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Optimal inventory management with supply backordering

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ABSTRACT

We study the inventory control problem of a retailer working under stochastic demand and stochastic limited supply. We assume that the unfilled part of the retailer's order is fully backordered at the supplier and replenished with certainty in the following period. As it may not always be optimal for the retailer to replenish the backordered supply, we also consider the setting in which the retailer has a right to either partially or fully cancel these backorders, if desired. We show the optimality of the base-stock policy and characterize the threshold inventory position above which it is optimal to fully cancel the replenishment of the backordered supply. We carry out a numerical analysis to quantify the benefits of supply backordering and the value of the cancelation option, and reveal several managerial insights.

1. Introduction

With sourcing in remote offshore locations, uncertainties in supply have increased. For instance, in 2009, during the recovery period of the 2008 financial crisis, electronics parts were in short supply and suppliers could not deliver orders. For instance, Nortech issued a formal statement in March 2009 stating that it was "*dependent on suppliers for electronic components and may experience shortages, cost premiums and shipment delays that would adversely affect our customers and us*" (Nortech Systems, 2009). In such cases, backorders are delivered very late. With regard to the same period of electronic component shortages in 2009 and 2010, Pierson (2010) reports that lead times for certain components had increased from 10 to 20 weeks, and that customers were on allocation, implying that customers were unsure how much they would receive. However, many buyers in fact later canceled their orders when they obtained new information on demand. The very late supply of backorders actually granted them a moral right to cancel the orders due to the very late delivery. When demand is uncertain, it is unclear to what extent it is best to actually cancel backorders and possibly place new orders instead.

In addition, the importance of time as a competitive weapon in supply chains has led suppliers to venture into lead time reduction projects to improve their ability to meet the customers' expectations for shorter lead times. However, such undertaking often results in, at least in a short term, worse supply performance, characterized mainly by delayed and/or partial replenishment of orders. Furthermore, a supplier might even decide to adopt the supply strategy where he would deliver only a part of the customer's order depending on his current supply capacity availability, and would guarantee to replenish the rest of the order with a short delay. From a customers' perspective, apart from the demand uncertainties, companies need to consider the possibility of these delays in deciding when and how much to order from the supplier. Interestingly, in a stochastic lead time setting, Wang and Tomlin (2009) show that a customer may sometimes prefer a less reliable lead time if the delay distribution is not very variable. Therefore, it may be beneficial for a company to accept the possibility of the uncertain delivery and, given that the order is eventually replenished in full, adapt its ordering policy to take advantage of a shorter lead time.

In this paper we study the inventory control problem of a retailer working under stochastic demand from the market, where he tries to satisfy the demand by making orders with a supplier. The supply capacity available to the retailer is assumed to be limited and stochastic as a result of a supplier's changing capacity and capacity allocation policy. The order placed by the retailer might therefore not be delivered in full, depending on the currently available capacity. The unfulfilled part of the retailer's order is backordered at the supplier, which we denote as a supply backorder. As the supply backorder is a result of the supplier's inadequate supply service, this gives the retailer an option (a moral right) to decide to what extent he wants the supplier to replenish the backordered supply. Depending on the current requirements, the retailer can decide for partial replenishment of the backordered supply, or to fully cancel the replenishment if necessary. Therefore, in each period the retailer has to make two decisions. Apart from the regular ordering decision to the supplier, he needs to decide about the extent of the replenishment of the

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backordered supply from the previous period. We assume that the replenishment of the backordered supply is certain, meaning that it is delivered in full in the following period. The situation that would result in such supply conditions for the retailer is that of a supplier giving high priority to fulfilling backorders from previous periods, which is a common situation observed in practice. It is also safe to assume that this would often not affect the regular capacity available to cover retailer's order in the next period.

The focus of this paper is on establishing the optimal inventory policy that would allow the retailer to improve his inventory control by utilizing a backordered supply. We denote the case where the backordered supply can be partially replenished at the request from the retailer as the *Partial backordering* policy, and compare it to the *No backordering* policy, where there is no supply backordering, to establish the value of supply backordering. Apart from this base setting, we are interested in analyzing the two subpolicies: *Full backordering* policy, where the backordered supply is always replenished in full, and the *Cancelation option* policy where the retailer has the option to fully cancel the replenishment of the supply backorder. Although it is expected that the costs can be substantially reduced in comparison with a capacitated supply environment in which the unfilled part of the order is lost rather than backlogged, we are also interested in whether full supply backordering can be counterproductive in specific situations. For clarity, we present the set of possible decision strategies and corresponding abbreviations in Table 1.

We proceed with a review of the relevant literature on supply uncertainty models, where our setting fits within the scope of single-stage inventory models with random capacity. The way we model the supply availability is inline with the work of Ciarallo et al. (1994), Güllü et al. (1999), Khang and Fujiwara (2000) and Iida (2002), where the random supply/production capacity determines a random upper bound on the supply availability in each period. Their research is mainly focused on establishing the structure of the optimal policy. For a finite horizon stationary inventory model they show that the optimal policy is of an order-up-to type, where the optimal base-stock level is increased to account for possible, albeit uncertain, capacity shortfalls in future periods.

There have been several approaches to mitigate the supply capacity uncertainty suggested in the literature, where some have explored the benefits of an alternative supply source, or the means of decreasing the uncertainty of supply itself. Our model can also be interpreted as a dual sourcing system in which the stochastically capacitated primary supplier delivers part of the order with zero lead time, while the alternative supplier has ample capacity but is only able to deliver the remaining part of the initial order with a one period delay. Assuming deterministic lead time, several papers discuss the setting in which lead times of the two suppliers differ by a fixed number of periods (Fukuda, 1964; Veeraraghavan and Sheller-Wolf, 2008). However, they all assume infinite supply capacity or at most a fixed capacity limit on one or both suppliers. For an identical lead time situation as ours, albeit uncapacitated, Bulinskaya (1964) shows the optimality of the base-stock policy and derives its parameters. However, when there is uncertainty in the supply capacity, diversification through multiple sourcing has received very little attention. The exception to this are the papers by Dada et al. (2007) and Federgruen and Yang (2009), where they study a single period problem with multiple capacitated suppliers and develop the optimal policy to assign orders to each supplier. Our model differs from the abovementioned dual sourcing models due to the fact that the alternative replenishment directly corresponds to the unfilled part of the order placed to the primary supplier, and therefore it cannot be considered as an independent ordering decision.

A recent stream of research considers the case where the information on the availability of supply capacity for the near future is provided by the supplier. In Jakšič et al. (2011) and Atasoy et al. (2012) they show how this so called advance capacity information influences the structure of the optimal policy, which is shown to be a state-dependent base-stock policy. A general assumption in stochastically capacitated single sourcing inventory models is that the part of the order above the available supply capacity in a certain period is lost to the customer. We believe this might not hold in several situations observed in practice. However, the literature that assumes the possibility of backordering the unfilled part of the customer's order at the supplier is scarce. For a continuous review system (Moinzadeh and Lee, 1989) study the system where orders arrive in two shipments, the first shipment with only random part of the items ordered, while the rest of the items arrive in a second shipment. Assuming (Q,R) policy, they present the approximate cost function and compute its parameters. Anupindi and Akella (1993) study a dual unreliable supplier system, where they assume that the unfilled part of the order is delivered with a one period delay in their Model III. A non-zero lead time setting is assumed in Bollapragada et al. (2004), where the supplier guarantees delivery either within his quoted lead time, or at most one period later. They study the two-stage serial inventory system under the assumption that approximate installation base-stock policies are followed, and evaluate the benefits of guaranteed delivery over the system with unlimited supply backlog. This way of modeling the supply backorders corresponds to what we denote in this paper as a *Full backordering* policy.

However, the situation in which the customers whose orders were backordered at the supplier may cancel their orders is rarely considered in the literature. You and Hsieh (2007) assume a constant fraction of customers are canceling their backorders. Therefore they do not consider the cancelation of backorders as a decision variable, but as a preset system parameter, which effectively reduces the demand the supplier is facing. In this paper we include the option to either partially or fully cancel backorders on the supply side as an integral part of the optimal ordering decision policy.

The remainder of the paper is organized as follows. We present our dynamic programming model incorporating supply backordering and the cancelation option in Section 2. The structure of the optimal policy and its characteristics related to the option of canceling the replenishment of the backordered supply are derived in Section 3. In Section 4 we assess the benefits of supply backordering and the value of the cancelation option through a numerical study and we point out the relevant managerial insights.

Table 1
Labeling scheme for possible decision strategies.

Label	Strategy	Description
NB	*No backordering*	Retailer is only placing regular orders and there is no supply backordering
FB	*Full backordering*	Backordered supply is always replenished in full
PB	*Partial backordering*	Retailer decides to what extent the backordered supply should be replenished
CO	*Cancelation option*	Retailer decides whether to fully cancel the replenishment of the backordered supply or not

We study the trade-off between the supply uncertainty and the lead time delay in Section 5, and summarize our findings in Section 6.

2. Model formulation

In this section, we introduce the notation and present the dynamic programming model to formulate the problem under consideration. The model assumes a periodic-review inventory control system with non-stationary stochastic demand and limited non-stationary stochastic supply with a zero supply lead time. The supply capacity is assumed to be exogenous to the retailer and the exact capacity realization is only revealed upon replenishment. Unused capacity in a certain period is assumed to be lost. In the case when currently available supply capacity is insufficient to cover the whole order, the unfilled part of the supply is backordered at the supplier. In each period, the retailer has to make two decisions. He needs to decide to what extent he wants the supplier to replenish the supply backorder from the previous period and needs to consider placing a new regular order with the supplier.

Depending on the level of flexibility in determining the extent of supply backorder replenishment we distinguish between three different policies (observe the labeling scheme in Table 1), where the backorder parameter β_t denotes a share of the backordered supply b_t to be replenished: *PB* policy, where a retailer freely decides about the extent of the supply backorder replenishment ($\beta_t \in [0,1]$); *CO* policy, where a retailer decides to either fully replenish or fully cancel the replenishment of the backordered supply ($\beta_t \in \{0,1\}$); and the *FB* policy, where supply backorder is always replenished in full ($\beta_t = 1$).

The notation used throughout the paper is summarized in Table 2 and some is introduced when needed.

We assume the following sequence of events:

(1) At the start of period t, the decision maker reviews the inventory position x_t, and the ordering decision is made, which is composed of: supply backordering decision β_{t-1}, about the extent of the replenishment of the supply backorder b_{t-1} from the previous period, and the regular order z_t. Correspondingly the inventory position is raised to $y_t = x_t + \beta_{t-1} b_{t-1} + z_t$.

(2) The previous period's supply backorder $\beta_{t-1} b_{t-1}$ and the current period's regular order z_t are replenished. With this, the available supply capacity q_t for the current order is revealed, and the inventory position after the replenishment is corrected in the case of insufficient supply capacity to $y_t - b_t = x_t + \beta_{t-1} b_{t-1} + z_t - b_t$, where $b_t = [z_t - q_t]^+$ represents the new supply backorder.[1]

(3) At the end of the period the decision maker observes the previously backordered demand and the current period's demand d_t, and tries to satisfy it from the available inventory $y_t - b_t$. Unsatisfied demand is backordered, and inventory holding and backorder costs are incurred based on the end-of-period inventory position, $x_{t+1} = y_t - b_t - d_t$.

Observe that the decision whether to replenish the supply backorder b_{t-1} and to what extent is only relevant if it is taken after the demand d_{t-1} from the previous period has been realized. If that is not the case, the decision maker would always opt to replenish the supply backorder since the system remains in the same state it was in when the regular order z_{t-1} was initially made.

We proceed by writing the relevant cost formulations, which will allow us to evaluate the performance of the model. The system's costs consist of inventory holding and backorder costs charged on end-of-period on-hand inventory. Cost c_h is charged per unit of excess inventory, and backorders cost c_b per unit. We ignore any fixed cost related to ordering, both when making the initial order, as well as with the replenishment of the backordered supply. We also assume there are no cancelation costs related to not replenishing the backordered supply. In a practical setting the retailer would normally bear the fixed ordering costs associated with placing regular orders, while the supplier would account for the costs of replenishing the backordered supply. Under such conditions the retailer would be reluctant to cancel the replenishment of the backordered supply and instead place a regular order, as he would be charged with ordering costs. Under *PB* policy, it is intuitively clear that the retailer will always take the advantage of the backordered supply first and only rely to placing regular orders when backordered supply is insufficient, thus avoiding cancelation strategy. However, under a *CO* policy, where only full cancelation is allowed, the higher the fixed ordering costs the less likely it will be that the retailer will cancel the supply backorder replenishment. Inclusion of fixed ordering costs would be a natural extension to the model, however, we believe that the major insights related to supply backordering can be gained already through the study of a zero fixed ordering costs setting.

Table 2
Summary of the notation.

T: Number of periods in the finite planning horizon
c_h: Inventory holding cost per unit per period
c_b: Backorder cost per unit per period
α: Discount factor ($0 \le \alpha \le 1$)
x_t: Inventory position before decision making in period t
y_t: Inventory position after decision making in period t
z_t: Order size in period t
b_t: Supply backorder; backordered part of the order in period t
β_t: Part of the supply backorder b_t to be replenished in period $t+1$
D_t: Random variable denoting demand
d_t: Actual demand realization
$g_t(D_t)$: Probability density function of demand
$G_t(D_t)$: Cumulative distribution function of demand
Q_t: Random variable denoting the available supply capacity
q_t: Actual available supply capacity
$r_t(Q_t)$: Probability density function of supply capacity
$R_t(Q_t)$: Cumulative distribution function of supply capacity

The expected single-period cost charged at the end of period t is expressed as

$$C_t(y_t, z_t) = \alpha E_{Q_t, D_t} \tilde{C}_t(y_t - b_t - D_t). \tag{1}$$

The inventory holding and backorder costs are charged on the end-of-period inventory position $y_t - b_t - D_t$, which depends both on the capacity limiting the actual replenishment and the demand realization. The single period cost function is defined as $\tilde{C}(x) = c_h[x]^+ + c_b[x]^-$.

The dynamic programming formulation minimizing the relevant inventory costs over finite planning horizon T from time t onward and starting in the initial state (x_t, b_{t-1}) characterized by the inventory position before the decision making x_t and the backordered supply b_{t-1}, can be written as

$$f_t(x_t, b_{t-1}) = \min_{\beta_{t-1} \in [0,1], z_t \ge 0} \left[C_t(y_t, z_t) + \alpha E_{Q_t, D_t} f_{t+1}(y_t - b_t - D_t, b_t) \right],$$

for $1 \le t \le T$,

$$= \min_{\beta_{t-1} \in [0,1], z_t \ge 0} \left[C_t(x_t + \beta_{t-1} b_{t-1} + z_t, z_t) + \alpha E_{Q_t, D_t} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + z_t - b_t - D_t, b_t) \right], \text{ for } 1 \le t \le T, \tag{2}$$

where the ending condition is defined as $f_{T+1}(\cdot) \equiv 0$. The state space is described by the pair (x_t, b_{t-1}), where the optimal supply backordering and regular ordering decisions are made by searching over possible (y_t, z_t) pairs that describe the state of the system after the decision is made, so that the total costs are minimized.

[1] $[a]^+ = \max(a, 0)$.

Finally, for the system to end up in the state (y_t, z_t), we need to calculate the expectation of $f_{t+1}(x_{t+1}, b_t)$ over all possible demand D_t and supply capacity Q_t realizations in period t.

3. Characterization of the optimal policy

In this section, we focus on the optimal policy characterization of the inventory system with supply backordering. We show some properties of the relevant cost functions and the structure of the optimal policy as a solution of the dynamic programming formulation given in (2). In addition, we give some insights into the optimal base-stock level and characterize the threshold inventory position, the point at which the decision maker is ambivalent between fully replenishing the backordered supply and placing a regular order. We refer to the Appendix for proofs of the theorems in this section.

We start by showing that, when placing a regular order, there is an optimal inventory position to which we generally want to raise our current inventory position. However, attaining the optimal inventory level exactly might not be feasible due to the limited supply capacity constraining the replenishment of the regular order and possibly limited level of flexibility in replenishment of the supply backorder. Especially, the second observation is critical for establishing the optimal decision strategy under the *CO* policy. Due to this, the focus of this section is on exploring the effect the option to partially or fully replenish the backordered supply or cancel the replenishment has on the optimal strategy. It would be practical if the strategy were to have the properties of a base-stock policy, meaning that there exists an optimal y_t and that it is independent of the starting state (x_t, b_{t-1}). The difficulty in establishing the global minimum and the characterization of the optimal inventory position lies in the fact that the cost function $f_t(x_t, b_{t-1})$ is not convex. This is a property that is a consequence of the complexity of supply backordering having been added to the underlying stochastic capacitated inventory model.

We define the auxiliary cost function J_t as

$$J_t(y_t, z_t) = C_t(y_t, z_t) + \alpha E_{Q_t, D_t} f_{t+1}(y_t - b_t - D_t, b_t), \text{ for } 1 \le t \le T, \tag{3}$$

and correspondingly we rewrite the minimal expected cost function f_t from (2) as

$$f_t(x_t, b_{t-1}) = \min_{\beta_{t-1} \in [0,1], z_t \ge 0} J_t(y_t, z_t), \text{ for } 1 \le t \le T.$$

We first analyze the *PB* policy and the underlying *FB* policy in which we assume that supply backorders are either partially or fully replenished. Then we proceed with an analysis of the *CO* policy where we have an option to cancel the replenishment of the backordered supply in any given period.

In the following theorem we show that the properties of the cost functions needed to characterize the inventory policy under a partial and full backordering assumption as a base-stock policy with the optimal base-stock level $\hat{y}_t$.

Theorem 1. *Let $\hat{y}_t$ be the smallest minimizer of J_t and the starting state is (x_t, b_{t-1}), where full backordering is assumed, $\beta_{t-1} = 1$, for any t:*

1. *$f_t(x_t, b_{t-1})$ is convex in x_t for $b_{t-1} = 0$.*
2. *$J_t(y_t, z_t)$ and $J_t(x_t, b_{t-1}, z_t)$ are quasiconvex in z_t.*
3. *Under the optimal PB policy, $y_t(x_t, b_{t-1})$ is given by*
$$y_t(x_t, b_{t-1}) = \begin{cases} x_t, & \hat{y}_t \le x_t, & \beta_{t-1} = 0, z_t = 0, \\ x_t + \beta_{t-1} b_{t-1} = \hat{y}_t, & x_t < \hat{y}_t \le x_t + b_{t-1}, & \beta_{t-1} \le 1, z_t > 0, \\ x_t + b_{t-1} + z_t = \hat{y}_t, & x_t + b_{t-1} < \hat{y}_t, & \beta_{t-1} = 1, z_t > 0, \end{cases} \tag{4}$$
4. *The optimal PB policy is a base-stock policy with the optimal base-stock level $\hat{y}_t$.*

Finding the optimal order size $\hat{z}_t$ requires searching for the global minimum of the auxiliary cost function $J_t(y_t, z_t)$, which exhibits a quasiconvex shape and thus has a unique minimum. The quasiconvexity is preserved through t as the underlying single-period cost function $C(y_t, z_t)$ is also quasiconvex in z_t and function $f_{t+1}(x_{t+1}, b_t)$ for $b_t = 0$ is convex in x_t. The latter holds due to the fact that the first partial derivative of J_t with regard to z_t is independent of b_t. This means that in its general form the problem equals the stochastic capacitated inventory problem studied by Ciarallo et al. (1994), where they show the optimality of the base-stock policy.

Correspondingly, in Parts 3 and 4, we show that the inventory policy which minimizes (2) is a base-stock policy characterized by the optimal inventory position after ordering $\hat{y}_t$. The optimal policy instructs the decision maker to place a regular order only if the inventory position after replenishment of the full supply backorder is lower than the base-stock level, $x_t + b_{t-1} < \hat{y}_t$, where the size of the optimal regular order is $z_t = \hat{y}_t - x_t - b_{t-1}$. If $x_t + b_{t-1} \ge \hat{y}_t$, the regular order should not be placed and the decision maker should only seek to partially replenish the backordered supply, so that $\hat{y}_t$ is met exactly. The result confirms the intuitive thinking that one should primarily rely to a reliable replenishment of the backordered supply to attain the optimal base-stock level, and resort to regular ordering up to uncertain supply capacity only if the backordered supply is insufficient. Observe that Parts 3 and 4 also hold for *FB* policy where due to full backordering it holds that $\beta_{t-1} = 1$.

For a single decision epoch problem, $\hat{y}_t^S$ is the minimizer of the single-period cost function C_t given in (1) and can be easily obtained as it does not depend on the capacity distribution, and is thus equal to the solution of an uncapacitated single-period newsvendor problem. The optimal base-stock level in the multiperiod full backordering case is lower than or equal to the situation in which supply backordering is unfeasible. This is because the optimal inventory level does not need to protect the system against full supply unavailability, as the backordered supply is replenished, although with a delay.

Now we move to the analysis of the supply backorder cancelation option. Observe that *CO* policy only makes sense in the case in which partial replenishment of the supply backorder is not possible. We show that it may not be optimal to replenish the backordered supply fully (through *FB* policy) if by so doing the inventory position is increased too far above the optimal base-stock level. We define a threshold inventory position $\overline{y}_t$ above which it is optimal to cancel the replenishment of the supply backorder. Given the starting state x_t, $\overline{y}_t$ can be attained by replenishment of the supply backorder $\overline{b}_{t-1} = \overline{y}_t - x_t$, which we denote as a threshold supply backorder level. Part 1 of Theorem 2 states that $\overline{y}_t$ represent the point at which the costs of either solely replenishing the supply backorder and only placing an optimal regular order up to the optimal base-stock level are equal. We give a graphical representation of the threshold inventory position $\overline{y}_t = x_t + \overline{b}_{t-1}$ in Fig. 1.

Theorem 2. *Let $\hat{y}_t$ be the smallest minimizer of $J_t(y_t, z_t)$ and the starting state is (x_t, b_{t-1}):*

1. *The threshold inventory position $\overline{y}_t = x_t + \overline{b}_{t-1} \ge \hat{y}_t$, where $\overline{b}_{t-1}$ is a threshold supply backorder level, is a solution to:*
$$J_t(x_t, \overline{b}_{t-1}, 0) = J_t(x_t, 0, \hat{z}_t), \tag{5}$$
where $\hat{z}_t = \hat{y}_t - x_t$ is the optimal order size.
2. *Under the optimal CO policy, $y_t(x_t, b_{t-1})$ is given by*
$$y_t(x_t, b_{t-1}) = \begin{cases} x_t, & \hat{y}_t \le x_t, & \beta_{t-1} = 0, z_t = 0, \\ x_t + z_t = \hat{y}_t, & x_t < \hat{y}_t < \overline{y}_t \le x_t + b_{t-1}, & \beta_{t-1} = 0, z_t > 0, \\ x_t + b_{t-1}, & x_t < \hat{y}_t \le x_t + b_{t-1} < \overline{y}_t, & \beta_{t-1} = 1, z_t = 0, \\ x_t + b_{t-1} + z_t = \hat{y}_t, & x_t + b_{t-1} < \hat{y}_t, & \beta_{t-1} = 1, z_t > 0, \end{cases} \tag{6}$$
3. *The optimal CO policy is a base-stock policy with the optimal base-stock level $\hat{y}_t$, which is equal to the optimal base-stock level of the PB policy and the FB policy, for any t.*

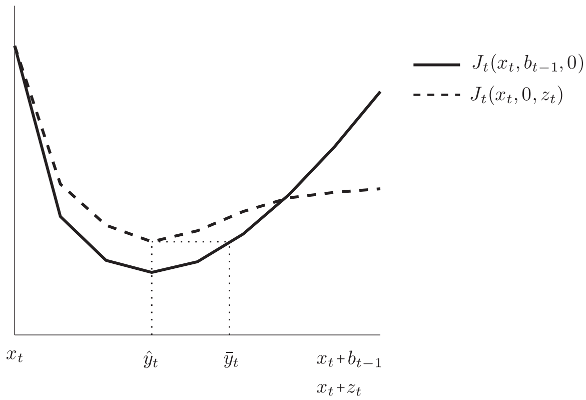

Fig. 1. The threshold inventory position $\bar{y}_t$.

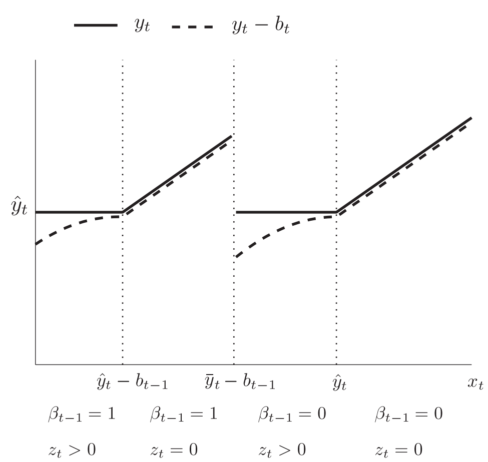

Fig. 2. Inventory position after decision making and replenishment.

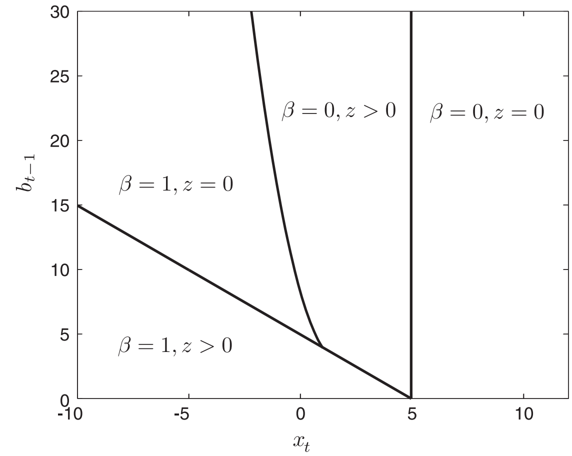

Fig. 3. Optimal ordering policy as a function of the starting state (x_t, b_{t-1}).

The optimal policy can thus be interpreted in the following way. Part 2 suggests that the optimal policy generally instructs the decision maker to increase the inventory position x_t to the optimal base-stock level $\hat{y}_t$. In the case where x_t exceeds $\hat{y}_t$, the decision maker should cancel the replenishment of the backordered supply and also not place a regular order ($\beta_{t-1} = 0, z_t = 0$). The opposite happens when the replenishment of the backordered supply is insufficient to raise the inventory position up to $\hat{y}_t$. In this situation it is optimal to replenish the supply backorder as it is replenished in full with certainty, and for the remainder place a regular order, which might be constrained depending on the available supply capacity ($\beta_{t-1} = 1, z_t > 0$).

The remaining two cases use either a regular order or replenishment of the supply backorder to increase the inventory position. Observe that placing a regular order generally results in the inventory position after replenishment $\hat{y}_t - b_t$, which is below the base-stock level due to potential capacity unavailability. On the other hand, replenishing the backordered supply overshoots the base-stock level. As Part 1 suggests, the decision maker should replenish the backordered supply if it is below the threshold size $b_{t-1} \le \bar{b}_{t-1}$, and thus the inventory position y_t will not overshoot the threshold inventory position $\bar{y}_t$. In the case where no regular order is placed ($\beta_{t-1} = 1, z_t = 0$). When $b_{t-1} > \bar{b}_{t-1}$ it is optimal to cancel the replenishment of the backordered supply and place a regular order up to the optimal base-stock level instead ($\beta_{t-1} = 0, z_t > 0$). We present an illustration of the resulting inventory positions after decision making y_t and replenishment $y_t - b_t$ in Fig. 2, and the optimal decision strategy as a function of the starting state (x_t, b_{t-1}) in Fig. 3.

In Part 3, we show that the optimal base-stock levels are the same for all three policies we have considered, which is a consequence of the fact that in all cases the optimal base-stock level is independent of the backordered supply. Therefore, the decision maker aims to target the same optimal base-stock level under all policies, where the *PB* policy through partial supply backordering offers the decision maker most flexibility. Limited “full or zero” flexibility is possible by using *CO* policy, while *FB* policy offers no flexibility in aligning the inventory position with the base-stock level. Such flexibility is worthwhile where replenishment of the backordered supply would result in excess inventory levels.

4. Benefits of supply backordering

To evaluate the benefits of supply backorder replenishment and the value of the option to cancel the backordered supply, we carried out a numerical analysis. Calculations were done by solving the dynamic programming formulation given in (2). The set of experiments was constructed based on the following base scenario: $T = 12$, $\alpha = 0.99$, and $c_b/c_h = 20$. A discrete uniform distribution was used to model stochastic demand and capacity with known independent distributions in each time period. We varied the following parameters: (1) system utilization[2] $Util = \{0.5, 0.67, 1, 2, \infty\}$; (2) the coefficient of variation of demand $CV_D = \{0, 0.14, 0.37, 0.61\}$ and supply capacity $CV_Q = \{0, 0.14, 0.37, 0.61\}$, where the CVs do not change over time[3]; and (3) the cost structure by changing the demand backorder to holding cost ratio $c_b/c_h = \{2, 20, 100\}$. Observe that the system utilization as defined above is assessed solely based on the supplier's ability to satisfy the regular order from the supplier quickly within the current period. Given that replenishment of the backordered supply is assumed to be certain, the supplier's capacity is sufficient to cover the retailers demand with a lead time of one period.

We start by exploring the dependence of the relevant system parameters on the value of supply backordering in Section 4.1. In Section 4.2, we determine the potential to decrease the costs of full backordering by having an option to partially or fully cancel the replenishment of the backordered supply.

[2] Defined as the sum of average demand over the sum of average supply capacity over the whole planning horizon.

[3] We give the approximate average CVs for demand and supply capacity distributions as it is impossible to come up with the exact same CVs for discrete uniform distributions with different means.

4.1. Value of supply backordering

The value of supply backordering is assessed based on a comparison of the capacitated inventory system where supply backordering is possible (*Partial backordering*) with the system in which the unfilled part of the supply is lost (*No backordering*). According to the labeling scheme in Table 1, we define the relative value of supply backorders $\%V_{PB}$ as the relative difference in cost of the *NB* policy and the *PB* policy, where the relative value is measured relative to the costs of the infinite capacity scenario $\%V_{Q=\infty}$. Here the uncapacitated system characterizes the best possible scenario, namely the one with the lowest possible costs, and it is therefore reasonable to assess the relative value of supply backordering $\%V_{PB}$ relative to the minimal costs of running the system:

$$\%V_{PB} = \frac{f_t^{NB} - f_t^{PB}}{f_t^{NB} - f_t^{Q=\infty}}, \tag{7}$$

where f_t^{NB}, f_t^{PB} and $f_t^{Q=\infty}$ represent the corresponding cost functions from (2) that apply to a chosen decision policy.

In addition, we define the absolute value of supply backordering ΔV_{PB} as the difference in costs between the *NB* policy and the *PB* policy:

$$\Delta V_{PB} = f_t^{NB} - f_t^{PB}. \tag{8}$$

Observe first that the optimal costs under both strategies we are comparing are increasing (or more precisely nondecreasing) with an increase in any chosen system parameter we vary: utilization, coefficient of variation of demand and capacity, and the demand backorder to holding cost ratio. Increasing c_b/c_h increases the value of supply backordering as it puts more stress on stockout avoidance, which we can achieve through use of the *PB* policy.

Looking at the results presented in Table 3 (the results for the value of supply backorders for $c_b/c_h = 20$ are presented), the predominant effect is that of system utilization, where both the relative and absolute value of supply backorders rises with an increase in utilization. $\%V_{PB}$ changes considerably over the set of experiments ranging from scenarios for some of the low utilization experiments denoted with "-", where the three strategies from (7) have the same costs, to practically 100% for high utilization. Due to supply capacity shortages the *NB* policy is unable to cope with the demand, which results in high cost mainly attributed to a high share of backordered demand. The replenishment of supply backorders effectively decreases the system's utilization through partial or full, albeit postponed, replenishment of orders. Observe that in the zero capacity situation ($Util = \infty$) the supply process under the *PB* policy is achieved solely through supply backordering.

In the case of low utilization the relative savings are still considerable at around 30–40%, particularly if demand and capacity uncertainty are also present. Here, the absolute savings are smaller, as the costs of the *NB* policy decrease to the same size class as the costs of the *PB* policy due to the less frequent capacity shortages.

While the value of supply backordering exhibits monotonic behavior with the change in the system's utilization, this is not the case when we consider the effect of demand and/or capacity uncertainty. When the utilization is high ($Util=2$), $\%V_{PB}$ decreases with the increase in demand uncertainty. The increased CV_D raises the costs of both strategies. Under the *NB* policy the primary contributor to the costs is a notorious lack of capacity that results in extended periods in which the inventory cannot be increased to the desired level. The additional negative effect of demand uncertainty is limited since the decision maker also decides on the new order size in each period depending on the actual demand realizations he observed at the end of the previous period. Under the *PB* policy the lack of capacity is tackled with supply backordering, although the ability to meet the desired optimal inventory target exactly is hampered by the high uncertainty in demand. It may happen that replenishing backordered supply might lead to too high inventory levels if the actual demand realization was low. This effect is even more profound when using *FB* policy. As the decision maker has no possibility to influence the extent of the replenishment of the backordered supply this results in too high inventory levels after the replenishment. For lower utilizations, $\%V_{PB}$ generally increases with CV_D. Here, more stockouts are the result of the target inventory level being insufficient to cover the unusually high demand, and not due to the capacity shortage. The supply backorders are smaller and it is therefore less likely that the replenishment of the supply backorder will be counterproductive in the low demand periods.

Table 3
The value of supply backordering.

			$\%V_{PB}$				ΔV_{PB}			
Util	CV_Q	CV_D	0.00	0.14	0.37	0.61	0.00	0.14	0.37	0.61
∞			100.0	100.0	100.0	100.0	638.5	652.2	696.3	761.7
2	0.00		100.0	100.0	99.8	98.3	319.2	335.1	381.5	436.7
2	0.14		96.9	96.9	98.3	97.6	326.8	331.4	379.3	435.9
2	0.37		91.6	92.6	94.6	95.0	341.5	341.9	379.7	434.3
2	0.61		87.4	88.6	90.8	91.9	361.9	359.7	387.9	437.6
1	0.00		–	85.8	86.9	88.0	0.0	39.3	99.5	160.9
1	0.14		75.7	71.4	81.8	85.4	45.7	50.5	102.9	162.2
1	0.37		71.5	71.4	73.8	78.5	114.9	110.4	133.1	179.6
1	0.61		70.4	71.7	72.6	74.3	183.5	178.4	186.2	215.4
0.67	0.00		–	–	42.3	73.2	0.0	0.0	0.3	18.1
0.67	0.14		–	–	34.4	62.0	0.0	0.0	2.2	21.5
0.67	0.37		34.3	37.7	46.9	56.6	19.0	18.8	29.4	51.6
0.67	0.61		53.3	54.8	56.0	58.8	94.8	90.2	91.1	107.2
0.5	0.00		–	–	–	–	0.0	0.0	0.0	0.0
0.5	0.14		–	–	–	20.1	0.0	0.0	0.0	0.2
0.5	0.37		12.3	21.4	31.1	41.5	2.5	2.9	5.9	14.0
0.5	0.61		38.7	41.0	43.1	46.5	54.0	51.6	51.7	60.7

In general, ΔV_{PB} increases as demand uncertainty increases. This is the case both in situations where capacity restrictions are prevalent, as well as when stockouts are occurring because of high demand volatility. In both cases, the *PB* policy resolves the stockout quickly, preventing it from extending into future periods.

The value of supply backordering is also sensitive to changes in capacity uncertainty. ΔV_{PB} is increased when capacity uncertainty rises. This monotonic behavior is intuitively clear as the probability of capacity shortages that can be resolved through supply backordering goes up with an increase in CV_Q. However, $\%V_{PB}$ exhibits non-monotonic characteristics by either increasing or decreasing with CV_Q, depending on the utilization, and a combination of demand and capacity uncertainty. For lower system utilizations, we observe that $\%V_{PB}$ is increasing with the uncertainty in capacity. We attribute this to the fact that the capacity shortages are a result of low capacity periods, which are more likely to occur when the uncertainty in capacity is high (as the expected capacity is generally high enough). On the contrary, when the system is highly utilized, high capacity uncertainty does not greatly contribute to the likelihood of a capacity shortage and therefore the relative benefit of the supply backordering diminishes.

We may conclude that the value of supply backordering is higher when the demand and supply capacity mismatches can be effectively resolved through use of the *PB* policy. The mismatches are caused by capacity unavailability and/or demand volatility, and since the occurrence of either of these greatly depends on the system parameters, we observe complex non-monotonic behavior of the value of supply backordering.

4.2. Value of the cancelation option

In this section, we want to analyze the potential to reduce the costs of the *FB* policy by allowing the decision maker to partially or fully cancel the replenishment of the backordered supply. It is reasonable to assume that by using either the *PB* policy or the *CO* policy the decision maker has more flexibility in attaining the target inventory levels, which would lead to better system performance.

Due to the optimality of the base-stock policy, the target is the optimal base-stock level, which is a function of the system parameters: future demand and capacity distributions, and the demand backorder to holding cost ratio. The cancelation option will be exercised if the replenishment of the supply backorder would place us above the optimal base-stock level in the case of the *PB* policy, and in the case of the *CO* policy when we would exceed the threshold inventory position.

In a non-stationary setting, the optimal base-stock level is changing depending on the future demand and capacity characteristics. The periods in which the optimal base-stock level decreases are those where cancelation of the supply backorder might be beneficial as it is likely that replenishment of the backordered supply could exceed the base-stock level. Note that in an infinite horizon stationary setting the optimal base-stock level is constant. Therefore, it is always rational to replenish the supply backorder as the replenished inventory never places us above the optimal base-stock level (even when there is no demand). However, in the finite horizon setting the optimal base-stock level is decreasing towards the end of the planning horizon to the newsvendor optimal base-stock level, giving the potential to reduce the costs by exercising the cancelation option.

In the same fashion as in Section 4.1, we define the relative value of the cancelation option based on the comparison of the *PB* policy and the *CO* policy to the *FB* policy:

$$\%V_{PB,CO} = \frac{f_t^{FB} - f_t^{PB,CO}}{f_t^{FB} - f_t^{Q=\infty}},$$

Table 4
The value of the cancelation option.

			$\%V_{PB}$				$\%V_{CO}$			
Util	CV_Q	CV_D	0.00	0.14	0.37	0.61	0.00	0.14	0.37	0.61
Inf			–	100.0	100.0	100.0	–	0.0	19.2	40.6
2	0.00		–	100.0	97.9	86.8	–	7.2	44.6	47.4
2	0.14		0.0	53.7	84.9	81.9	0.0	8.6	40.0	45.2
2	0.37		0.0	26.2	56.6	64.1	0.0	4.5	25.2	35.6
2	0.61		0.0	12.8	36.5	47.3	0.0	1.5	14.0	24.9
1	0.00		–	36.1	34.6	39.0	–	24.0	20.1	23.3
1	0.14		0.0	4.0	20.1	30.2	0.0	2.9	11.4	19.0
1	0.37		0.0	1.3	4.8	13.6	0.0	0.4	2.8	8.1
1	0.61		0.0	0.4	3.2	5.9	0.0	0.1	1.3	3.6
0.67	0.00		–	–	2.0	16.2	–	–	2.0	10.3
0.67	0.14		–	–	0.0	2.5	–	–	0.0	2.5
0.67	0.37		0.0	0.0	0.1	1.6	0.0	0.0	0.0	1.3
0.67	0.61		0.0	0.0	0.6	1.5	0.0	0.0	0.2	0.8
0.5	0.00		–	–	–	–	–	–	–	–
0.5	0.14		–	–	–	0.0	–	–	–	0.0
0.5	0.37		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.61		0.0	0.0	0.1	0.5	0.0	0.0	0.0	0.2

We present the results of a numerical experiment in Table 4, and based on those we make the following observations.

The higher the system utilization the higher the relative value of the cancelation option. For the *PB* policy, $\%V_{PB}$ reaches up to 100% for the two high utilization scenarios, where we see that the performance approaches that of an infinite capacity setting. $\%V_{CO}$ reaches up to 50%, which means the cost difference between the *FB* policy and that of running the uncapacitated system, is halved by using the *CO* policy. As the period-to-period optimal base-stock levels need to be high enough to cover potential future supply shortages, they are sensitive to changes in future period-to-period utilization. The sensitivity is increased for higher average system utilization, thus the swings in the base-stock levels become larger. This gives more potential to reduce the costs through implementation of the policy that allows supply backorder cancelations. In addition, the higher the utilization, the more the optimal base-stock level decreases towards the end of the planning horizon, which increases the probability that the full replenishment of the supply backorder is not rational.

Similar behavior is observed when one considers the effect of demand uncertainty on the value of the cancelation option. The higher the demand uncertainty the higher the value of the cancelation option. When demand volatility is high, low demand periods are more likely which may subsequently lead to situations where replenishment of the supply backorder would place us above the optimal base stock level in the case of the *PB* policy, or above the threshold inventory position in the case of the *CO* policy, which would result in unwanted inventory holding costs.

According to the above, we could anticipate that, through its effect on changes in the period-to-period optimal base-stock levels, the capacity uncertainty would also increase the value of the cancelation option. However, the opposite can be observed in Table 4 for higher utilizations in general and also in the low utilization region when demand uncertainty is sufficiently high. Although the potential to exercise the cancelation option is higher due to changes in the optimal base-stock level, it becomes less likely that the decision maker will actually exercise it when capacity uncertainty is high. This is because the regular replenishment becomes highly uncertain, making the system vulnerable to demand backorders. Due to this, the optimal base stock level and the threshold inventory position are increased, making it more likely that the decision maker will replenish the backordered supply and thus not exercise the cancelation option. When we have a combination of low utilization and low demand uncertainty the costs are either equal to the costs of the *FB* policy ($\%V_{PB,CO} = 0$), or the costs of the two strategies equal those of an uncapacitated system ($\%V_{PB,CO} = '' - ''$).

To summarize, the system parameters that describe the setting in which the highest cost reduction through partial or full cancelation of backordered supply can be attained are high utilization, high demand uncertainty and low capacity uncertainty.

5. Trade-off between the supply uncertainty and lead time delay

We base the discussion on the trade-off between supply uncertainty and lead time delay on a comparison of the *FB* policy and the so called *Early ordering* policy, which we will denote briefly as the *EO* policy. *EO* policy is a special case of the *FB* policy, where no supply capacity is available, $Q_t = 0$ for all t. As full backordering is assumed, the supply is backordered in full and replenished with a delay, in the next period. In fact, it is easy to see that the *EO* policy behaves equivalently to the uncapacitated inventory system with a lead time of one period.

The relevant question we pose is whether it is better to accept one-period lead time with certain delivery, or use the *FB* policy where the order might not be fully replenished in the current period. If, in certain settings, the *EO* policy is superior to *FB*, one could conclude that lower supply capacity availability leads to better performance. While this is counterintuitive at first, there is a simple explanation of this phenomenon when we consider a deterministic supply backordering setting. Due to the perfect information on future demand realizations, the retailer is able to implement optimal ordering policy that results in zero end-of-period net stock and therefore zero costs. To avoid demand backorders, he copes with the periods of limited supply by prebuilding stock in advance, however this results in inventory holding costs. Therefore, a better option for him would be if a supplier delivers the full order with a delay of one period, as this means that the supplier incurs the inventory holding cost when waiting for the inventory to accumulate to the full order size. Accordingly, a retailer that deals with a responsive yet unreliable supplier is worse off. As the future demand is fully predictable, it is rational for the retailer to negotiate an arrangement with the supplier that would allow him to use the optimal early ordering strategy.

To assess which policy performs better in which settings, we performed a numerical analysis using the same set of system parameters as in Section 4. We present the results in Fig. 4.

Based on the above arguments, it is reasonable to assume that the *EO* policy incurs lower costs when demand uncertainty is low. When CV_D is sufficiently increased the *FB* policy becomes favorable. The exposure of the *EO* policy to demand uncertainty is higher with a longer lead time as we are only able to react to demand changes with a delay.

With the *FB* policy's regular ordering, fast reaction to demand changes is possible, but the extent of the reaction is constrained by the available capacity. The ability to react to changes in demand by correspondingly adjusting the inventory position is limited due to the constant supply shortages. Therefore, when CV_Q is increasing, a delayed certain replenishment guaranteed by the early ordering policy is a better option because the fulfillment of regular orders becomes more uncertain. With early ordering the target inventory levels are easily attained as there is no uncertainty in supply. But, as we observed above, when also faced with demand uncertainty these targets do not correspond that well to the state of the system since recent demand information is not revealed yet.

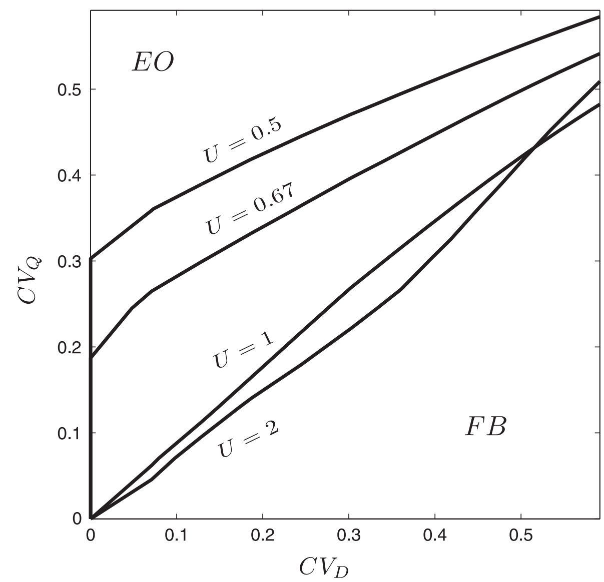

Fig. 4. Optimal decision policy as a function of system parameters.

When the combined effect of demand and capacity uncertainty is considered, we believe that the more chaotic the system becomes, the more likely it is that the *EO* policy will incur lower costs. We attribute this to the fact that in such a system the supply shortages result in frequent high supply backorders which, together with volatile demand, result in situations in which replenishment of the backordered supply might be counterproductive. This effect can be mitigated to some extent by applying the *PB* or *CO* policy instead of full backordering, but the cost reduction potential is limited for high capacity uncertainty.

With increasing system utilization the performance of the *FB* policy compared to the *EO* policy deteriorates because increasing utilization enhances the likelihood of capacity shortages and this causes problems in effectively managing the inventory levels. However, when utilization is increased further, the *FB* policy essentially faces the same conditions we have assumed the *EO* policy applies to.

6. Conclusions

In this paper we establish the optimal inventory control policies for a finite horizon stochastic capacitated inventory system in which the unfilled part of an order is backordered at the supplier and delivered either partially or in full in the following period; a concept we denote as supply backordering. We developed a dynamic programming model that incorporates the notion of supply backordering for the partial supply backordering policy. We have further analyzed the two sub-policies, full backordering policy in which full replenishment of the backordered supply is assumed, and the policy in which the option to cancel the replenishment of the supply backorder is available. For all policies, we show that the structure of the optimal inventory policy is a base-stock policy where the base-stock levels are equal in all three cases. The optimal base-stock levels are thus independent of the starting inventory position and the backordered supply from the previous period. In addition, we characterize the threshold inventory position above which it is optimal to fully cancel the replenishment of the backordered supply and place a new order instead.

By means of numerical analysis we quantify the cost reduction potential through supply backordering and develop managerial insights. In general, the causes of the demand and capacity mismatches lie in capacity unavailability, either due to an inherent lack of capacity or its variability, and/or demand volatility. The possibility of replenishing the unfilled part of the orders effectively increases the capacity availability and enables the decision maker to better cope with periods of high capacity utilization. We show that the relative cost savings can be substantial, ranging from around 30% for moderate system utilization to practically 100% in the case of a highly utilized system. The nature of the causes of the mismatches also depends greatly on the extent of demand and capacity uncertainty, where we observe the complex non-monotonic behavior of the value of supply backordering. The observed complex interaction of the system parameters requires the decision maker to consider them in an integrated manner.

We show that the supply backorder needs to be canceled if the threshold inventory position would be exceeded by replenishing the supply backorder fully. Supply backorder cancelation is a rational decision particularly when the current period's demand is relatively small and replenishment of the supply backorder would lead to unnecessary high inventory levels, and correspondingly to high inventory holding costs. We establish the following conditions in which exercising the cancelation option is optimal: high system utilization, high demand uncertainty and low capacity uncertainty.

Finally, we give some insights into the trade-off between supply uncertainty and lead time delay, which corresponds to a managerial dilemma of whether to rely more heavily on regular orders or on delayed albeit full order replenishment. Contrary to what one might think, an overutilized system may actually lead to a better performance as it enables the decision maker to reap the benefits of the certain delivery of the backordered supply by placing orders early. On the other hand, regular ordering is exposed to the capacity shortages particularly when capacity uncertainty is high. Due to the early placement of the order, the latest demand information is not yet revealed, when the early order is placed. Thus the cost reduction potential by using the early ordering strategy is higher in the presence of low demand uncertainty and high capacity uncertainty.

A natural extension to our model would be to include fixed ordering costs associated with placing regular orders and to assume non-zero lead time setting. We anticipate that the optimality results under fixed ordering costs would correspond to the results for random capacity setting without supply backordering, where it was shown that the optimal policy is of an (s,S) type. In addition, the supply backorder cancelation policy would need to be adapted by correspondingly increasing the threshold supply backorder level over which it is optimal to cancel its replenishment. Based on the insights from the dual sourcing models, we can predict that the optimal policy for the case of positive lead times that differ for only one period $(L, L+1)$ is of an order-up-to type, and remains relatively simple without complex state dependency.

Appendix A

In Lemma 1 we provide insights into the single decision epoch problem and build some prerequisites to study the multiperiod case. We elect to suppress subscript t in the state variables for clarity reasons.

Lemma 1. *Let $\hat{y}^S$ be the smallest minimizer of (1) and the starting state is (x, b_{-1}), where $x+b_{-1} < \hat{y}^S$ and correspondingly the optimal order size is $\hat{z}^S = \hat{y}^S - x - \beta_{-1} b_{-1}$:*

1. *$C(x, b_{-1})$ is convex in x and in b_{-1}.*
2. *$C(x, b_{-1}, z)$ is quasiconvex in z, where:*
$$\frac{\partial^2}{\partial z^2} C(x, b_{-1}, z) \geq 0, \text{ for } z \leq \hat{z}^S,$$
$$\frac{\partial}{\partial z} C(x, b_{-1}, z) \geq 0, \text{ for } z > \hat{z}^S.$$
3. *$\hat{y}^S$ is the optimal myopic base-stock level, where $\hat{y}^S = G^{-1}(c_b/(c_b+c_h))$.*

Proof. We first rewrite the single decision epoch cost function $C_t(y_t, z_t)$ given in (1), as a function of the inventory position before decision making x, supply backorder b_{-1} and the regular order z:

$$\begin{aligned} C(x, b_{-1}, z) = (1-R(z)) \bigg[c_b \int_{x+\beta_{-1}b_{-1}+z}^{\infty} (D - x - \beta_{-1} b_{-1} - z) g(D) \mathrm{d}D \\ + c_h \int_0^{x+\beta_{-1}b_{-1}+z} (x+\beta_{-1}b_{-1}+z-D) g(D) \mathrm{d}D \bigg] \\ + c_b \int_0^z \int_{x+\beta_{-1}b_{-1}+Q}^{\infty} (D - x - \beta_{-1} b_{-1} - Q) g(D) \mathrm{d}D r(Q) \mathrm{d}Q \\ + c_h \int_0^z \int_0^{x+\beta_{-1}b_{-1}+Q} (x+\beta_{-1}b_{-1}+Q-D) g(D) \mathrm{d}D r(Q) \mathrm{d}Q. \end{aligned} \tag{A.1}$$

The first of the three terms represents the case where the supply capacity is higher than the order size, meaning that the supply capacity is not limiting the order placed, while the last two terms account for the case where the capacity constrains the order. To prove Part 1, we derive the first partial derivative of (A.1) with respect to x, b_{-1}, and the sum $x+b_{-1}$:

$$\begin{aligned} \frac{\partial}{\partial x} C(x, b_{-1}, z) = \frac{\partial}{\partial b_{-1}} C(x, b_{-1}, z) = \frac{\partial}{\partial (x+b_{-1})} C(x, b_{-1}, z) = \\ = (c_b + c_h)(1-R(z)) \left[G(x+\beta_{-1}b_{-1}+z) - \frac{c_b}{c_b+c_h} \right] \end{aligned} \tag{A.2}$$

$$+ (c_b + c_h) \int_0^z G(x+\beta_{-1}b_{-1}+Q) r(Q) \mathrm{d}Q - c_b R(z), \tag{A.3}$$

and the second partial derivative:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} C(x, b_{-1}, z) = \frac{\partial^2}{\partial b_{-1}^2} C(x, b_{-1}, z) = \frac{\partial^2}{\partial (x+b_{-1})^2} C(x, b_{-1}, z) \\ = (c_b + c_h) \bigg[\int_0^z g(x+\beta_{-1}b_{-1}+Q) r(Q) \mathrm{d}Q \end{aligned} \tag{A.4}$$

$$+ (1-R(z)) g(x+\beta_{-1}b_{-1}+z) \bigg]. \tag{A.5}$$

Since all terms in (A.5) are nonnegative, Part 1 holds.

Similarly for Part 2, we obtain the first two partial derivatives of (1) with respect to z:

$$\frac{\partial}{\partial z} C(x, b_{-1}, z) = (c_b + c_h)(1-R(z)) \left[G(x+\beta_{-1}b_{-1}+z) - \frac{c_b}{c_b+c_h} \right], \tag{A.6}$$

$$\begin{aligned} \frac{\partial^2}{\partial z^2} C(x, b_{-1}, z) = (c_b + c_h) \bigg[(1-R(z)) g(x+\beta_{-1}b_{-1}+z) \\ - r(z) \left(G(x+\beta_{-1}b_{-1}+z) - \frac{c_b}{c_b+c_h} \right) \bigg]. \end{aligned} \tag{A.7}$$

Observe first that setting (A.6) to 0 proves Part 3. For $z \leq \hat{z}^S$, $G(x+\beta_{-1}b_{-1}+z) \leq c_b/(c_b+c_h)$ holds in the second term of (A.7). Since the first term is always nonnegative, the function $C(x, b_{-1}, z)$ is convex on the respected interval. For $z > \hat{z}^S$ this does not hold, although we see from (A.6) that due to $G(x+\beta_{-1}\ b_{-1}+z) > c_b/(c_b+c_h)$ and $1-R(z) \to 0$ with z, the function $C(x, b_{-1}, z)$ is nondecreasing, which proves Part 2. Due to this, the $C(x, b_{-1}, z)$

has a quasiconvex form, which is sufficient for $\hat{y}$ to be its global minimizer. □

To keep the remaining proofs more concise, we elect to suppress the time indices related to demand and capacity probability distributions and their realizations, although we keep them for the description of the state variables.

Proof of Theorem 1:. We rewrite the auxiliary cost function $J_t(y_t, z_t)$ defined in (3) as

$$J_t(x_t, b_{t-1}, z_t) = C_t(x_t, b_{t-1}, z_t) + \alpha \int_0^\infty \int_0^{z_t} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + Q - D, b_t) r(Q)\, dQ g(D)\, dD + \alpha(1 - R(z_t)) \int_0^\infty f_{t+1}(x_t + \beta_{t-1} b_{t-1} + z_t - D, 0) g(D)\, dD. \quad \text{(A.8)}$$

Observe here that the last term denotes the expected optimal costs of period $t+1$ in the case where supply capacity is sufficient to cover order z_t, which in turn means there is no supply being backordered, $b_t = 0$.

The first partial derivative of (A.8) with respect to regular order z_t is

$$\frac{\partial}{\partial z_t} J_t(x_t, b_{t-1}, z_t) = \frac{\partial}{\partial z_t} C_t(x_t, b_{t-1}, z_t) + \alpha(1 - R(z_t)) \times \int_0^\infty \frac{\partial}{\partial x_{t+1}} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + z_t - D, 0) g(D) dD, \quad \text{(A.9)}$$

where $\partial x_{t+1}/\partial z_t \partial/\partial x_{t+1} f_{t+1} = \partial/\partial x_{t+1} f_{t+1}$ for $b_t = 0$. By setting $\partial/\partial z_t J_t$ to 0 and using (A.6), we obtain the expression for the optimal regular order $\hat{z}_t$:

$$(1 - R(\hat{z}_t)) \left[(c_b + c_h) \left(G(x_t + \beta_{t-1} b_{t-1} + \hat{z}_t) - \frac{c_b}{c_b + c_h} \right) + \alpha \int_0^\infty \frac{\partial}{\partial x_{t+1}} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + \hat{z}_t - D, 0) g(D) dD \right] = 0. \quad \text{(A.10)}$$

The second partial derivative is as follows:

$$\frac{\partial^2}{\partial z_t^2} J_t(x_t, b_{t-1}, z_t) = \frac{\partial^2}{\partial z_t^2} C_t(x_t, b_{t-1}, z_t) + \alpha(1 - R(z_t)) \int_0^\infty \frac{\partial^2}{\partial x_{t+1}^2} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + z_t - D, 0) g(D) dD - \alpha r(z_t) \int_0^\infty \frac{\partial}{\partial x_{t+1}} f_{t+1}(x_t + \beta_{t-1} b_{t-1} + z_t - D, 0) g(D) dD. \quad \text{(A.11)}$$

Note that $\partial/\partial z_t J_t$ is not a function of b_t, and therefore obtaining $\hat{z}_t$ requires solving (A.10) at $b_t = 0$. This is important since the cost function $f_{t+1}(x_{t+1}, b_t)$ given in (2) is not convex but, as Part 1 suggests, $f_{t+1}(x_{t+1}, 0)$ is convex, which allows us to find its global minimizer easily.

Looking at the expressions (A9) and (A11) we have developed based on the analysis of the *PB* policy, one can see that they correspond in form to their equivalents in a stochastic capacitated inventory problem analyzed by Ciarallo et al. (1994), where the inventory position before ordering is expressed as the sum $x_t + \beta_{t-1} b_{t-1}$.

For further details of the proofs for Parts 2 and 3, we direct the reader to their paper. □

Lemma 2.

1. *$J_t(x_t, b_{t-1}, z_t)$ is convex in x_t, b_{t-1} and $x_t + b_{t-1}$.*
2. *$\hat{y}_t$, a solution to (A.10), is also a solution to $\partial/\partial b_{t-1} J_t(x_t, b_{t-1}, 0) = 0$ for $\beta_{t-1} = 1$ and $z_t = 0$.*

Proof. The proof of Part 1 follows the same lines as the proof of Part 1 of Theorem 1 and the relevant steps from Ciarallo et al. (1994).

In Part 2 we need to establish that $\hat{y}_t$ is also the optimal base-stock level in the case where the supply backorder is replenished $\beta_{t-1} = 1$, and no order is placed $z_t = 0$. In this instance, the first partial derivative of (A.10) with respect to b_{t-1} is as follows:

$$\frac{\partial}{\partial b_{t-1}} J_t(x_t, b_{t-1}, 0) = \frac{\partial}{\partial b_{t-1}} C_t(x_t, b_{t-1}, 0) + \alpha \int_0^\infty \frac{\partial}{\partial x_{t+1}} f_{t+1}(x_t + b_{t-1} - D, 0) g(D)\, dD, \quad \text{(A.12)}$$

where $\partial x_{t+1}/\partial b_{t-1} \partial/\partial x_{t+1} f_{t+1} = \partial/\partial x_{t+1} f_{t+1}$ for $b_t = 0$. By applying (A.3) for $z_t = 0$ and setting $\partial/\partial b_{t-1} J_t$ to 0, we obtain the following first order optimality condition:

$$\left[(c_b + c_h) \left(G(x_t + b_{t-1}) - \frac{c_b}{c_b + c_h} \right) + \alpha \int_0^\infty \frac{\partial}{\partial x_{t+1}} f_{t+1}(x_t + b_{t-1} - D, 0) g(D)\, dD \right] = 0, \quad \text{(A.13)}$$

where $\hat{b}_{t-1}$ is the size of the supply backorder by which x_t is increased to $\hat{y}_t$. Observe that the solution to (A.13) is the same as a solution to (A.10): $\hat{y}_t = x_t + \hat{b}_{t-1} = x_t + b_{t-1} + \hat{z}_t$. This means that we always want to target the same optimal base-stock level $\hat{y}_t$, irrespective of the size of the supply backorder from the previous period. To finally show that the optimal policy is a base-stock policy we need to repeat the same steps as in Part 3 of Theorem 1. □

Proof of Theorem 2. Part 1 states that the threshold inventory position $\overline{y}_t$ represents the point at which the costs of exercising the supply backorder $\overline{b}_{t-1}$ and placing an optimal regular order $\hat{z}_t$ are equal. Except for the case of $J_t(x_t, b_{t-1}, 0)$ and $J_t(x_t, 0, z_t)$ being equal for all y_t, we show that two such points exist where $J_t(x_t, \overline{b}_{t-1}, 0) = J_t(x_t, 0, \hat{z}_t)$ (Fig. 1). We are interested in finding the one that lies above the base-stock level, as the one below the base-stock level can be improved through placement of the regular order.

We show this by studying the auxiliary cost functions $J_t(x_t, b_{t-1}, 0)$ and $J_t(x_t, 0, z_t)$ for the cases where $y_t < \hat{y}_t$ and $y_t \geq \hat{y}_t$.

Case 1 for $y_t < \hat{y}_t$: From Part 1 of Lemma 2 we know that $J_t(x_t, b_{t-1}, 0)$ is convex in b_{t-1} and from Part 2 of Theorem 1 $J_t(x_t, 0, z_t)$ is convex in z_t on the respective interval. Looking at their first partial derivatives given in (A12) and (A9), we see that both are decreasing up to $\hat{y}_t$, where they both attain the minimum due to Part 2 of Lemma 2. In addition, we see that $\partial/\partial b_{t-1} J_t(x_t, b_{t-1}, 0) \leq \partial/\partial z_t J_t(x_t, 0, z_t)$ due to $1 \geq 1 - R(z_t)$, which means that $J_t(x_t, b_{t-1}, 0)$ is decreasing at a higher rate. This, and the fact that both functions have the same value $J_t(x_t, 0, 0)$ at $y_t = x_t$, are sufficient conditions to conclude that $J_t(x_t, b_{t-1}, 0) \leq J_t(x_t, 0, z_t)$ on $y_t < \hat{y}_t$.

Case 2 for $y_t \geq \hat{y}_t$: Part 1 of Lemma 2 states that $J_t(x_t, b_{t-1}, 0)$ is convex and Part 2 of Theorem 1 that $J_t(x_t, 0, z_t)$ is quasiconvex. Both functions are therefore nondecreasing functions on the respective interval. This, together with the findings for Case 1 above, is sufficient to show that $J_t(x_t, b_{t-1}, 0)$, at some $y_t \geq \hat{y}_t$, increases above the minimum of $J_t(x_t, 0, \hat{z}_t)$, attained at $\hat{y}_t$. We have denoted this y_t as the threshold inventory position $\overline{y}_t$.

In Part 3 we need to show first that the structure of the optimal policy is that of a base-stock policy. From Part 2 of Lemma 2, which also holds for general z_t, we follow the lines of proof from Part 3 of Theorem 1 to establish a general structure of the policy. The equivalence of the base-stock levels for the *FB*, *PB* and *CO* policies results from the abovementioned generalization of Part 2 of Lemma 2. Part 2 directly follows from this and Part 1 above. □

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