A colour problem for infinite graphs and a problem in the theory of relations

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MATHEMATICS

A COLOUR PROBLEM FOR INFINITE GRAPHS AND A PROBLEM IN THE THEORY OF RELATIONS

BY

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Theorems 1, 3 and 4 of this paper were announced in a previous paper of one of us [1]. As related problems were discussed there, and references were given, we present our theorems without any introduction.

The Axiom of Choice is adopted throughout the paper.

§ 1. A graph G is called k-colourable if to each vertex one of a given set of k colours can be attached in such a way that on each edge the two end-points get different colours.

Theorem 1. *Let k be a positive integer, and let the graph G have the property that any finite subgraph is k-colourable. Then G is k-colourable itself.*

Our original proof was simplified by SZEKERES. Later, a simple proof, based on Tychonoff's theorem that the cartesian product of a family of compact sets is compact, was indicated by RABSON and A. STONE. We suppress these proofs here, since theorem 1 can be considered as a special case of a theorem of R. RADO which appeared meanwhile [3], and a topological proof for Rado's theorem was given by GOTTSCHALK [2].

Theorem 2 (RADO). *Let M and M_1 be arbitrary sets. Assume that to any $\nu \in M_1$ there corresponds a finite subset A_ν of M. Assume that to any finite subset $N \subset M_1$ a choice function $x_N(\nu)$ is given, which attaches an element of A_ν to each $\nu \in N$:*

$$x_N(\nu) \in A_\nu.$$

Then there exists a choice function $x(\nu)$ defined for all $\nu \in M_1$ ($x(\nu) \in A_\nu$ if $\nu \in M_1$) with the following property. If K is any finite subset of M_1, then there exists a finite subset $N (K \subset N \subset M_1)$, such that, as far as K is concerned, the function $x(\nu)$ coincides with $x_N(\nu)$:

$$x(\nu) = x_N(\nu) \qquad (\nu \in K)$$

We now deduce theorem 1 from theorem 2. Let M be the set of k colours, and let M_1 be the set of all vertices of G. We always choose $A_\nu = M$. To any finite $N (N \subset M_1)$ there corresponds a finite subgraph of G, consisting of the vertices belonging to N, and all connections between these vertices as far as these belong to G. This subgraph is assumed to be k-colourable, and so we have a function $x_N(\nu)$, defined for $\nu \in N$, taking its values in M. Now the function $x(\nu)$ defines a colouration of the whole graph G. In order to show that opposite ends of any edge get different colours, we consider an arbitrary edge e, and we denote the set of its two end-points ν_1, ν_2 by K. Let N be a finite set satisfying $K \subset N \subset M_1$, $x(\nu) = x_N(\nu)$ $(\nu \in K)$. To N there corresponds a finite graph G_N which is k-colourable by the function $x_N(\nu)$; G_N contains e. Therefore $x_N(\nu_1) \neq \neq x_N(\nu_2)$, and so $x(\nu_1) \neq x(\nu_2)$. This proves theorem 1.

As to Rado's theorem one could raise the following question. In the statement of theorem 2 the words "finite subset" occur four times. Is it allowed to replace these simultaneously by "subset of power $< m$", where m is an infinite cardinal? Naturally we may take $m = \aleph_0$, but we may not take $m = \aleph_1$. A counterexample is readily obtained from the ingenious counterexample which Specker [4] gave to a problem of Sikorski.

§ 2. We shall apply theorem 1 to a problem in the theory of relations. Let S be a set, and assume that to every element $b \in S$ a subset $f(b) \subset S - b$ is given. $|f(b)|$ denotes the number of elements of $f(b)$. Two elements b and c $(b \in S,\ c \in S)$ are called independent if $b \in S - f(c)$ and $c \in S - f(b)$ both hold. A subset S_1 of S is called an independent set if any two elements of S_1 are independent. S_1 is also called independent if $|S_1| = 0$ or 1.

Theorem 3. *Let k be a non-negative integer, and assume that $|f(b)| \leqslant k$ for each $b \in S$. Then S is the union of $2k + 1$ independent sets.*

Proof. First assume S to be finite. We proceed by induction with respect to $|S|$. The case $|S| = 1$ is trivial. Assume the theorem to be true for $|S| = m - 1$; next consider $|S| = m$.

Construct a graph G whose vertices are the elements of S. The vertices b and c are connected in G if $b \in f(c)$, and also if $c \in f(b)$.

The number of edges is at most km, and so there exists a vertex d which is connected with less than $2k + 1$ vertices. By the induction hypothesis, $S - d$ is the union of $2k + 1$ independent sets. It follows that d is independent of all elements of at least one of these independent sets; hence d can be added to that set without disturbing independence. This proves the theorem for finite $|S|$.

The division of S into $2k + 1$ independent sets can be interpreted as $(2k + 1)$-colourability of the graph G, and vice versa. Now theorem 1 immediately shows that theorem 3 holds true if S is infinite.

Theorem 4. *If $f(b)$ is finite for each $b \in S$, then S is the union of a countable number of independent sets.*

Proof. Define S_k as the set of all $b \in S$ for which $|f(b)| = k$. Then $S = S_0 + S_1 + S_2 + \ldots$, and to each S_k we can apply theorem 3.

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