A nonexistence proof for 4-error-correcting codes

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A Nonexistence Proof for 4-Error-Correcting Codes

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1. Definitions

Let $S$ be a set of $q$ symbols and $V := S^n$. For $x \in V$, $y \in V$ define the Hamming distance $d(x,y)$ to be the number of coordinates in which $x$ and $y$ differ.

Let

$$S_e(x) := \{ z \in V | d(z,x) \leq e \}.$$

A perfect $e$-error-correcting code is a subset $C \subseteq V$ such that the $S_e(x)$ ($x \in C$) form a partition of $V$.

2. Conditions

Necessary conditions for the existence of perfect codes are:

a) the sphere packing condition:

$$\left( 1 + n(q - 1) + \binom{n}{2}(q - 1)^2 + \ldots + \binom{n}{e}(q - 1)^e \right) | q^n$$

b) the polynomial condition (see [1], [2]):

$$P_e(x) = \sum_{i=0}^{e} (-1)^i \frac{n-x}{e-i} (x-1)^{e-i} (q - 1)^{e-i}$$

has $e$ different integral zeros among $1, 2, 3, \ldots, n$.

3. Previous results

It was proved by A. Tietävainen (see [5], [6]) that if a perfect $e$-code on $q$ symbols exists and $e \geq 3$, then $q$ is divisible by at least three distinct primes.

In this paper we reduce $P_4(X)$ to a cubic polynomial which can be treated in a way like Van Lint did in [3] and I did in [4] to prove that there is no unknown perfect 3-code.
4. The case \( e = 4 \)

In the following we shall prove that there does not exist a perfect four-error-correcting code, except maybe if \((n, q)\) belongs to a certain set of pairs, with cardinality 100 or so, some of whose elements \((n, q)\) maybe satisfy the sphere packing condition. In Lemma 3 we suppose that this is not true.

**Lemma 1.** Let \( P(Z) := Z^4 + pZ^2 + rZ + s \) be a polynomial in \( \mathbb{C}[Z] \) with four integral zeros. Then \( Q(Z) \) has three integral zeros, where

\[
Q(Z) := Z^3 - pZ^2 - 4sZ + 4ps - r^2.
\]

**Proof.** Let \( P(Z) \) have integral zeros \( z_1, z_2, z_3, z_4 \). Then we can write \( p, r \) and \( s \) as the symmetric expressions:

\[
\begin{align*}
p &= z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 \\
r &= z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 \\
s &= z_1 z_2 z_3 z_4
\end{align*}
\]

Now define

\[
Q(Z) := (Z - y_1)(Z - y_2)(Z - y_3),
\]

where

\[
\begin{align*}
y_1 &= z_1 z_2 + z_3 z_4 \\
y_2 &= z_1 z_3 + z_2 z_4 \\
y_3 &= z_2 z_3 + z_1 z_4
\end{align*}
\]

Then it is straightforward to show that

\[
\begin{align*}
y_1 + y_2 + y_3 &= p \\
y_1 y_2 + y_1 y_3 + y_2 y_3 &= -4s \\
y_1 y_2 y_3 &= r^2 - 4ps
\end{align*}
\]

Moreover, \( y_1, y_2 \) and \( y_3 \) are integers. \[\square\]
Theorem. A perfect four-error-correcting code does not exist.

Proof. Assume that there exists such a code, with parameters $n$, $c$, $q$.

Then by $\theta := qx - n(q - 1)$ and $z := 2\theta + 3q - 8$ the Lloyd polynomial $P_4(x)$ is transformed into $T_4(z) := z^4 + pz^2 + rz + s$, where $p$, $r$ and $s$ shall not be mentioned.

Following Lemma 1, we find that $Q_3(z)$ must have three integral zeros, where

$$Q_3(z) := z^3 - pz^2 - 4sz + 4ps - r^2.$$ 

Since the coefficient of $(n - 4)^3$ in $Q_3(z)$ is independent of $z$ we substitute

$$2y := z + 24(q - 1)(n - 4)$$

and find that $F(Y)$ must have three integral zeros, where

$$F(Y) := a_2(Y)(n - 4)^2 + a_1(Y)(n - 4) + a_0(Y),$$

and

$$a_2(Y) := 3Y + 11q^2 + 16q - 16$$
$$a_1(Y) = -24(q - 1)(Y + 5q^2)(Y + q^2 + 4q - 4)$$
$$a_0(Y) = (Y - 3q^2)(Y + 3q^2)(Y + 5q^2).$$

Now if $Y_0 = -\frac{1}{3}(11q^2 + 16q - 16)$, we have: $a_2(Y_0) = 0$ and

$$a_2(Y_0 - \frac{1}{3}) = -32(q - 1)^2,$$

and

$$72q^4(q - 1) < a_1(Y) < 88q^4(q - 1)$$

for $Y = Y_0$ and $Y = Y_0 - \frac{1}{3}$, and

$$0 < q_0(Y) < 8q^6 \text{ for } Y = Y_0 \text{ and } Y = Y_0 - \frac{1}{3}.$$ 

Then we find:

$$F(Y_0) > 72q^4(q - 1)(n - 4) > 0$$
$$F(Y_0 - \frac{1}{3}) < -32(q - 1)^2(n - 4)^2 + 88(q - 1)q^4(n - 4) + 8q^6.$$
So

\[ F(Y_0 - \frac{1}{3}) < 0 \text{ if } n - 4 \geq \frac{14}{5} \frac{q^4}{q - 1}. \]

Therefore, if \( n - 4 \geq \frac{14}{5} \frac{q^4}{q - 1} \), there must be an integral zero of \( F(Y) \) in the open interval \((Y_0 - \frac{1}{3}, Y_0)\).

Since this interval does not contain an integer we find that \( n - 4 < \frac{14}{5} \frac{q^4}{q - 1} \).

Now we shall see in the following Lemmas 2 and 3 that this is impossible too. Hence we proved the theorem.

Lemma 2. Suppose that there exists a perfect four-error-correcting code with word length \( n - 4 < \frac{14}{5} \frac{q^4}{q - 1} \), and let \( q = 2^k 3^\ell q' \), and \( \gcd(q', 6) = 1 \).

Then we have the following diagram of possibilities:

\[
\ell \begin{array}{c|cccc}
\ell = 0 & k = 0 & k = 1 & k = 2 & k \geq 3 \\
\ell = 1 & q < 4 & q < 46 & q < 718 & q < 7 \\
\ell \geq 1 & q < 10 & q < 136 & q < 2152 & q < 18
\end{array}
\]

Proof. Let \( x_1, x_2, x_3, x_4 \) be the zeros of the Lloyd polynomial \( P_4(x) \).

Then the following expressions must be integers:

(i) \( x_1 + x_2 + x_3 + x_4 = \frac{4(n - 4)(q - 1)}{q} + 10 \)

(ii) \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{4(n - 4)^2 + 20(n - 4) + 30}{q^2} - \frac{4(n - 4)}{q^2}((2q - 1)(n - 3) + 4) \)

(iii) \( x_1^3 + x_2^3 + x_3^3 + x_4^3 = \frac{4(n - 4)^3 + 30(n - 4)^2 + 90(n - 4) + 100}{q^3} - \frac{n - 4}{q^3}((n - 4)^2(12q^2 - 12q + 4) + (n - 4)(24q^2 + 42q - 36) + (12q^2 + 54q + 24)) \)

(iv) \( P_4(1) = (n - 1)(n - 2)(n - 3)(n - 4)(\frac{q - 1}{q})^4 \)

Then if \( 3 \mid q \) we see from (i) that \( 3 \mid n - 4 \).

So

\[ \{ (n - 4)^2(12q^2 - 12q + 4) + (n - 4)(24q^2 + 42q - 36) + (12q^2 + 54q + 24) \} \]
has exactly one factor 3.
Then, since $27 | q^3$, we see from (iii) that $9 | n - 4$.

Furthermore, if $8 | q$, we see from (i) that $2 | n - 4$.
So $((2q - 1)(n - 3) + 4)$ is odd.
Then, since $64 | q^2$, we see from (ii) that $4 | n - 4$.

Hence, since from (i) and (iv)
$q | 4(n - 4)$ and
$q^4 | (n - 1)(n - 2)(n - 3)(n - 4)$,
we can make the following diagram of possibilities, with $A$ chosen in such a way that $q^4 | A(n - 4)$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$= 0$</td>
<td>$A = 2$</td>
<td>$A = 16$</td>
<td>$A = 256$</td>
</tr>
<tr>
<td>$A$</td>
<td>$A = 3$</td>
<td>$A = 48$</td>
<td>$A = 768$</td>
<td>$A = 16$</td>
</tr>
</tbody>
</table>

Now is each of the cases listed in this diagram we have the condition

$$\frac{q^4}{A} \leq n - 4 < \frac{14q^4}{5(q-1)}$$

So we have the diagram of possibilities listed in the lemmas statement.

**Lemma 3.** The several values of $q$ listed in the statement of Lemma 2 are impossible too.

**Proof.** This can be checked by computer.
First of all, the $q$ with only one or two prime divisors are excluded (since these are impossible following the results of Tietavainen).
Second, with each $q$ is associated the (often unique) $n$ such that

$$q^4 | A(n - 4)$$

and $n - 4 < \frac{14q^4}{5(q-1)}$.

Finally for the remaining pairs $(n, q)$ it has to be verified that $n$ and $q$ do not satisfy the sphere packing condition.
Acknowledgements

Thanks to prof. Van Lint for some valuable suggestions, and for checking the calculations.

References


[6] A. Tietävainen; Nonexistence of nontrivial perfect codes in case $q = p_1^{r_1} p_2^{r_2}$, $e \geq 3$.
Submitted to Discrete Mathematics.