Inductance Calculation Nearby Conducting Material

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This paper presents the modeling of inductances nearby conducting material. A 3-D magnetic model is derived based on Fourier analysis in which the diffusion equation is incorporated. The model is applied to calculate the mutual inductance in a contactless energy transfer system between four primary coils and a single secondary coil for different positions of the secondary coil. The calculated values are compared with finite element simulations and measurements. A difference of maximum 7% is obtained among the different methods.

Index Terms—Inductance, magnetic fields, magnetic vector potential, modeling.

I. INTRODUCTION

The inductance of a coil is strongly depending on the environment surrounding the coil. If, for example, conducting materials are located in the neighborhood, eddy currents are generated in those materials by the time-varying field. These eddy currents generate a reaction field which counteracts the magnetic fields of a coil and influences the inductance of a coil. Examples in which the influence of the eddy-current reaction field may not be ignored are the modeling of the end-winding in linear induction machines and inductances in a contactless energy transfer (CET) system by means of an inductive coupling in machines [1], [2]. All examples which have a 3-D geometrical structure where a 2-D modeling method is not sufficient and 3-D one needs to be applied.

Due to complexity of the 3-D structures and the appearance of different materials, finite element analysis (FEA) is an often applied method to model the magnetic fields, including the eddy-current reaction field [3], [4]. For a planar structure, FEA is a time-consuming method because of the required dens mesh, where in the conducting material several mesh-layers per skin-depth need to be applied to model the eddy currents. The 3-D Fourier analysis is an analytical alternative to FEA. Comparable with FEA, it is possible to model coils, magnets, slots, and cavities with 3-D Fourier analysis [5]. Fourier analysis divides a geometry into regions, a volume parallel to the direction of the current, which deviate from each other with respect to different material properties and current sources [6]. For each region, an expression of the magnetic vector potential is obtained, which is solved by applying the boundaries between the regions. By incorporating the diffusion equation in the solution in terms of the magnetic vector potential, the range of applications for the 3-D model can be extended; both the properties of conducting materials and the eddy-current reaction field can be taken into account in problems with either coils or permanent magnets as the source of the magnetic fields. Up to now, this technique is only applied in combination with 2-D Fourier analysis. Since this modeling technique assumes a finite length in one direction, it is not suited to model end-windings and planar structures [7], [8].

This paper starts with a derivation of the 3-D magnetic vector potential, by including the diffusion equation in the Fourier analysis. To validate the model, the mutual inductance between the primary and secondary coils of a CET system integrated in a planar motor, as shown in Fig. 1, is calculated. The obtained values are compared with FEA and measurements on the similar system.

II. 3-D MAGNETIC MODEL

The magnetic flux density \( \mathbf{B} \) can be described by using the magnetic vector potential \( \mathbf{A} \)

\[
\mathbf{B} = \nabla \times \mathbf{A}. \tag{1}
\]

Combining Ampere’s law and Faraday’s law, the following differential equation for the 3-D magnetic vector potential is obtained:

\[
\nabla^2 \mathbf{A} = -\mu \mathbf{J}_{\text{coil}} + \mu \sigma \frac{\partial \mathbf{A}}{\partial t} \tag{2}
\]

where \( \mathbf{J}_{\text{coil}} \) is the current density distribution of a coil.

To obtain a solution for the magnetic vector potential, it is assumed that the imposed current inside a coil and the induced currents in the conducting material only flow in tangential, \( x, y \), direction. Based on these assumptions and the Coulomb Gauge condition, \( \nabla \cdot \mathbf{A} = 0 \), the following two relations for...
the magnetic vector potential are derived:

\[ A_x = 0 \]  \hspace{1cm} (3)  \\
\[ A_y = - \int A_x \, dy \]  \hspace{1cm} (4)

which yields to the following set of differential equations:

\[ \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} - \mu \sigma \frac{\partial A_x}{\partial t} = -\mu J_{\text{coil}} \]  \hspace{1cm} (5)  \\
\[ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} - \mu \sigma \frac{\partial A_y}{\partial t} = -\mu J_{\text{coil}} \]  \hspace{1cm} (6)

The differential equations are solved by applying the method of separation of variables, which results in a multiplication of four functions, each depending on \( x \), \( y \), \( z \), or \( t \). Because of the periodicity in \( x \) and \( y \), double Fourier series are used to describe the harmonics in the both directions. Complex exponential functions are obtained for \( z \) and \( t \), such that the Poisson and Laplace equation are satisfied for a time-varying magnetic source, and the Coulomb Gauge condition remains fulfilled. The resulting magnetic vector potential is equal to

\[ A_x = \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ \frac{\omega_m}{\gamma} a_1(z) + C_{x_n} \right] \cos(\omega_n x) \sin(\omega_m y) \]

\[ + \left( \frac{\omega_m}{\gamma} a_2(z) + C_{x_2} \right) \sin(\omega_n x) \cos(\omega_m y) \]

\[ + \left( \frac{\omega_n}{\gamma} a_3(z) + C_{x_3} \right) \sin(\omega_n x) \cos(\omega_m y) \]

\[ + \left( \frac{\omega_n}{\gamma} a_4(z) + C_{x_4} \right) \cos(\omega_n x) \cos(\omega_m y) \] \[ e^{j\omega t} \]  \hspace{1cm} (7)

\[ A_y = \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ \frac{-\omega_m}{\gamma} a_1(z) + C_{y_n} \right] \sin(\omega_n x) \cos(\omega_m y) \]

\[ + \left( \frac{\omega_m}{\gamma} a_2(z) + C_{y_2} \right) \cos(\omega_n x) \cos(\omega_m y) \]

\[ + \left( \frac{-\omega_n}{\gamma} a_3(z) + C_{y_3} \right) \cos(\omega_n x) \sin(\omega_m y) \]

\[ + \left( \frac{\omega_n}{\gamma} a_4(z) + C_{y_4} \right) \sin(\omega_n x) \sin(\omega_m y) \] \[ e^{j\omega t} \]

where

\[ a_1(z) = c_1 e^{y z} + c_2 e^{-y z} \]  \hspace{1cm} (9)  \\
\[ a_2(z) = c_3 e^{y z} + c_4 e^{-y z} \]  \hspace{1cm} (10)  \\
\[ a_3(z) = c_5 e^{y z} + c_6 e^{-y z} \]  \hspace{1cm} (11)  \\
\[ a_4(z) = c_7 e^{y z} + c_8 e^{-y z} \]  \hspace{1cm} (12)  \\
\[ \gamma = \sqrt{\omega_n^2 + \omega_m^2 + j\omega \mu \sigma} \]  \hspace{1cm} (13)  \\
\[ \omega_n = \frac{n \pi}{\tau_x} \]  \hspace{1cm} (14)  \\
\[ \omega_m = \frac{m \pi}{\tau_y} \]  \hspace{1cm} (15)

where \( C_{x_n} - C_{y_n} \) are constants related to the current density distribution, \( \omega_n \) and \( \omega_m \) are the spatial harmonics in the \( x \)- and \( y \)-direction, respectively, and \( c_1 - c_8 \) are unknown coefficients which are solved by applying the boundary conditions.

Similar to the magnetic vector potential, the current density distribution of a coil is modeled by means of a double Fourier series, such that an identical periodic expression is obtained. A single coil, constructed from Litz-wire with an imposed current, is, therefore, split into straight and corner segments. Since the method of separation of variables does not allow to model round sections independently in \( x \)-direction and \( y \)-direction, and, therefore, the corners of the coil cannot be taken into account. Resulting that a coil is modeled by four straight bars, as shown in Fig. 2, where each bar is defined by its inner, \( c_{iw} \), outer width, \( c_{ou} \), and length

\[ c_{ouw} = 0.5(c_{iw} + c_{ou}) \]  \hspace{1cm} (16)

By overlapping the four bars in direction parallel to the current, the error related to the ignorance of the round corners of a coil is minimized [5]. The current density distribution is described by

\[ J_x = \sum_{n=0}^{N} \sum_{m=1}^{M} (j_{x_n} \cos(\omega_n x) \cos(\omega_m y) \]

\[ + j_{y_n} \cos(\omega_n x) \sin(\omega_m y) + j_{x_n} \sin(\omega_n x) \cos(\omega_m y) \]

\[ + j_{y_n} \sin(\omega_n x) \sin(\omega_m y) \] \[ e^{j\omega t} \]  \hspace{1cm} (17)

\[ J_y = \sum_{n=1}^{N} \sum_{m=0}^{M} (j_{x_n} \cos(\omega_n x) \cos(\omega_m y) \]

\[ + j_{y_n} \cos(\omega_n x) \sin(\omega_m y) + j_{x_n} \sin(\omega_n x) \cos(\omega_m y) \]

\[ + j_{y_n} \sin(\omega_n x) \sin(\omega_m y) \] \[ e^{j\omega t} \]  \hspace{1cm} (18)

where the coefficients \( j_{x_n} - j_{y_n} \) can be found in [5]. Based on this current density distribution, the constants \( C_{x\beta} \) and \( C_{y\beta} \) are found as

\[ C_{x\beta} = \frac{-\mu J_{\text{coil}}}{\omega_n^2 + \omega_m^2} \]  \hspace{1cm} (19)  \\
\[ C_{y\beta} = \frac{-\mu J_{\text{coil}}}{\omega_n^2 + \omega_m^2} \]  \hspace{1cm} (20)

where the subscript \( \beta \) represents the different combinations of cosine and sine functions of the double Fourier series, i.e., \( cc, ss, cs, \) and \( xc \).

Finally, based on the derived magnetic vector potential, the flux density, eddy-currents, forces, and self- and mutual
inductances can be calculated. The mutual inductance between the primary, \( p \), and secondary, \( s \), coil is calculated based on the solution of the magnetic vector potential and the current density distribution

\[
M_{ps} = \frac{N_p N_s}{I_p I_s} \int_{D} \mathbf{A}_p \cdot \mathbf{J}_s \, dD
\]

(21)

**III. Model Validation**

To validate the derived expression for the 3-D magnetic vector potential, Fourier analysis is applied to calculate the mutual inductances of a CET system located above the aluminum cooling plate of a planar motor and compared with FEA and measurements. A cross section of the CET system is shown in Fig. 3 [2]. The system consists of an array of primary coils (Region III) and single secondary coil (Region V), which overlaps four primary coils. Underneath the primary coils, the aluminum cooling plate of the motor (Region I) is located, in which eddy-currents are generated. Between the aluminum and the coil a layer of epoxy (Region II) is present, which is modeled as air. Similar, the layer of epoxy above the primary coils and the airgap between the coils are combined in Region IV and modeled as air. Finally, the area above the secondary coil (Region VI) is modeled as air and has an infinite height. The model parameters are listed in Table I.

**TABLE I  
MODEL PARAMETERS**

<table>
<thead>
<tr>
<th>Par.</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2r_p )</td>
<td>54.0</td>
<td>mm</td>
<td>Coil pitch in the x-direction</td>
</tr>
<tr>
<td>( 2r_s )</td>
<td>54.0</td>
<td>mm</td>
<td>Coil pitch in the y-direction</td>
</tr>
<tr>
<td>( h_I )</td>
<td>40.0</td>
<td>mm</td>
<td>Height of Region I</td>
</tr>
<tr>
<td>( h_{II} )</td>
<td>15.0</td>
<td>mm</td>
<td>Height of Region II</td>
</tr>
<tr>
<td>( h_{III} )</td>
<td>11.4</td>
<td>mm</td>
<td>Height of the primary coils</td>
</tr>
<tr>
<td>( h_{IV} )</td>
<td>1.50</td>
<td>mm</td>
<td>Height of Region IV</td>
</tr>
<tr>
<td>( h_{V} )</td>
<td>1.30</td>
<td>mm</td>
<td>Height of the secondary coil</td>
</tr>
<tr>
<td>( c_{w_p} )</td>
<td>25.5</td>
<td>mm</td>
<td>Outer length of a primary coil</td>
</tr>
<tr>
<td>( c_{w_s} )</td>
<td>4.40</td>
<td>mm</td>
<td>Inner length of a primary coil</td>
</tr>
<tr>
<td>( c_{w_{II}} )</td>
<td>57.5</td>
<td>mm</td>
<td>Outer length of the secondary coil</td>
</tr>
<tr>
<td>( c_{w_{II}} )</td>
<td>30.2</td>
<td>mm</td>
<td>Inner length of the secondary coil</td>
</tr>
<tr>
<td>( N_p )</td>
<td>240</td>
<td>turns</td>
<td>Number of turns of a primary coil</td>
</tr>
<tr>
<td>( N_s )</td>
<td>19</td>
<td>turns</td>
<td>Number of turns of the secondary coil</td>
</tr>
<tr>
<td>( \mu_{al} )</td>
<td>1</td>
<td></td>
<td>Relative permeability of aluminium</td>
</tr>
<tr>
<td>( \sigma_{al} )</td>
<td>26-10^6</td>
<td>Sm^-1</td>
<td>Conductivity of aluminium</td>
</tr>
<tr>
<td>( N = M )</td>
<td>15 harmonics</td>
<td></td>
<td>Number of harmonics</td>
</tr>
</tbody>
</table>

**IV. Mutual Inductance**

For each region, an expression for the magnetic vector potential is obtained, identical to (7) and (8). The different expressions are coupled by means of boundary conditions, to be able to solve the unknown coefficients in each region. At the boundary between neighboring regions, the normal component of the magnetic flux density and the tangential components of the magnetic field strength must be equal to each other. A Neumann boundary condition is applied at the bottom of Region I. Because of the limited height of the region, this condition is only valid in case of frequencies above 1 kHz, where the skin-depth of the magnetic fields is four times smaller than the height of the region. The height of Region VI is modeled as infinite, and, therefore, the Neumann boundary condition may be applied at the top of the region.

An over-constrained problem is obtained since the number of boundary condition is larger as the number of unknown coefficients, which need to be solved. To reduce the number of boundary conditions, the conditions related to the two tangential terms of the magnetic field strength are rewritten into a scalar potential

\[
\varphi = -\int H_x \, dx = -\int H_y \, dy.
\]

(22)

Identical to the magnetic field strength, the scalar potential must be equal at both sides of the boundary between two regions. This reduction may be applied, since locally at the boundary between two regions no surface charges are present, and, therefore, the components of the magnetic field strength obtained from the scalar potential must be equal to ones obtained from the magnetic vector potential.

Summarized, the following set of boundary conditions is applied to obtain an expression for the magnetic fields:

\[
\begin{align*}
A_z &= 0 & \text{for } z &= 0 \\
\varphi_I &= \varphi_{III} & \text{for } z &= 0 \\
B_{zI} &= B_{zIII} & \text{for } z &= h_I \\
\vdots \\
\varphi_V &= \varphi_{VII} & \text{for } z &= h_V \\
B_{zV} &= B_{zVII} & \text{for } z &= h_V \\
\varphi_{VI} &= 0 & \text{for } z &= h_{VI}.
\end{align*}
\]

(23)
at different positions with respect to the array of primary coils. The values are obtained along four lines parallel to the y-axis, for \( x_a = 2.80 \, \text{mm}, x_b = 17.9 \, \text{mm}, x_c = 35.9 \, \text{mm}, \) and \( x_d = 52.8 \, \text{mm}, \) as shown in Fig. 4. The simulations are conducted by steady-state FEA using Flux 3-D [9]. The measured values are obtained by the open-circuit voltage across the secondary coil, while the primary coils are connected in series to a high-frequency sinusoidal current source, and using the following relation:

\[
V_s = \omega M_{\text{meas}} I_p. \tag{24}
\]

The calculated, simulated, and measured values are shown in Fig. 5(a)–(d) for the different lines. The analytical curve in each graph has been calculated within 42 s. The calculated inductances deviate at maximum 7% with the measured values and the results obtained from FEA, ignoring the measurements for \( y \geq 0.03 \, \text{mm} \). The large deviation in the measurements for \( y \geq 0.03 \, \text{mm} \) compared the calculated and simulated values is caused by the relative low measured amplitude of the open circuit voltage with respect to the other measured points.

Fig. 6 shows the mutual inductance as a function of the frequency, where the center of the secondary coil is aligned with the center of the set of primary coils. As the induced eddy-current density, \( J_{\text{eddy}} \)

\[
J_{\text{eddy}} = \sigma E = -\sigma \frac{\partial A}{\partial t} \tag{25}
\]

increases with respect to the frequency of the primary current, the amplitude of the reaction field, which counteracts the flux linkage from the primary to the secondary coil, increases as well. This results, in this particular case, in a decrease in mutual inductance of 16% for a frequency equal to 500 kHz with respect to the case with no conducting plate, i.e., \( f = 0 \, \text{Hz} \). The eddy-current density in the aluminum plate is shown in Fig. 7. The figure clearly shows the location of the four current caring coils and the distribution of the eddy currents.

V. CONCLUSION

This paper extends the 3-D Fourier analysis by including conducting-materials and eddy-current reaction fields, which makes the technique able to give an accurate prediction of the inductances in complex 3-D structures. A solution for the 3-D magnetic vector potential has been presented and the method is verified by calculating the mutual inductances of a CET system integrated in a planar motor. A difference of at maximum 7% has been obtained between the calculated, measured, and inductances obtained from FEA, which validates the proposed solution for the magnetic vector potential.

REFERENCES


