

# A colour problem for infinite graphs and a problem in the theory of relations

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MATHEMATICS

A COLOUR PROBLEM FOR INFINITE GRAPHS AND A PROBLEM  
IN THE THEORY OF RELATIONS

BY

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(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of November 24, 1951)

Theorems 1, 3 and 4 of this paper were announced in a previous paper of one of us [1]. As related problems were discussed there, and references were given, we present our theorems without any introduction.

The Axiom of Choice is adopted throughout the paper.

§ 1. A graph  $G$  is called  $k$ -colourable if to each vertex one of a given set of  $k$  colours can be attached in such a way that on each edge the two end-points get different colours.

*Theorem 1. Let  $k$  be a positive integer, and let the graph  $G$  have the property that any finite subgraph is  $k$ -colourable. Then  $G$  is  $k$ -colourable itself.*

Our original proof was simplified by SZEKERES. Later, a simple proof, based on Tychonoff's theorem that the cartesian product of a family of compact sets is compact, was indicated by RABSON and A. STONE. We suppress these proofs here, since theorem 1 can be considered as a special case of a theorem of R. RADO which appeared meanwhile [3], and a topological proof for Rado's theorem was given by GOTTSCHALK [2].

*Theorem 2 (RADO). Let  $M$  and  $M_1$  be arbitrary sets. Assume that to any  $v \in M_1$  there corresponds a finite subset  $A_v$  of  $M$ . Assume that to any finite subset  $N \subset M_1$  a choice function  $x_N(v)$  is given, which attaches an element of  $A_v$  to each  $v \in N$ :*

$$x_N(v) \in A_v.$$

*Then there exists a choice function  $x(v)$  defined for all  $v \in M_1$  ( $x(v) \in A_v$  if  $v \in M_1$ ) with the following property. If  $K$  is any finite subset of  $M_1$ , then there exists a finite subset  $N(K \subset N \subset M_1)$ , such that, as far as  $K$  is concerned, the function  $x(v)$  coincides with  $x_N(v)$ :*

$$x(v) = x_N(v) \quad (v \in K)$$

We now deduce theorem 1 from theorem 2. Let  $M$  be the set of  $k$  colours, and let  $M_1$  be the set of all vertices of  $G$ . We always choose  $A_v = M$ . To any finite  $N(N \subset M_1)$  there corresponds a finite subgraph of  $G$ , consisting of the vertices belonging to  $N$ , and all connections between

these vertices as far as these belong to  $G$ . This subgraph is assumed to be  $k$ -colourable, and so we have a function  $x_N(v)$ , defined for  $v \in N$ , taking its values in  $M$ . Now the function  $x(v)$  defines a colouration of the whole graph  $G$ . In order to show that opposite ends of any edge get different colours, we consider an arbitrary edge  $e$ , and we denote the set of its two end-points  $v_1, v_2$  by  $K$ . Let  $N$  be a finite set satisfying  $K \subset N \subset M_1$ ,  $x(v) = x_N(v)$  ( $v \in K$ ). To  $N$  there corresponds a finite graph  $G_N$  which is  $k$ -colourable by the function  $x_N(v)$ ;  $G_N$  contains  $e$ . Therefore  $x_N(v_1) \neq x_N(v_2)$ , and so  $x(v_1) \neq x(v_2)$ . This proves theorem 1.

As to Rado's theorem one could raise the following question. In the statement of theorem 2 the words "finite subset" occur four times. Is it allowed to replace these simultaneously by "subset of power  $< m$ ", where  $m$  is an infinite cardinal? Naturally we may take  $m = \aleph_0$ , but we may not take  $m = \aleph_1$ . A counterexample is readily obtained from the ingenious counterexample which SPECKER [4] gave to a problem of SIKORSKI.

§ 2. We shall apply theorem 1 to a problem in the theory of relations. Let  $S$  be a set, and assume that to every element  $b \in S$  a subset  $f(b) \subset S - b$  is given.  $|f(b)|$  denotes the number of elements of  $f(b)$ . Two elements  $b$  and  $c$  ( $b \in S, c \in S$ ) are called independent if  $b \in S - f(c)$  and  $c \in S - f(b)$  both hold. A subset  $S_1$  of  $S$  is called an independent set if any two elements of  $S_1$  are independent.  $S_1$  is also called independent if  $|S_1| = 0$  or 1.

**Theorem 3.** *Let  $k$  be a non-negative integer, and assume that  $|f(b)| \leq k$  for each  $b \in S$ . Then  $S$  is the union of  $2k + 1$  independent sets.*

*Proof.* First assume  $S$  to be finite. We proceed by induction with respect to  $|S|$ . The case  $|S| = 1$  is trivial. Assume the theorem to be true for  $|S| = m - 1$ ; next consider  $|S| = m$ .

Construct a graph  $G$  whose vertices are the elements of  $S$ . The vertices  $b$  and  $c$  are connected in  $G$  if  $b \in f(c)$ , and also if  $c \in f(b)$ .

The number of edges is at most  $km$ , and so there exists a vertex  $d$  which is connected with less than  $2k + 1$  vertices. By the induction hypothesis,  $S - d$  is the union of  $2k + 1$  independent sets. It follows that  $d$  is independent of all elements of at least one of these independent sets; hence  $d$  can be added to that set without disturbing independence. This proves the theorem for finite  $|S|$ .

The division of  $S$  into  $2k + 1$  independent sets can be interpreted as  $(2k + 1)$ -colourability of the graph  $G$ , and vice versa. Now theorem 1 immediately shows that theorem 3 holds true if  $S$  is infinite.

**Theorem 4.** *If  $f(b)$  is finite for each  $b \in S$ , then  $S$  is the union of a countable number of independent sets.*

*Proof.* Define  $S_k$  as the set of all  $b \in S$  for which  $|f(b)| = k$ . Then  $S = S_0 + S_1 + S_2 + \dots$ , and to each  $S_k$  we can apply theorem 3.

## REFERENCES

1. ERDÖS, P., Some remarks on set theory. Proc. Amer. Math. Soc. **1**, 127—141 (1950).
2. GOTTSCHALK, W. H., Choice functions and Tychonoff's theorem. Proc. Amer. Math. Soc. **2**, 172 (1951).
3. RADO, R., Axiomatic treatment of rank in infinite sets. Canad. J. Math. **1**, 337—343 (1949).
4. SPECKER, E., Sur un problème de Sikorski. Colloquium Mathematicum **2**, 9—12 (1949).

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