A 64-dimensional counterexample to Borsuk's conjecture

Jenrich, T.; Brouwer, A.E.

Published in:
The Electronic Journal of Combinatorics

Published: 01/01/2014

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal ?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 15. Dec. 2018
A 64-dimensional counterexample to Borsuk’s conjecture

Thomas Jenrich Andries E. Brouwer

thomas.jenrich@gmx.de aeb@cwi.nl

Submitted: Feb 3, 2014; Accepted: Oct 28, 2014; Published: Nov 6, 2014
Mathematics Subject Classifications: 51M25, 05E30

Abstract

Bondarenko’s 65-dimensional counterexample to Borsuk’s conjecture contains a 64-dimensional counterexample. It is a two-distance set of 352 points.

1 Introduction

In 1933 Karol Borsuk [2] asked whether each bounded set in the $n$-dimensional Euclidean space (containing at least two points) can be divided into $n + 1$ parts of smaller diameter. (The diameter of a set $X$ is the least upper bound of the distances of pairs of points in $X$.) The hypothesis that the answer to that question is positive became famous under the name Borsuk’s conjecture.

The first counterexamples were given by Jeff Kahn and Gil Kalai [7] who showed that Borsuk’s conjecture is false for $n = 1325$ and gave an exponential lower bound $c\sqrt{n}$ with $c = 1.2$ for the number of parts needed for large $n$. Subsequently, several authors found counterexamples in lower dimensions.

In 2013 Andriy V. Bondarenko [1] constructed a 65-dimensional two-distance set $S$ of 416 vectors that cannot be divided into fewer than 84 parts of smaller diameter. That was not just the first known two-distance counterexample to Borsuk’s conjecture but also a considerable reduction of the lowest known dimension the conjecture fails in in general.

This article presents a 64-dimensional subset of $S$ of size 352 that cannot be divided into fewer than 71 parts of smaller diameter, thus producing a two-distance counterexample to Borsuk’s conjecture in dimension 64.

2 Euclidean representation of strongly regular graphs

We very briefly repeat the basic facts. More details can be found in [1] and [3].

A finite graph $\Gamma$ without loops or multiple edges is called a srg($v, k, \lambda, \mu$), where srg abbreviates ‘strongly regular graph’, when it has $v$ vertices, is regular of valency $k$, where
0 < k < v − 1, and any two distinct vertices x, y have λ common neighbours when x and y are adjacent (notation: x ∼ y), and μ common neighbours otherwise (notation: x ̸∼ y).

The adjacency matrix A of Γ is the matrix of order v defined by A_\text{xy} = 1 if x ∼ y and A_\text{xy} = 0 otherwise. Let I be the identity matrix of order v, and let J be the matrix of order v with all entries equal to 1. Then A is a symmetric matrix with zero diagonal such that AJ = JA = kJ and A^2 = kI + λA + μ(J - I - A). It follows that the eigenvalues of A are k, r, s, with k ≥ r ≥ 0 > s, where r, s are the two solutions of \(x^2 + (μ - λ)x + μ - k = 0\), so that \((A - rI)(A - sI) = μJ\). The multiplicities of k, r, s are 1, f, g (respectively), where \(1 + f + g = v\) and \(k + fr + gs = 0\).

The matrix \(M = A - sI - \frac{k-μ}{v}J\) has rank f, so that the map \(x \rightarrow \overline{x}\) that sends each vertex x to row x of M is a representation of Γ in \(\mathbb{R}^f\), and the inner product \((\overline{x}, \overline{y})\) depends only on whether \(x = y\), \(x ∼ y\) or \(x ̸∼ y\).

3 The \(G_2(4)\) graph

There exists a graph Γ that is a srg(416,100,36,20) with automorphism group \(G_2(4):2\) acting rank 3, with point stabilizer \(J_2:2\), see, e.g., Hubaut [4], pp. 370, 372. Here \(v = 416\), \(k = 100\), \(r = 20\), \(s = -4\) and \(f = 65\), \(g = 350\), so that \(M = A + 4I - \frac{1}{4}J\) and we have \(M^2 = 24M = 24A + 96I - 6J\). This means that

\[
(\overline{x}, \overline{y}) = \begin{cases} 
90 & \text{if } x = y \\
18 & \text{if } x ∼ y \\
-6 & \text{if } x ̸∼ y
\end{cases}
\]

and \(\|\overline{x} - \overline{y}\|^2 = 144\) when \(x ∼ y\), and \(\|\overline{x} - \overline{y}\|^2 = 192\) when \(x ̸∼ y\).

This graph Γ has maximal clique size 5 (because each point neighbourhood is a srg(100,36,14,12), that has point neighbourhoods srg(36,14,4,6), which has bipartite point neighbourhoods).

Bondarenko’s example S is the image of Γ in \(\mathbb{R}^{65}\). Any subset of smaller diameter corresponds to a clique and therefore has size at most 5. Since \(|S| = 416\), at least 84 subsets of smaller diameter are needed to cover the set.

4 Structure of the \(G_2(4)\) graph

The graph Γ occurs as point neighbourhood in the Suzuki graph Σ, which is a srg(1782, 416,100,96) (cf. [4]). For two nonadjacent vertices a, b of Σ, we can identify the set of 416 neighbours of a with the vertex set X of Γ, and then the 96 common neighbours of a and b form a 96-subset B of X.

The graph Σ has a triple cover 3 · Σ constructed by Leonard Soicher [8]. It is distance-transitive with intersection array \(\{416, 315, 64, 1; 1, 32, 315, 416\}\) on 5346 vertices.
We see that the 96-subset $B$ is the union of three mutually nonadjacent subsets $B_1$, $B_2$ and $B_3$ of size 32. Put $C = X \setminus B$ so that $|C| = 320$. Since $3 \cdot \Sigma$ is tight (cf. [6]), the partition $\{B_1, B_2, B_3, C\}$ of $X$ is regular (a.k.a. equitable) with diagram

\[ \begin{array}{c}
80 \\
32 \\
80 \\
32 \\
80 \\
20
\end{array} \]

(that is, each vertex in $B_1$ has 20 neighbours in $B_1$, none in $B_2$, $B_3$, and 80 in $C$, etc.).

Now we define $T = \{ \overline{x} \mid x \in B_1 \cup C \} \subseteq \mathbb{R}^{65}$. Let $u$ be the vector

\[ u = \sum_{y \in B_2} \overline{y} - \sum_{y \in B_3} \overline{y}. \]

Then $u$ is a vector in our $\mathbb{R}^{65}$, and for all $x \in T$ we have $(u, x) = 0$. On the other hand, $(u, u) = 64 \cdot 576 \neq 0$. It follows that $T$ lies in the hyperplane $u^\perp$, a copy of $\mathbb{R}^{64}$. Because any subset of smaller diameter contains at most 5 vectors, we proved

**Theorem 1** There is a 2-distance set $T$ of size 352 in $\mathbb{R}^{64}$ such that any partition of $T$ into parts of smaller diameter has at least 71 parts.

**Remark** For more explicit constructions and a corresponding computer program, see [5].

**References**


