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On the \(\mathcal{NP}\)-hardness of solving Bilinear Matrix Inequalities and simultaneous stabilization with static output feedback

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Abstract
In this paper, it is shown that the problem of checking the solvability of a bilinear matrix inequality (BMI), is \(\mathcal{NP}\)-hard. A matrix valued function, \(F(X,Y)\), is called bilinear if it is linear with respect to each of its arguments, and an inequality of the form, \(F(X,Y) > 0\) is called a bilinear matrix inequality. Recently, it was shown that, the static output feedback problem, fixed order controller problem, reduced order \(H^\infty\) controller design problem, and several other control problems can be formulated as BMIs \([4, 3]\). Based on the results of \([4, 3]\), BMIs seem to be a potentially powerful tool for the formulation of some important control problems. But the main result of this paper shows that the problem of checking the solvability of BMIs is \(\mathcal{NP}\)-hard, and hence it is rather unlikely to find a polynomial time algorithm for solving general BMI problems. As an independent result, it is also shown that simultaneous stabilization with static output feedback is an \(\mathcal{NP}\)-hard problem, namely given \(n\) plants, the problem of checking the existence of a static gain matrix, \(K\), which stabilizes all of the \(n\) plants, is \(\mathcal{NP}\)-hard.

1. Introduction
In this paper, it is shown that the problem of checking the solvability of a general bilinear matrix inequality is \(\mathcal{NP}\)-hard (See \([2]\) for the definition of the complexity class \(\mathcal{NP}\)-hard and related issues). A matrix valued function \(F(X,Y)\) is called bilinear if it is linear with respect to each of its arguments, and an inequality of the form \(F(X,Y) > 0\) is called a bilinear matrix inequality. The main result of this paper is the \(\mathcal{NP}\)-hardness of the problem of checking the existence of \(X\), \(Y\) such that \(F(X,Y) > 0\). Results of \([4, 3]\) show that some important control problems, including the static output feedback problem, fixed order controller problem, and reduced order \(H^\infty\) controller design problems, can be formulated using BMIs, and hence BMIs seem to be a potentially powerful tool for solving these control problems. But the \(\mathcal{NP}\)-hardness of checking the solvability of BMIs imply that, it is rather unlikely to find a polynomial time solution procedure for general BMI problems. This result does not eliminate the possibility of conservative approaches or efficient solution procedures for some special type of BMI problems, but shows that the general BMI problem is \(\mathcal{NP}\)-hard. As an independent result, it is also shown that simultaneous stabilization with static output feedback is \(\mathcal{NP}\)-hard, namely given \(n\) plants, the problem of checking the existence of a static gain matrix, \(K\), which stabilizes all of the \(n\) plants, is \(\mathcal{NP}\)-hard. This result does not give much information about the complexity of the static output feedback problem of a single plant, but shows that the this rather reliably version of simultaneous stabilization problem \([1, 5]\) is \(\mathcal{NP}\)-hard.

2. Solvability of BMIs
In this section, it is shown that the following version of the Knapsack problem \([2]\), is \(\mathcal{NP}\)-hard.

**Lemma 1:** Given \(n\) integers \(a_1, ..., a_n \in \mathbb{Z}\), the problem of checking the existence of \(n\) real numbers \(x_1, ..., x_n \in \mathbb{R}\), such that

\[
1 - \alpha < |x_k| < 1 + \alpha, \quad k = 1, ..., n, \quad \text{and} \quad |a_1 x_1 + ... + a_n x_n| < \beta,
\]

where \(\alpha = \frac{1}{5n(1+\sum_{k=1}^n |a_k|)}\), and \(\beta = \frac{1}{5}\), is \(\mathcal{NP}\)-hard.

**Proof:** For a given vector, \(z = [a_1, ..., a_n]^T \in \mathbb{Z}^n\), the above problem has a solution if there exists a vector, \(x \in \{-1, +1\}^n\) satisfying \(a^T x = 0\).

Conversely, if the above problem has a solution, then

\[
x = \bar{x} + \Delta x, \quad \text{where} \quad \bar{x}_k \in \{-1, 1\}, \quad \|\Delta x\|_\infty < \alpha.
\]

Note that, \(a^T \bar{x}\) is an integer, \(|a^T \bar{x}| \leq |a^T x| + |a^T \Delta x|\), and

\[
|a^T \Delta x| \leq \|a\|_2 \|\Delta x\|_2 \leq \|a\|_\infty \|\Delta x\|_\infty \leq \frac{1}{5},
\]

\[
|a^T x| \leq \frac{1}{5},
\]

therefore \(a^T \bar{x} = 0\).

This shows that the above problem has a solution iff the integer Knapsack problem \([2]\) has a solution. By the \(\mathcal{NP}\)-hardness of the integer Knapsack problem, \(\mathcal{NP}\)-hardness of the above problem follows.

The following lemma shows that, given an affine matrix subspace, the problem of checking the existence of a stable matrix in that affine subspace, is \(\mathcal{NP}\)-hard.

**Lemma 2:** Given \((n+1)\) real matrices, \(M_0, ..., M_n\), the problem of checking the existence of \(n\) real numbers \(x_1, ..., x_n\), such that \(M_0 + x_1 M_1 + ... + x_n M_n\) is stable, is \(\mathcal{NP}\)-hard.

**Proof:** For a given vector, \(a = [a_1, ..., a_n]^T \in \mathbb{Z}^n\), define \(\alpha = \frac{1}{5n(1+\sum_{k=1}^n |a_k|)}\), and \(\beta = \frac{1}{5}\),

\[
A_{1,k}(x) = -(1 + \alpha) - x_k, \quad A_{2,k}(x) = -(1 + \alpha) + x_k,
\]

\[
A_{3,k} = \begin{bmatrix}
\frac{3}{5} (1-\alpha)^2 & -x_k \\
-x_k & -2
\end{bmatrix}, \quad k = 1, ..., n,
\]

and \(A_r(x) = \text{diag}(A_{1,r}(x), ..., A_{n,r}(x))\), \(r = 1, 2, 3\). Note that, the existence of a real vector, \(x\), such that

\[
M(x) = \text{diag}(A_1(x), A_2(x), A_3(x)), -\beta - a^T x, -\beta + a^T x),
\]

is stable, is equivalent to the solvability of the problem of Lemma 1. Since \(M(x)\) characterizes an affine matrix subspace, by the \(\mathcal{NP}\)-hardness result of...
Lemma 1, it follows that the problem of checking the existence of a stable matrix in a given affine matrix subspace, is an \(NP\)-hard problem. □

**Remark 1:** Given \(n\) real matrices, \(M_1, \ldots, M_n\), the problem of checking the existence of \(n\) real numbers \(x_1, \ldots, x_n\), such that \(x_1M_1 + \cdots + x_nM_n\) is stable, is also \(NP\)-hard. Because, given \((n+1)\) matrices \(M_0, \ldots, M_n\), define \(N_k = \text{diag}(M_k, -k, 0)\), for \(k = 0, \ldots, n\). Then, there exists \(n\) real numbers \(x_1, \ldots, x_n\), such that \(M_0 + x_1M_1 + \cdots + x_nM_n\) is stable, if and only if there exists \((n+1)\) real numbers \(x_0\), such that \(x_0N_0 + \cdots + x_nN_n\) is stable. By the \(NP\)-hardness result of Lemma 2, it follows that, given a matrix subspace, the problem of checking the existence of a stable matrix in that subspace, is \(NP\)-hard too.

**Theorem 1:** Given a matrix valued bilinear form \(F(\cdot, \cdot)\), the problem of checking the existence of \((n)\) matrices \(X, Y\) such that \(F(X, Y) > 0\), is \(NP\)-hard.

**Proof:** Given \((n+1)\) matrices \(M_0, \ldots, M_n\), define \(M(x) = M_0 + x_1M_1 + \cdots + x_nM_n\), \(N(x) = x_0M_0 + \cdots + x_nM_n\), \(S(P) = (P + PT)/2\), and
\[
F(x, P) = \begin{bmatrix} -N(x)^T S(P) - S(P)N(x) & 0 \\ 0 & x_0S(P) \end{bmatrix}
\]
Then if there exists a real \(x\) and \(P\) such that \(F(x, P) > 0\), then \(x_0 \neq 0\), and \(\frac{1}{x_0}N(x)\) is stable, therefore there exist \(n\) real numbers \(x_1, \ldots, x_n\), such that \(M_0 + x_1M_1 + \cdots + x_nM_n\) is stable. Conversely, if there exist \(n\) real numbers, then the BMI, \(F(x, P) > 0\), is solvable. This shows that the problem only reduces to the problem of solvability of BMIs, therefore checking the solvability of a BMI is an \(NP\)-hard problem.

3. **Simultaneous stabilization with static output feedback**

In this section, it is shown that given \(n\) plants, the problem of checking the existence of a (real) static gain matrix, \(K\), which stabilizes all of the \(n\) plants, is \(NP\)-hard. Simultaneous stabilizability of \(n\) plants with static output feedback, is rationally decidable [1, 5], but Theorem 2 shows that this problem is \(NP\)-hard, namely, although the problem can be solved (possibly in doubly exponential amount of time) using only rational operations, it is rather unlikely to find a polynomial time solution procedure for that problem. This result does not give information about the computational complexity of the static output feedback problem of a single plant, but shows that the simultaneous version of the problem is \(NP\)-hard.

**Theorem 2:** Given \(n\) plants with state space realizations, \((A_k, B_k, C_k)\), \(k = 1, \ldots, n\), the problem of checking the existence of a (real) static gain matrix, \(K\), such that \(A_k + BK\) is stable for \(k = 1, \ldots, n\), is \(NP\)-hard.

**Proof:** For a given vector, \(a = [a_1 \ldots a_n]^T \in \mathbb{Z}^n\), define \(\alpha = \frac{a_1}{\sqrt{n+1+\sum_{k=1}^n |a_k|^2}}\), and \(\beta = \frac{1}{\alpha}\),
\[
A_{1,k}(x) = -(1 + \alpha)x_k, A_{2,k}(x) = -(1 + \alpha) + x_k,
A_{3,k}(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\beta x_k & -1 - x_k & 1 - x_k \end{bmatrix},
\]
for \(k = 1, \ldots, n\), and \(A_4(x) = -\beta - x^T x, A_5(x) = -\beta + x^T x\). Note that \(A_{3,k}\) is stable if \(0 < x_k < \frac{1}{\alpha}\) or \(x_k > 2\). Therefore, there exists a real vector, \(x\), such that
\[
A_{1,k}(x), A_{2,k}(x), A_{3,k}(\frac{3}{4(1-\alpha)}x + \frac{5}{4}), A_{4,k}(x), A_{5,k}(x),
\]
are stable for \(k = 1, \ldots, n\), if the problem of Lemma 1 has a solution. Furthermore, the above \(A(x)\) functions are of the form \(A + BK\) for some real \(A, B, C\), and \(K = x\). Therefore, given \(n\) plants, the problem of checking the existence of a static gain matrix which stabilizes all of the \(n\) plants, is \(NP\)-hard.

4. **Concluding remarks**

In this paper, it was shown that the problem of checking the solvability of a general bilinear matrix inequality, is \(NP\)-hard. Although BMIs can be used to formulate some important control problems [4, 3], and hence seems to be a potentially powerful tool, the main result of this paper shows that it is rather unlikely to find both efficient and nonconservative solution procedures for general BMI problems. As an independent result, it was also shown that, simultaneous stabilization with static output feedback problem is also \(NP\)-hard. This shows that simultaneous version of the static output feedback problem is \(NP\)-hard.

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