Coverings by rook domains

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0. Abstract

The following inequalities and values for coverings by rook domains are proved:

(i) \[ \sigma(1 + t \frac{q^{r-1} - 1}{q - 1}, q) \leq (q - t + 1)q^{k-r}; \]

here \( q \) is a prime and \( k = 1 + t \frac{q^{r-1} - 1}{q - 1} \).

(ii) \[ \sigma(n,kt) \leq \sigma(n,k)t^{n-1} \]

for any \( n, k \) and \( t \).

(iii) \[ \sigma(q+1,qt) = q^{q-1} t^q \]

for any prime power \( q \) and any \( t \).

1. Introduction

Let \( V = (V^n_k,d) \) denote the metric space of all \( n \)-tuples \((a_1,a_2,\ldots,a_n)\) with \( a_i \in \{1,2,\ldots,k\} \) provided with the Hamming distance:

\[ d(a,b) = |\{i \mid a_i \neq b_i\}|. \]

A subset \( W \) of \( V \) is called a covering (by rook-domains) if each point of \( V \) is at distance \( \leq 1 \) from some point in \( W \).

We are interested in bounds on the number of points in a minimal covering of \( V \), to be denoted by \( \sigma(n,k) \). Points of \( W \) will be called rooks, the sphere of radius 1 around a rook a rook-domain. Since each rook-domain contains \( 1 + n(k-1) \) points we get \( \sigma(n,k) \geq \frac{k^n}{1 + n(k-1)} \). Equality can be
obtained if $k$ is a prime power and $1 + n(k-1)|k$. E. Rodemich [1] proved that this bound can be improved to $\sigma(n,k) \geq \frac{k^{n-1}}{n-1}$ in the case $k \geq n$.

2. A generalization of the bounds of van Lint and Kamps

A trivial observation is that $\sigma(n+1,k) \leq k\sigma(n,k)$. This observation, combined with $\sigma(4,3) = 3^2$ yields $\sigma(13,3) \leq 3^{11}$, but actually $\sigma(13,3) = 3^{10}$. It is natural therefore to study the behaviour of $\sigma(n,k)$ in between. In [2] J.H. van Lint and H.J.L. Kamps proved $\sigma(9,3) \leq 2 \cdot 3^6$. We will now demonstrate a technique which generalizes their construction.

Let $A = (a_1, a_2, \ldots, a_k)$ be a matrix with $k$ columns and $r$ linearly independent rows, with $a_i \in \mathbb{F}_q^r$ where $q$ is a prime. Let $S$ be a set of points in $\mathbb{F}_q^r$ such that $\{s + \alpha a_i | s \in S, \alpha \in \mathbb{F}_q, 1 \leq i \leq k\} = \mathbb{F}_q^r$.

**Lemma.** $W := \{w \in \mathbb{F}_q^k | Aw \in S\}$ is a covering of $V_q^k = \mathbb{F}_q^k$ and $|W| = |S| \cdot q^{k-r}$.

**Proof.** Take $x \in \mathbb{F}_q^k$, then $Ax \in \mathbb{F}_q^r$, so we may write $Ax = s + \alpha a_i$. Let $e_i = (0,0,\ldots,1,0\ldots0)$ denote the $i^{th}$ unit vector in $\mathbb{F}_q^k$, then $A(x - \alpha e_i) = s \in S$ hence $x - \alpha e_i \in W$, and $d(x,W) \leq 1$.

**Application**

$$
A = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 1 & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & t
\end{pmatrix}
$$

$$
S = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
0 & 0 & 0 \\
0 & 1 & q-t
\end{pmatrix}
$$
the columns of $A$ are all projective $(r-1)$-vectors over $\mathbb{F}_q$, repeated $t$ times, with last coordinates $1,2,\ldots,t$ together with the vector $(0,0,\ldots,0,1)^T$, so $k = 1 + t \frac{q^{r-1} - 1}{q - 1}$.

It is easily checked, using the pigeonhole principle, that the pair $A,S$ satisfies the conditions, hence

$$\sigma(k,q) \leq (q - t + 1)q^{k-r}.$$ 

4. A sequence of cases meeting the Rodemich bound

**Theorem.** $\sigma(n,kt) \leq \sigma(n,k)t^{n-1}$. 

**Proof.** Let $W$ be a covering of $V^t_n$. Regard $V^t_n$ as obtained from $V^n_k$ by replacing each point by $V^t_n$ and give $V^t_n$ coordinates as follows:

For $a = (a_1,a_2,\ldots,a_n) \in V^n_k$ and $b = (b_1,b_2,\ldots,b_n) \in V^t_n$ the point in position $b$ of the set $V^t_n$ replacing $a$ gets coordinates

$$(a_1 - 1)t + b_1, (a_2 - 1)t + b_2,\ldots,(a_n - 1)t + b_n).$$

Now for each rook in $W$ fill the corresponding set $V^t_n$ with $t^{n-1}$ rooks placed at the points $(x_1,x_2,\ldots,x_n)$ satisfying $x_1 + x_2 + \ldots + x_n \equiv 0 \pmod{t}$. It is easy to verify that the set of rooks thus defined covers $V^n_k$.

**Corollary.** If $q$ is a prime power then $\sigma(q+1,qt) = q^{q-1}t^q$.

**Proof.** Since $\sigma(q+1,q) = q^{q-1}$ by the Hamming bound we get $\sigma(q+1,qt) \leq q^{q-1}t^q$. Rodemich's equality however, gives $\sigma(q+1,qt) \geq q^{q-1}t^q$. 
References
