Practical synchronization in networks of diffusively coupled non-identical systems

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Supplementary material:
Practical synchronization in networks of
diffusively coupled non-identical systems

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This document is attached as supplementary material to the manuscript
entitled “Training a network of electronic neurons for control of a mobile robot”.
In this document a general theoretical framework for practical synchronization
in networks of diffusively coupled non-identical systems is described.

1 Introduction

We consider a network of \( i = 1, 2, \ldots, k \) systems

\[
\begin{align*}
\dot{x}_i &= f_i(x_i) + B_i u_i \\
y_i &= C_i x_i
\end{align*}
\]

that interact via diffusive coupling functions

\[
u_i = -\sigma \sum_{j=1, j \neq i}^{k} \gamma_{ij} (y_i - y_j),
\]

with constants \( \sigma > 0 \) and \( \gamma_{ij} \geq 0 \) denoting the coupling strength and interaction
weights, respectively, \( x_i \in \mathbb{R}^n \) is the state of system \( i \), \( u_i \in \mathbb{R}^m \) with \( 1 \leq m \leq n \)
is its input, \( y_i \in \mathbb{R}^m \) its output, a sufficiently smooth vectorfield \( f_i : \mathbb{R}^n \to \mathbb{R}^n \)
and constant matrices \( B_i \) and \( C_i \) are of appropriate dimension. It is assumed that

- \( f_i(s) = f(s) + \Delta f_i(s) \) with \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( \Delta f_i : \mathbb{R}^n \to \mathbb{R}^n \) sufficiently
  smooth;
- \( B_i = B + \Delta B_i \) with \( \text{rank}(B_i) = \text{rank}(B) = m \);
- \( C_i = C + \Delta C_i \) with \( \text{rank}(C_i) = \text{rank}(C) = m \);
- \( C_i B_i \) and \( CB \) are similar to a positive definite matrix of rank \( m \);
- the matrix
  \[
  \Gamma = \begin{pmatrix}
  \sum_{j=2}^{k} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1k} \\
  -\gamma_{21} & \sum_{j=1, j \neq 2}^{k} \gamma_{2j} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & -\gamma_{(k-1)1} \\
  -\gamma_{k1} & \cdots & -\gamma_{k(k-1)} & \sum_{j=1, j \neq k}^{k} \gamma_{kj}
  \end{pmatrix}
  \]
  is irreducible.
We remark that \( \Gamma \) is an irreducible matrix if and only if it is the (weighted) Laplacian matrix of a strongly connected digraph \([2]\).

Note that the Hindmarsh-Rose model is written in the form (1.1) and satisfies the assumptions on the system matrices and the coupling functions.

## 2 Ultimately bounded solutions

**Definition 1.** The system
\[
\begin{cases}
\dot{x} = g(x, u) \\
y = h(x)
\end{cases}
\tag{2.1}
\]
with state \( x \in \mathbb{R}^n \), inputs and outputs \( u, y \in \mathbb{R}^m \), \( 1 \leq m \leq n \), sufficiently smooth functions \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}^m \), is called strictly semi-passive with storage function \( S : \mathbb{R}^n \to \mathbb{R}_+ \) if there exist strictly increasing functions \( s_0, s_1, s_2 : \mathbb{R}_+ \to \mathbb{R}_+ \), \( s_j(0) = 0 \) and \( \lim_{r \to \infty} s(r) = \infty, j = 0, 1, 2 \), such that
\[
s_0(\|x\|) \leq S(x) \leq s_1(\|x\|)
\]
and
\[
\dot{S}_{(2.1)}(x) \leq y^\top u - H(x)
\]
where \( H : \mathbb{R}^n \to \mathbb{R}_+ \) satisfies
\[
H(x) \geq s_2(\|x\|) - M
\]
for some constant \( M > 0 \).

The Hindmarsh-Rose model of the dynamics of the membrane potential of a neuron is a strictly semi-passive system \([7]\).

**Lemma 1.** Suppose that each system (1.1) is strictly semi-passive with storage function \( S : \mathbb{R}^n \to \mathbb{R}_+ \), then the solutions of the coupled systems (1.1), (1.2) are uniformly ultimately bounded.

**Proof.** We let
\[
x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}.
\]
Note that \( u = -(\Gamma \otimes I_m)y \) with \( I_m \) the \( m \times m \) identity matrix. For any irreducible \( \Gamma \) there is a vector \( \nu \) with strictly positive entries \( \nu_i, i = 1, \ldots, k \), such that \( \nu^\top \Gamma = 0 \). (This follows immediately from the Perron-Frobenius theorem, cf. \([2]\).) Consider the function
\[
V(x) = \nu_1 S(x_1) + \ldots + \nu_k S(x_k).
\]
Then the strict semi-passivity property of the systems implies that
\[
\dot{S}_{(1.1)}(x_i) = \left( \frac{\partial S}{\partial x_i} \right)^\top (f_i(x_i) + B_i u_i) \leq y_i^\top u_i - H_i(x_i)
\]
such that
\[
\dot{V}(x) \leq -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k) - y^\top (\text{diag}(\nu) \otimes I_m)u
= -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k) - \sigma y^\top (\text{diag}(\nu) \Gamma \otimes I_m)y,
\]
for some constant \( \sigma > 0 \).
where $\text{diag}(\nu)$ is a diagonal matrix with the entries of $\nu$ on its diagonal. It is not difficult to show that

$$\text{diag}(\nu)\Gamma + \Gamma^\top\text{diag}(\nu)$$

is positive semi-definite, hence

$$\dot{\mathcal{V}}(x) \leq -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k).$$

The properties of $H_i$ imply that

$$\dot{\mathcal{V}}(x) \leq s_3(\|x\|)$$

for $\|x\| > R$ for some $R > 0$ and $s_3 \in \mathcal{K}_\infty$. An application of Theorem 4.1.16 of [1] proves the lemma. \hfill \Box

For a constant $c > 0$ we let

$$\Omega_c := \{w \in \mathbb{R}^n | S(w) < c\}.$$

Lemma 2. Suppose that each system (1.1) is strictly semi-passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_+$. Let constant $c_0$ be such that for all $i = 1, 2, \ldots, k$

$$H_i(x_i) > 0$$
on $\mathbb{R}^n \setminus \Omega_{c_0}$. Then the solutions of the coupled systems (1.1), (1.2) converge to the compact set

$$\Omega_{c_0}^k := \underbrace{\Omega_{c_0} \times \ldots \times \Omega_{c_0}}_{k \text{ times}}.$$

Proof. Let

$$c_1 := \sup_{w \in \Omega_{c_0}} S(w)$$

and

$$\tilde{S}(x_i) := \begin{cases} 0 & \text{if } x_i \in \Omega_{c_0}, \\ S(x_i) - c_1 & \text{otherwise}. \end{cases}$$

Consider

$$\dot{\mathcal{V}}(x) = \nu_1 \tilde{S}(x_1) + \ldots + \nu_k \tilde{S}(x_k).$$

It follows from the arguments of the proof of Lemma 1 that there exists a continuous function $W : \mathbb{R}^{kn} \to [0, \infty)$ that is positive definite with respect to the set\footnote{We refer to [1] for a definition of a function to be positive definite with respect to a set.} $\Omega_{c_0}^k$ such that

$$\dot{\tilde{S}}(x) \leq -W(x).$$

An application of Theorem 4.1.18 of [1] proves the lemma. \hfill \Box
3 Practical network synchronization

Let, for some constant $\epsilon > 0$,

$$\mathcal{M}_\epsilon = \{x = \text{col}(x_1, x_2, \ldots, x_k) \in \mathbb{R}^{kn} | \|x_i - x_j\| \leq \epsilon \text{ for all } i, j = 1, 2, \ldots, k\}.$$  

The set $\mathcal{M}_\epsilon$ is the practical synchronization manifold. The set $\mathcal{M}_0$ is the synchronization manifold; For any solution of (1.1), (1.2) on $\mathcal{M}_0$ the corresponding solutions of the individual systems are indistinguishable.

**Definition 2.** Let $\phi(\cdot; t_0, x_0)$ denote the unique solution of (1.1), (1.2) through $x_0 \in \mathbb{R}^{kn}$ at $t = t_0$ defined on the interval $[t_0, t_1)$, $t_1 > t_0$. The coupled systems (1.1), (1.2) practically synchronize with bound $\epsilon$ if for each $\epsilon > \epsilon$ there is a $T = T(\epsilon)$, $T < t_1 - t_0$, such that

$$\phi(t; t_0, x_0) \in \mathcal{M}_\epsilon \quad \forall t \geq t_0 + T$$

Since each $C_iB_i$ is similar to a positive definite matrix, there is a linear, invertible change of variables

$$x_i \mapsto (z_i, y_i)$$

with $z_i \in \mathbb{R}^{n-m}$ and $y_i \in \mathbb{R}^m$. See [6] for the details. In new coordinates the systems’ dynamics read

$$\begin{cases}
\dot{z}_i = q_i(z_i, y_i) \\
\dot{y}_i = a_i(z_i, y_i) + C_iB_i u_i
\end{cases} \tag{3.1}$$

with sufficiently smooth functions $q_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$ and $a_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$. Since $f_i(x_i) = f(x_i) = \Delta f_i(x_i)$ with $f$ and $\Delta f_i$ sufficiently smooth, $q_i(z_i, y_i) = q(z_i, y_i) + \Delta q_i(z_i, y_i)$ and $a_i(z_i, y_i) = a(z_i, y_i) + \Delta a_i(z_i, y_i)$ with $q, \Delta q_i, a, \Delta a_i$ sufficiently smooth. Without loss of generality we assume that $C_iB_i$ is a diagonal matrix, hence a diagonal matrix with strictly positive entries. In addition we assume that:

**Assumption 1.** There are sets $\mathcal{Z} \subset \mathbb{R}^{n-m}$ and $\mathcal{Y} \subset \mathbb{R}^m$ such that

$$z_i(t) \in \mathcal{Z}, \quad y_i(t) \in \mathcal{Y}, \quad \forall t \geq t_0 \quad \forall i.$$ 

It follows from Lemmas 1 and 2 that this assumption is satisfied, possibly after re-defining $t_0$, if the systems are strictly semi-passive with a common storage function. In addition we assume that:

**Assumption 2.** There exist a positive definite matrix $P \in \mathbb{R}^{(n-m) \times (n-m)}$ such that the symmetric matrix

$$Q(z_i, y_i) := \left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right)^\top P + P \left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right)$$

is uniformly negative definite on $\mathbb{R}^{n-m} \times \mathcal{Y}^m$, i.e. for any $(z_i, y_i) \in \mathbb{R}^{n-m} \times \mathcal{Y}^m$ the eigenvalues of $Q(z_i, y_i)$ are negative and separated away from zero.

It can easily be shown, cf. [5], that this assumption implies the existence of a positive constant $\alpha$ such that

$$(z_i - z_j)^\top P[q(z_i, y_i) - q(z_j, y_j)] \leq -\alpha \|z_i - z_j\|^2 \quad \forall z_i, z_j \in \mathbb{R}^{n-1} \quad \forall y_i \in \mathcal{Y}.$$ 

Assumption 2 holds for the Hindmarsh-Rose neuron with $P$ the identity matrix [7]. Thus both assumptions hold for Hindmarsh-Rose neurons.
Theorem 1. Consider the network of diffusively coupled systems (3.1), (1.2) and suppose that assumptions 1 and 2 hold. There exists a constant $\sigma > 0$ such that for any $\sigma \geq \sigma_0$ the network of diffusively coupled systems (3.1), (1.2) practically synchronizes with bound $\epsilon$.

Proof. Let 
\[
\Gamma = \text{diag}(C_1B_1, \ldots, C_kB_k)(\Gamma \otimes I_m)
\]
and note that $\Gamma$

- is irreducible since $\Gamma$ is irreducible and each $C_iB_i$ diagonal;
- has $m$ zero eigenvalues since $\Gamma$ has a simple zero eigenvalue, hence $\Gamma \otimes I_m$ has $m$ zero eigenvalues;
- has all non-zero eigenvalues in $\mathbb{C}_+$, the open right half-plane of the complex plane.

The second property of $\Gamma$ follows from the fact that $\Gamma$ is the Laplacian matrix of a strongly connected weighted digraph and the multiplicity of the zero eigenvalue equals the number of strongly connected components. The third property of $\Gamma$ is a direct consequence of Gershgorin’s Disc theorem, cf. [2].

Let 
\[
U = \begin{pmatrix} 1 & 0^T \\ 1 & -I_{k-1} \end{pmatrix} \in \mathbb{R}^{k \times k}
\]
where $0$ is a vector of appropriate dimension with all entries equal 0. Then 
\[
(U \otimes I_m)\tilde{\Gamma}(U \otimes I_m)^{-1} = \begin{pmatrix} 0 & * \\ 0 & \tilde{\Gamma} \end{pmatrix}
\]
with $\tilde{\Gamma} \in \mathbb{R}^{(k-1)m \times (k-1)m}$ and $*$ denotes some $m \times (k-1)m$ matrix. It is straightforward that the eigenvalues of $\tilde{\Gamma}$ are the non-zero eigenvalues of $\Gamma$. Let 
\[
\tilde{z}_j = z_1 - z_j, \quad \tilde{y}_j = y_1 - y_j, \quad j = 2, \ldots, k.
\]

Then 
\[
\begin{pmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_{k-1} \end{pmatrix} = \begin{pmatrix} q(z_1, y_1) - q(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ q(z_1, y_1) - q(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} + \tilde{q}(z_1, y_1, \tilde{z}, \tilde{y})
\]
and 
\[
\begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_{k-1} \end{pmatrix} = \begin{pmatrix} a(z_1, y_1) - a(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ a(z_1, y_1) - a(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} - \tilde{\Gamma} \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_{k-1} \end{pmatrix} + \tilde{a}(z_1, y_1, \tilde{z}, \tilde{y})
\]
with $\tilde{z} = \text{col}(\tilde{z}_1, \ldots, \tilde{z}_{k-1})$, $\tilde{y} = \text{col}(\tilde{y}_1, \ldots, \tilde{y}_{k-1})$,
\[
\tilde{q}(z_1, y_1, \tilde{z}, \tilde{y}) = \begin{pmatrix} \Delta q_1(z_1, y_1) - \Delta q_2(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ \Delta q_1(z_1, y_1) - \Delta q_k(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix}
\]
Consider the positive definite function

\[ V(\tilde{z}, \tilde{y}) = \tilde{z}^T (I_k \otimes P) \tilde{z} + \tilde{y}^T P_1 \tilde{y} \]

with \( P \) as in Assumption 1 and positive definite matrix \( P_1 \) is such that \( \|P_1\| = 1^2 \) and

\[-\tilde{\Gamma} P_1 - P_1 \tilde{\Gamma}^T \leq -\eta I \]

for some positive constant \( \eta \). The existence of such matrix \( P_1 \) is guaranteed by the fact the \(-\tilde{\Gamma}\) is a stable (Hurwitz) matrix.

Assumption 1 and smoothness of the functions \( q_i, a_i \) implies the existence of positive constants \( b_q \) and \( b_a \) such that

\[ \|(I_k \otimes P) \tilde{q}(z_1, y_1, \tilde{z}, \tilde{y})\| \leq b_q, \quad \|P \tilde{a}(z_1, y_1, \tilde{z}, \tilde{y})\| \leq b_a. \]

Assumptions 1, 2 and smoothness of the function \( q \) implies the existence of positive constants \( \kappa_0, \kappa_1 \) such that

\[ \tilde{z}^T (I_k \otimes P) \begin{pmatrix} q(z_1, y_1) - q(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ q(z_1, y_1) - q(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} \leq -\kappa_0 \|\tilde{z}\|^2 + \kappa_1 \|\tilde{z}\| \|\tilde{y}\|. \]

Assumption 1 and smoothness of the function \( a \) implies the existence of positive constants \( \kappa_2, \kappa_3 \) such that

\[ \tilde{y}^T P_1 \begin{pmatrix} a(z_1, y_1) - a(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ a(z_1, y_1) - a(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} \leq \kappa_2 \|\tilde{z}\| \|\tilde{y}\| + \kappa_3 \|\tilde{y}\|^2. \]

Thus

\[ \dot{V}(\tilde{z}, \tilde{y}) \leq -\kappa_0 \|\tilde{z}\|^2 + \kappa_1 \|\tilde{z}\| \|\tilde{y}\| + \kappa_2 \|\tilde{z}\| \|\tilde{y}\| \\
+ (\kappa_3 - \sigma \eta) \|\tilde{y}\|^2 + b_q \|\tilde{z}\| + b_a \|\tilde{y}\| \\
= -\left( \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix}^T \right) W \left( \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix} \right) + b_q \|\tilde{z}\| + b_a \|\tilde{y}\| \]

with

\[ W = \begin{pmatrix} \kappa_0 & -\kappa_1 \kappa_2 \\ -\kappa_1 \kappa_2 & \kappa_2 \sigma \eta - \kappa_3 \end{pmatrix}. \]

Note that \( W \) is positive definite for

\[ \sigma > \bar{\sigma} = \frac{1}{\eta} \left( \kappa_3 + \frac{(\kappa_1 + \kappa_2)^2}{4\kappa_0} \right). \]

\(^2\)Given a matrix \( A, \|A\| = \max\|x\| = \|Ax\| \).
Thus there is a positive constant $\mu = \mu(\sigma) = \lambda_{\min}(W)$ such that for $\sigma > \bar{\sigma}$
\[
\dot{V}(\bar{z}, \bar{y}) \leq -\mu(\|\bar{z}\|^2 + \|\bar{y}\|^2) + B_y \|\bar{z}\| + B_a \|\bar{y}\|
\]
such that
\[
\dot{V}(\bar{z}, \bar{y}) \leq -\mu(1 - \rho)(\|\bar{z}\|^2 + \|\bar{y}\|^2), \quad \rho \in (0, 1),
\]
if
\[
\|\bar{z}\| = \frac{B_a}{\rho \mu} \quad \text{and} \quad \|\bar{y}\| = \frac{B_a}{\rho \mu}.
\]

Standard techniques, cf. Lemma 9.2 of [3], imply that for any $\sigma > \bar{\sigma}$ the systems practically synchronize with bound
\[
\epsilon = \sqrt{\frac{\lambda_{\max}(\text{diag}(I_k \otimes P), P_1) B_q^2 + B_a^2}{\lambda_{\min}(\text{diag}(I_k \otimes P), P_1)} \rho^2 \mu^2},
\]
where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the smallest, respectively, largest eigenvalue of a symmetric matrix. \qed

4 Practical output-synchronization of two coupled systems

Consider two systems (1.1) that interact via coupling
\[
\begin{align*}
    u_1 &= \sigma(1 - \eta)[y_2 - y_1] \\
    u_2 &= \sigma \eta[y_1 - y_2]
\end{align*}
\]
with constant $\eta \in (0, 1)$.

**Definition 3.** Let $y_i(t; t_0, x_0)$, $i = 1, 2$, denote the unique output solution of (1.1), (4.1) through $x_0 \in \mathbb{R}^{2n}$ at $t = t_0$ defined on the interval $[t_0, t_1)$, $t_1 > t_0$. The coupled systems (1.1), (4.1) practically output-synchronize with bound $\epsilon_y$ if for each $\epsilon_y > \epsilon_y$ there is a $T = T(\epsilon_y)$, $T < t_1 - t_0$, such that
\[
|y_1(t; t_0, x_0) - y_2(t; t_0, x_0)| < \epsilon_y, \quad \forall t \geq t_0 + T.
\]

Of course, whenever the conditions of Lemma 1 and Theorem 1 are satisfied the two coupled systems (1.1), (4.1) practically output-synchronize with some bound $\epsilon_y$, provided that $\sigma > \bar{\sigma}$.

We assume for the systems (1.1)
\[
C_1 B_1 = C_2 B_2
\]
and let
\[
\begin{align*}
    \tilde{z} &= z_1 - z_2 \\
    \tilde{y} &= y_1 - y_2.
\end{align*}
\]
Then we get
\[
\begin{align*}
    \dot{\tilde{z}} &= q_1(z_1, y_1) - q_2(z_2, y_2) \\
    \dot{\tilde{y}} &= a_1(z_1, y_1) - a_2(z_2, y_2) - \sigma \tilde{y}.
\end{align*}
\]
We observe that these dynamics, which describe the synchronization error, do not depend on the weight parameter $\eta$. Consider

$$V(\tilde{y}) = \frac{1}{2} \tilde{y}^2.$$  

Using similar reasoning as in the proof of Theorem 1, we deduct that

$$\dot{V}(\tilde{y}) \leq c_1 + c_2 \tilde{y} - \sigma \tilde{y}^2$$

where constants $c_1$ and $c_2$ depend on the functions $q_1(z_1, y_i) - q_2(z_2, y_2)$ and $a_1(z_1, y_i) - a_2(z_2, y_2)$ given the bounds of the solutions $y_i(\cdot)$ and $z_i(\cdot)$. A direct application of LaSalle’s invariance principle, cf. [4], shows that we can find a $\tilde{\sigma} > 0$ such that if $\sigma > \tilde{\sigma}$ then $\dot{V}(\tilde{y}) < 0$ for all $\|\tilde{y}\| > \epsilon_y = \epsilon_y(\sigma)$ and, moreover, the larger $\sigma$ the smaller $\epsilon_y$ with

$$\lim_{\sigma \to \infty} \epsilon_y(\sigma) = 0.$$  

References


