(u_\delta))dt \leq C. \tag{3.54}
\]
Testing again (2.10) with \( \psi = \partial_t T_\delta(u_\delta) \), we have
\[
\int_0^{T_\delta} (\partial_t u_\delta, \partial_t T_\delta(u_\delta))dt - \int_0^{T_\delta} (k_{\alpha\delta} \nabla p_{\delta}, \nabla \partial_t T_\delta(u_\delta))dt + \int_0^{T_\delta} (\nabla \vartheta(u_\delta), \nabla \partial_t T_\delta(u_\delta))dt = 0. \tag{3.55}
\]
Choosing now \( \psi = p_{\delta} + \partial_t T_\delta(u_\delta) \) in (2.11) gives
\[
\int_0^{T_\delta} (k_{\alpha\delta} \nabla p_{\delta}, \nabla u_\delta)dt + \int_0^{T_\delta} (k_{\alpha\delta} \nabla p_{\delta}, \nabla T_\delta(u_\delta))dt + \int_0^{T_\delta} \left\| \sqrt{k_{\alpha\delta}} \nabla (p_{\delta} + \partial_t T_\delta(u_\delta)) \right\|^2 dt = 0. \tag{3.56}
\]
Adding the equations (3.55) and (3.56), and using (3.53), we find
\[
\int_0^{T_\delta} \left\| \sqrt{k_{\alpha\delta}} \partial_t u_\delta \right\|^2 dt + \int_0^{T_\delta} \left\| \sqrt{k_{\alpha\delta}} \nabla p_{\delta} \right\|^2 dt + \int_0^{T_\delta} \left\| \sqrt{k_{\alpha\delta}} \nabla (p_{\delta} + \partial_t T_\delta(u_\delta)) \right\|^2 dt \leq \int_0^{T_\delta} \left| (\nabla \vartheta(u_\delta), \nabla \partial_t T_\delta(u_\delta)) \right| dt \leq C, \tag{3.57}
\]
which concludes the proof. \( \Box \)

Furthermore, by (A1), from (3.11), one gets
\[
\int_0^{T_\delta} \left\| -p'_{\alpha\delta}(u_\delta) \nabla u_\delta \right\|^2 dx dt + \int_0^{T_\delta} \left\| \sqrt{\tau_\delta(u_\delta)} \nabla u_\delta \right\|^2 dx dt \leq C. \tag{3.58}
\]
With the notation:
\[
[u_\delta]_{0-\delta} = \max(0, \min(1 - \delta, u_\delta)).
\]
Since \( \nabla \int_0^{|u|^\delta} \sqrt{-p_\alpha(z)} dz = \sqrt{-p_\alpha(u)} \nabla u \in L^2(Q) \), recalling that (A5) implies \( \lambda > 2 \) for all \( d \), and
\[
\int_0^{[u_\delta]^\delta} (z + \delta)^{-1/2} dz = \frac{1}{1 - \lambda/2} \big( ([u_\delta]^\delta + \delta)^{1/2 - \delta^{1/2}} - \delta^{1/2} \big),
\]
we have
\[
\nabla ([u_\delta]^\delta + \delta)^{1/2 - \delta^{1/2} \in L^2(Q),}
\]
and by (3.58) it is bounded uniformly w.r.t. \( \delta \). Further, since the trace of \([u_\delta]^\delta + \delta \) on \( \partial \Omega \) is \( C_D + \delta \), applying the Poincaré’s inequality for \( ([u_\delta]^\delta + \delta)^{1/2} - (C_D + \delta)^{1/2} \), one immediately obtains that
\[
\|(u_\delta)^{1/2} + \delta)^{1/2} \|_{L^2(0,T;W^{1,2}(\Omega))} \leq C,
\]
for some \( \delta \)-independent \( C \).

By Sobolev Embedding Theorem, one obtains
\[
([u_\delta]^\delta + \delta)^{1/2} \in L^2(0,T;C(\bar{\Omega})), \quad \text{if } d = 1,
\]
\[
([u_\delta]^\delta + \delta)^{1/2} \in L^2(0,T;L^r(\Omega)), \quad \text{for any } r \in (1, +\infty), \quad \text{if } d = 2,
\]
\[
([u_\delta]^\delta + \delta)^{1/2} \in L^2(0,T;L^{\frac{2r}{r-2}}(\Omega)), \quad \text{if } d > 2,
\]
and the respective norms are bounded uniformly w.r.t. \( \delta \).

Similarly, for
\[
[u_\delta]^\delta := \max(\delta, \min(1, u_\delta)],
\]
one has
\[
(1 - [u_\delta]^\delta + \delta)^{1-\omega/2} \in L^2(0,T;C(\bar{\Omega})), \quad \text{if } d = 1,
\]
\[
(1 - [u_\delta]^\delta + \delta)^{1-\omega/2} \in L^2(0,T;L^r(\Omega)), \quad \text{for any } r \in (1, +\infty), \quad \text{if } d = 2,
\]
\[
(1 - [u_\delta]^\delta + \delta)^{1-\omega/2} \in L^2(0,T;L^{\frac{2r}{r-2}}(\Omega)), \quad \text{if } d > 2,
\]
and the functions \( k_\omega([u_\delta]^\delta) \), \( k_\omega([u_\delta]^\delta) \), and \( \tau_\delta([u_\delta]^\delta) \) are in \( L^1(Q) \) and have uniformly bounded norms w.r.t. \( \delta \).

**Lemma 3.3.** For \( \gamma_u, \gamma_n, \gamma_t \), chosen appropriately, the functions \( k_\omega([u_\delta]^\delta) \), \( k_\omega([u_\delta]^\delta) \), and \( \tau_\delta([u_\delta]^\delta) \) are in \( L^1(Q) \) and have uniformly bounded norms w.r.t. \( \delta \).

**Proof.** We detail the proof for \( k_\omega \) and \( \tau_\delta \), the arguments for \( k_\omega \) being identical to those in [27]. To do so, we consider the cases \( d = 1, 2, \) and \( 3 \) separately (\( d > 3 \) being similar to \( d = 3 \)). We start with the case \( d = 3 \), and choose \( \gamma_n = \frac{1}{\beta} (\frac{2\gamma}{\lambda} + \frac{\gamma}{2} - \frac{3}{2}) \). By (A5), one gets \( \gamma_n > 1 \). Applying H"older inequality, for \( p = 3, q = \frac{3}{2} \), one gets for a.e. \( t \):
\[
\int_\Omega (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} dx dt \leq \left( \int_\Omega (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} dx \right)^{1/\gamma_n} \left( \int_\Omega (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} dx \right)^{2/\gamma_n},
\]
Due to (3.9) and (3.67), we have \( (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} \in L^\infty(0,T;L^1(\Omega)) \) and \( (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} \in L^2(0,T;L^\frac{2r}{r-2}(\Omega)) \), and the norms are bounded uniformly w.r.t. \( \delta \).

This implies:
\[
\int_0^t \int_\Omega (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} dx dt \leq \int_0^t \left( \int_\Omega (1 - [u_\delta]^\delta + \delta)^{-\gamma_n} dx \right)^{1/\gamma_n} \cdot \left( \max_{\delta \leq \delta \leq T} \int_\Omega (1 - [u_\delta]^\delta + \delta)^{2-\gamma} dx \right)^{2/\gamma_n} \leq C.
\]

20
With $γ_n = \frac{1}{θ} \left( \frac{2θ}{3} + \frac{5θ}{4} - \frac{10}{r} \right)$, the estimate for $\left( τ_δ([u_δ])_n^1 \right)^γ_n$ follows similarly.

For $d = 2$, we choose any $r > \max(2σ - 4)/λ, (e - 2)/(ω - 2), (2e - 4)/(e - 4)$ and define

\[
γ_n = \frac{1}{3} \left( 4 - \frac{4}{r} - ω(1 - 2) \right),
\]

respectively,

\[
γ_r = \frac{1}{ω} \left( 4 - \frac{4}{r} - ω(1 - 2) \right),
\]

and we apply Hölder inequality for $p = \frac{r}{2}$ and $q = \frac{r}{r-2}$ to obtain for a.e. $t$

\[
\int_Ω (1 - [u_δ]^1_0 + δ)^{-γ_r} \cdot dx \cdot dt \leq \left( \int_Ω (1 - [u_δ]^1_0 + δ)^{1-ω/2} dx \right)^{2r/2} \cdot \left( \int_Ω (1 - [u_δ]^1_0 + δ)^{2-ω} dx \right)^{1-2r/2},
\]

and

\[
\int_Ω (1 - [u_δ]^1_0 + δ)^{-γ_r} \cdot dx \cdot dt \leq \left( \int_Ω (1 - [u_δ]^1_0 + δ)^{1-ω/2} dx \right)^{2r/2} \cdot \left( \int_Ω (1 - [u_δ]^1_0 + δ)^{2-ω} dx \right)^{1-2r/2}.
\]

Then, the proof continues as before.

Finally, for $d = 1$, we take $γ_n = (ω + e - 4)/β$ and $γ_r = (ω + e - 4)/ω$. Similarly, by assumption (A5), we have $γ_n, γ_r > 1$. Then, using (3.9), (3.65) to estimate for a.e. $t$

\[
\int_Ω (1 - [u_δ]^1_0 + δ)^{-γ_r} dx \leq \left( \max(1 - [u_δ]^1_0 + δ)^{1-ω/2} \right)^{2r} \int_Ω (1 - [u_δ]^1_0 + δ)^{2-ω} dx \in L^1(Ω, T_M),
\]

and

\[
\int_Ω (1 - [u_δ]^1_0 + δ)^{-γ_r} dx \leq \left( \max(1 - [u_δ]^1_0 + δ)^{1-ω/2} \right)^{2r} \int_Ω (1 - [u_δ]^1_0 + δ)^{2-ω} dx \in L^1(Ω, T_M),
\]

and the proof follows again as before.

Now we obtain further estimates for $(u_δ, p_δ)$.

**Lemma 3.4.** Let $d = 1, 2, 3$ and assume (A1), (A2) and (A5). There exist $r_1, r_2, r_3 \in (1, 2)$ and $C > 0$ independent on $δ$, such that the weak solution pair $(u_δ, p_δ)$ satisfies for all $δ > 0$

\[
||\partial_δ T_δ(u_δ)||_{L^{r_1}(Q)} + ||\nabla p_δ||_{L^{r_2}(Q)} + ||\nabla (p_δ + \partial_δ T_δ(u_δ))||_{L^{r_3}(Q)} \leq C.
\]

**Proof.** The proof uses the estimates in Lemma 3.3 and distinguishes as before three cases, $d = 1, 2$, and 3. We start with the latter.

By (3.43), one has $\frac{1}{\sqrt{t_δ}} \partial_δ T_δ(u_δ) \in L^2(Ω)$. So here we show for any $r_1 \in (1, 2)$, one has $\partial_δ T_δ(u_δ) \in L^{r_1}(Q)$. Moreover, for appropriately chosen $r_1$, the corresponding norm is bounded uniformly w.r.t. $δ$. To see this, we apply Hölder inequality to get

\[
\int_0^T \int_Ω |\partial_δ T_δ(u_δ)|^{r_1} dx \cdot dt = \int_0^T \int_Ω \frac{1}{\sqrt{t_δ}} |\partial_δ T_δ(u_δ)|^{r_1/2} |\partial_δ T_δ(u_δ)|^{r_1/2} dx \cdot dt \cdot \int_0^T \int_Ω |\partial_δ T_δ(u_δ)|^{r_1/2} dx \cdot dt \leq \left( \int_0^T \int_Ω |\partial_δ T_δ(u_δ)|^{r_1/2} dx \cdot dt \right)^{r_1/2} \cdot \left( \int_0^T \int_Ω |\partial_δ T_δ(u_δ)|^{r_1/2} dx \cdot dt \right)^{1-r_1/2}.
\]
The first integral on the right hand side is bounded by (3.43), the second we recall Lemma 3.3 and choose \( r_1 \) such that
\[
\frac{1}{r_1} = \gamma, \quad \gamma = \frac{1}{2} \left( \frac{2}{3} + \frac{3}{2m} - \frac{3}{4} \right),
\]
which, by (A5), \( r_1 = \frac{2\alpha + 5r_2 - \omega}{5r_2 - \omega} \), satisfies \( r_1 \in (1, 2) \).
Similarly, one also has the estimate
\[
\int_0^{T_\omega} \int_\Omega |\nabla p_\delta|^2 dxdt = \int_0^{T_\omega} \int_\Omega |\nabla p_\delta|^{3/2} dxdt \leq \left( \int_0^{T_\omega} \int_\Omega |k_{\delta\omega} |\nabla p_\delta|^2 dxdt \right)^{r_2/2} \cdot \left( \int_0^{T_\omega} \int_\Omega |k_{\delta\omega}|^{-r_2/(r_2 - 3)} dxdt \right)^{1-r_2/2}
\]
We obtain
\[
r_2 = \frac{2(3\omega + 2e - 10)}{(5e - 10)},
\]
and following (A5), one has \( r_2 \in (1, 2) \).
Similarly, one has
\[
\int_0^{T_\omega} \int_\Omega |\nabla(p_\delta + \partial_T\delta(u_\delta))|^\alpha dxdt = \int_0^{T_\omega} \int_\Omega |\sqrt{k_{\delta\omega}} |\nabla(p_\delta + \partial_T\delta(u_\delta))|^{\alpha/2} (\sqrt{k_{\delta\omega}})^{-r_2/(r_2 - 3)} dxdt \leq \left( \int_0^{T_\omega} \int_\Omega |k_{\delta\omega} |\nabla(p_\delta + \partial_T\delta(u_\delta))^2 dxdt \right)^{r_2/2} \cdot \left( \int_0^{T_\omega} \int_\Omega |k_{\delta\omega}|^{-r_2/(r_2 - 3)} dxdt \right)^{1-r_2/2}
\]
then for this case, we have
\[
r_3 = \frac{2(3\lambda + 2\alpha - 10)}{3\lambda + 5\alpha - 10} \in (1, 2),
\]
when \( \alpha > 5 \) and \( \lambda > 10/3 + \alpha/3 \), this implies \( \nabla(p_\delta + \partial_T\delta(u_\delta)) \in L^\alpha(Q) \).

Case 2: \( d = 2 \)
Similarly, for two dimension case, we show for any \( r_1 \in (1, 2) \), one has \( \partial_T\delta(u_\delta) \in L^\alpha(Q) \) and the corresponding norm is bounded uniformly w.r.t. \( \delta \). Since one has
\[
\int_0^{T_\omega} \int_\Omega |\partial_T\delta(u_\delta)|^\alpha dxdt \leq \left( \int_0^{T_\omega} \int_\Omega \left| \frac{1}{T_\delta} |\partial_T\delta(u_\delta)|^2 dxdt \right|^{r_2/2} \cdot \left( \int_0^{T_\omega} \int_\Omega |\partial_T\delta(u_\delta)|^{-r_2/(r_2 - 3)} dxdt \right)^{1-r_2/2}
\]
Then, for any \( r > 2(e - 2)/(2\omega + e - 4) \), we solve
\[
r_1 = \frac{2(\omega + e - 2(e - 2)/r - 4)}{2\omega + e - 2(e - 2)/r - 4} \in (1, 2),
\]
which implies \( \partial_T\delta(u_\delta) \in L^\alpha(Q) \).
Using the same way, we also get \( \nabla p_\delta \in L^\alpha(Q) \) for
\[
r_2 = \frac{2(\omega + e - 2(e - 2)/r - 4)}{\omega + e + \beta - 2(e - 2)/r - 4},
\]
and \( \nabla(p_\delta + \partial_T\delta(u_\delta)) \in L^\alpha(Q) \) for
\[
r_3 = \frac{2(4 - \lambda - \alpha + 2(\alpha - 2)/r)}{4 - \lambda - 2\alpha + 2(\alpha - 2)/r}.
\]

Case 3: \( d = 1 \)
The proof follows as before, we have \( \partial_T\delta(u_\delta) \in L^\alpha(Q) \) for
\[
r_1 = \frac{2(\omega + e - 4)}{2\omega + e - 4} \in (1, 2),
\]
\( \nabla p_\delta \in L^\alpha(Q) \), for
\[
r_2 = \frac{2(\omega + e - 4)}{\omega + e + \beta - 4} \in (1, 2),
\]
and furthermore, $\nabla (p_\delta + \partial_t T_\delta (u_\delta)) \in L^\gamma (Q)$ for
\[
r_3 = \frac{2(4 - \lambda - \alpha)}{(4 - \lambda - 2\alpha)} \in (1, 2).
\]

Then we have the estimates
\[
\| \partial_t T_\delta (u_\delta) \|_{L^2 (Q)} \leq C, \quad \text{with } r_1 \in (1, 2),
\]
\[
\| \nabla p_\delta \|_{L^2 (Q)} \leq C, \quad \text{with } r_2 \in (1, 2),
\]
\[
\| \nabla (p_\delta + \partial_t T_\delta (u_\delta)) \|_{L^3 (Q)} \leq C, \quad \text{with } r_3 \in (1, 2).
\]

\[\square\]

With $r^* = \min \{ r_1, r_2, r_3 \}$, by Lemma 3.1, 3.2 and 3.4, one obtains the existence of a subsequence $\delta \searrow 0$ (still denoted by $\delta$) and of $u \in W^{1,2} (Q), \ T^* \in W^{1,\lambda} (0, T_M; W^{1,\lambda} (\Omega))$ and $p \in L^2 (0, T_M; W^{1,\lambda} (\Omega))$, such that
\[
u_\delta \longrightarrow u \text{ strongly in } L^2 (Q),
\]
\[
\partial_t \nu_\delta \rightarrow \partial_t u \text{ weakly in } L^2 (Q),
\]
\[
\nabla \nu_\delta \rightarrow \nabla u \text{ weakly in } L^2 (Q),
\]
\[
T_\delta \rightarrow T^* \text{ weakly in } W^{1,\lambda} (Q),
\]
\[
\tilde{T}_\delta \rightarrow T^* \text{ strongly in } L^\lambda (Q),
\]
\[
\nabla \partial_t T_\delta \rightarrow \nabla \partial_t T^* \text{ weakly in } L^\gamma (Q),
\]
\[
p_\delta \rightarrow p \text{ weakly in } W^{1,\lambda} (Q).
\]

where $q = +\infty$, if $d = 1$, $q = \frac{d\gamma}{d - \gamma}$, if $d = 2$ or $d = 3$ (see [17]).

In the remaining, we prove that $T^* = T (u)$ a.e., and that $(u, p)$ is a solution pair to Problem P. But before doing so, we also prove that the limit $u$ above is essentially bounded by 0 and 1.

**Theorem 3.1.** The limit $u \in W^{1,2} (Q)$ satisfies $0 \leq u \leq 1$ a.e. in $Q$.

**Proof.** Given $t \in (0, T_M]$, let $\Omega_{-\delta} (t)$ be the support of $[u_\delta (t, \cdot) + \epsilon]_-$ (the negative cut of $u_\delta (t, \cdot) + \epsilon$). As follows from Lemma 3.1, a $C > 0$ exists such that, for all $\delta > 0$, one has
\[
\int_{\Omega} \tilde{E}_\delta (u_\delta) dx = \int_{\Omega} \int_{C_D} \int_{k_\delta} \frac{1}{k_\delta} (z) dz dv dx + \int_{\Omega} \frac{(C_D + \delta)^{2-a}}{(1 - \alpha)(2 - \alpha)} - \frac{(C_D + \delta)^{1-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} C_D dx \leq C,
\]
\[
\int_{\Omega} \tilde{E}_\delta (u_\delta) dx = \int_{\Omega} \int_{C_D} \int_{k_\delta} \tau_\delta (z) dz dv dx + \int_{\Omega} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} dx \leq C.
\]

Since the constant arguments in the last two integrals are positive, this gives
\[
\int_{\Omega} \int_{C_D} \int_{k_\delta} \frac{1}{k_\delta} (z) dz dv dx + \int_{\Omega} \int_{C_D} \int_{k_\delta} \tau_\delta (z) dz dv dx \leq C.
\]

Then we get
\[
C \geq \int_{\Omega} \int_{0}^{u_\delta} \frac{1}{k_\delta} (z) dz dv dx
\]
\[
= \int_{\Omega} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} \frac{\delta^{2-a}}{(2 - \alpha)} dx.
\]

(3.94)
Let now $\delta \searrow 0$, this immediately implies

$$\text{meas}(\Omega_{0,\delta}(t)) = 0,$$

(3.95)

with $\Omega_{0,\delta}(t)$ having the same definition as $\Omega_{\delta}(t)$, but now for the function $u$. Since $u_{\delta} \to u$ in $C((0,T_{M});L^{2}(\Omega))$ (by (3.85), (3.86) and the compact embedding see [1] and Theorem 4.4). Thus $u_{\delta} \to u$ a.e. in $\Omega$, for all $t$. This holds for every $\epsilon > 0$, hence $u \geq 0$. Similarly, if $u_{\delta} > 1 + \epsilon$, use the bounds on $\int_{\Omega} \int_{1}^{\infty} \frac{\tau_{x}}{k_{n}} dx dv dx$, we obtain $u \leq 1$.

\[ \square \]

**Remark 4.** For two phase flow model, $0 \leq u \leq 1$ means that the saturation remains in the physical range. Note that this only holds due to the degeneracy encountered for $u = 0$ or $u = 1$.

Finally, we obtain the existence of a solution for Problem P.

**Theorem 3.2.** Under the assumptions (A1), (A2) and (A5), there exists a solution pair $(u, p)$ for Problem P.

**Proof.** We start by identifying $T^{\ast}$ as $T(u)$. To do so, define $[v]^{1\ast} = \min [1,v]$ and let $T(f) = T^{\ast}$ a.e.. We proceed as in [28] and consider first the inverse function of $T^{-1}$. According to the definition of $T$ and $T_{\delta}$, one has

$$\begin{align*}
T^{-1}(T^{\ast}) &= \begin{cases} 
T^{\ast} - \frac{1}{\omega^{-1}}, & \text{if } T^{\ast} < \frac{1}{\omega^{-1}}, \\
1 - \left(\frac{1}{\omega^{-1}}\right)^{1/\gamma}, & \text{if } T^{\ast} \geq \frac{1}{\omega^{-1}}.
\end{cases}
\end{align*}$$

(3.96)

Clearly,

$$[u_{\delta}]^{1\ast} = T_{\delta}^{-1}(T_{\delta}) = \begin{cases} 
\tilde{T}_{\delta} - \frac{1}{\omega^{-1}} + \delta, & \text{if } \tilde{T}_{\delta} < \frac{1}{\omega^{-1}}, \\
1 - \left(\frac{1}{\omega^{-1}}\right)^{1/\gamma} + \delta, & \text{if } \tilde{T}_{\delta} \geq \frac{1}{\omega^{-1}}.
\end{cases}$$

(3.97)

Now we prove that $[u_{\delta}]^{1\ast} \to f$ strongly in $L^{2}(Q)$, and hence a.e. in $Q$. Since $T^{-1}(\cdot)$ is Lipschitz continuous, by (3.97) one has

$$\int_{Q} |f - [u_{\delta}]^{1\ast}| dx dt = \int_{Q} |T^{-1}(T^{\ast}) - T^{-1}(T_{\delta}) + T^{-1}(T_{\delta}) - T_{\delta}^{-1}(T_{\delta})| dx dt$$

$$\leq C \int_{Q} |T^{\ast} - \tilde{T}_{\delta}| dx dt + \int_{Q} |T^{-1}(T_{\delta}) - T_{\delta}^{-1}(T_{\delta})| dx dt.$$  

(3.98)

Clearly, the first integral above approaches 0 as $\delta \searrow 0$. For the second we note that, by (3.96) and (3.97), one has $T_{\delta}^{-1}(T_{\delta}) - T^{-1}(T_{\delta}) = \delta$, for any argument $\tilde{T}_{\delta}$. With this, the second integral also approaches 0 as $\delta \searrow 0$. This means that

$$[u_{\delta}]^{1\ast} \to f \quad \text{a.e. in } Q.$$  

(3.99)

However $u_{\delta} \to u$ a.e. in $Q$ (by the strong convergence in $L^{2}(Q)$). This immediately gives $[u_{\delta}]^{1\ast} \to [u]^{1\ast}$ a.e.. In the view of Theorem 3.1, we also have $[u]^{1\ast} = u$ a.e.. Therefore, $u = f$ a.e., and consequently, we have

$$T(u) = T^{\ast}.$$  

(3.100)

Having identified $T^{\ast}$ by $T(u)$, (3.89) gives $\nabla \tilde{T}_{\delta}(u_{\delta}) \to \nabla \tilde{T}(u)$.

Then, according to $k_{n}(\cdot)$ Lipschitz continuous, one has

$$k_{n}(u_{\delta}) \to k_{n}(u) \quad \text{a.e. in } Q.$$  

Further, since $k_{n}(u)$ converges pointwise to $k_{n}(u)$, and

$$|k_{n}(u)| \leq C,$$
uniformly with w.r.t. $\delta$. Then by Dominated convergence theorem, we obtain

$$k_{\alpha}(u) \to k_\alpha(u) \quad a.e. \text{ in } Q.$$ 

Therefore, we have

$$k_{\alpha}(u_0) \to k_\alpha(u) \quad \text{strongly in } L^2(Q).$$

Similarly, we also have $\sqrt{k_{\alpha}(u_0)}$ converges to $\sqrt{k_\alpha(u)}$ strongly in $L^2(Q)$. In the same fashion, since $k_\alpha$, $k$ and $\theta$ are Lipschitz continuous, on also obtains

$$k_{\alpha}(u_0) \to k_\alpha(u),$$

$$k_{\alpha}(u) \to k(u),$$

$$\theta_{\alpha}(u_0) \to \theta(u),$$

strongly in $L^2(Q)$.

By the (3.44), a $g \in L^2(Q)$ exists, such that

$$\sqrt{k_{\alpha}(u_0)}\nabla(p_\delta + \partial T_\alpha(u_0)) \to g \quad \text{weakly in } L^2(Q), \quad \text{for } \delta \searrow 0. \quad (3.101)$$

To identify $g$, we consider $\phi \in C_0^\infty(\Omega)$ arbitrarily, and note that $\phi \sqrt{k_{\alpha}(u_0)} \to \phi \sqrt{k_\alpha(u)}$ strongly in $L^q(Q)$, for any $q \in [1, \infty)$. This follows as above, using the uniform boundness of $k_{\alpha}$, $k_\alpha$, and the pointwise convergence of $k_{\alpha}(u_0)$ to $k_\alpha(u)$. Further, by (3.89) and (3.90), one gets $\nabla(p_\delta + \partial T_\alpha(u_0)) \to \nabla(p + \partial T(u))$ weakly in $L^r(Q)$. Taking $q$ such that $1/q + 1/r^* = 1$ gives by weak-strong convergence argument

$$\int_0^{T_u} (k_{\alpha}\nabla(p_\delta + \partial T_\alpha(u_0)), \nabla\phi)dt \to \int_0^{T_u} (k_\alpha\nabla(p + \partial T(u)), \nabla\phi)dt. \quad (3.102)$$

This is sufficient to identify $g = \sqrt{k_\alpha(u)}\nabla(p + \partial T(u))$. We have now all the ingredients to pass to the limit ($\delta \searrow 0$) in the integrals appearing in Problem $P_\delta$. Using the convergence results above, it is straightforward to show that $(u, p)$ solves Problem $P$. 

4. Conclusions

In this paper, we prove the existence of a weak solution to degenerate elliptic parabolic system modeling two-phase flow in porous media, and including dynamic effects in the capillary pressure. The major difficulty is the degeneracy of the non-linear third order derivative term. We get the estimate for the third order derivative term by applying the structures of relative permeabilities, capillary pressure and the dynamic damping factor. By compactness arguments, we show the existence of a solution for the original problem.

Acknowledgments

The work of X. Cao is supported by CSC (China Scholarship Council). This support is gratefully acknowledged. The authors are members of the International Research Training Group NUPUS funded by the German Research Foundation DFG (GRK 1398) and The Netherlands Organization for Scientific Research NWO (DN 81-754).

References


### PREVIOUS PUBLICATIONS IN THIS SERIES:

<table>
<thead>
<tr>
<th>Number</th>
<th>Author(s)</th>
<th>Title</th>
<th>Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>14-33</td>
<td>E.N.M. Cirillo</td>
<td>Residence time estimates for asymmetric simple exclusion dynamics on stripes</td>
<td>Nov. '14</td>
</tr>
<tr>
<td></td>
<td>A. Muntean</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>R.A. van Santen</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A. Sengar</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14-34</td>
<td>B.S. van Lith</td>
<td>Existence and uniqueness of solutions to Liouville's equation and the associated flow for Hamiltonians of bounded variation</td>
<td>Dec. '14</td>
</tr>
<tr>
<td></td>
<td>J.H.M. ten Thije</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Boonkamp</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W.L. IJzerman</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>T.W. Tukker</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15-01</td>
<td>M.E. Hochstenbach</td>
<td>A Golub-Kahan-type reduction method for matrix pairs</td>
<td>Jan. '15</td>
</tr>
<tr>
<td></td>
<td>L. Reichel</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>X. Yu</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15-02</td>
<td>J.H.M. Evers</td>
<td>From continuum mechanics to SPH particle systems and back: Systematic derivation and convergence</td>
<td>Febr. '15</td>
</tr>
<tr>
<td></td>
<td>I.A. Zisis</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B.J. van der Linden</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>M.H. Duong</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15-03</td>
<td>X. Cao</td>
<td>Degenerate two-phase flow in porous media including dynamic effects in the capillary pressure: existence of a weak solution</td>
<td>Febr. '15</td>
</tr>
<tr>
<td></td>
<td>I.S. Pop</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>