Coverings by rook domains

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0. Abstract

The following inequalities and values for coverings by rook domains are proved:

(i) \( \sigma(1 + t \frac{q-1}{q-1}, q) \leq (q - t + 1)q^{k-r} \); 
here \( q \) is a prime and \( k = 1 + t \frac{q-1}{q-1} \).

(ii) \( \sigma(n, k) \leq \sigma(n, k)t^{n-1} \) for any \( n, k \) and \( t \).

(iii) \( \sigma(q+1, qt) = q^{q-1}t^q \) for any prime power \( q \) and any \( t \).

1. Introduction

Let \( V = (V^n_k, d) \) denote the metric space of all \( n \)-tuples \((a_1, a_2, \ldots, a_n)\) with \( a_i \in \{1, 2, \ldots, k\} \) provided with the Hamming distance:
\( d(a, b) = |\{i \mid a_i \neq b_i\}|. \) A subset \( W \) of \( V \) is called a covering (by rook-domains) if each point of \( V \) is at distance \( \leq 1 \) from some point in \( W \).

We are interested in bounds on the number of points in a minimal covering of \( V \), to be denoted by \( \sigma(n, k) \). Points of \( W \) will be called rooks, the sphere of radius 1 around a rook a rook-domain. Since each rook-domain contains \( 1 + n(k-1) \) points we get \( \sigma(n,k) \geq \frac{k^n}{1 + n(k-1)} \). Equality can be
obtained if \( k \) is a prime power and \( 1 + n(k-1)/k \). E. Rodemich [1] proved that this bound can be improved to \( \sigma(n,k) \geq \frac{k^{n-1} - n - 1}{n-1} \) in the case \( k \geq n \).

2. A generalization of the bounds of van Lint and Kamps

A trivial observation is that \( \sigma(n+1,k) \leq k \sigma(n,k) \). This observation, combined with \( \sigma(4,3) = 3^2 \) yields \( \sigma(13,3) \leq 3^4 \), but actually \( \sigma(13,3) = 3^0 \). It is natural therefore to study the behaviour of \( \sigma(n,k) \) in between. In [2] J.H. van Lint and H.J.L. Kamps proved \( \sigma(9,3) \leq 2 \cdot 3^6 \). We will now demonstrate a technique which generalizes their construction.

Let \( A = (a_1, a_2, \ldots, a_k) \) be a matrix with \( k \) columns and \( r \) linearly independent rows, with \( a_i \in \mathbb{F}_q \) where \( q \) is a prime. Let \( S \) be a set of points in \( \mathbb{F}_q^r \) such that \( \{s + \alpha a_i | s \in S, \alpha \in \mathbb{F}_q, 1 \leq i \leq k\} = \mathbb{F}_q^r \).

**Lemma.** \( W := \{\mathbf{w} \in \mathbb{F}_q^k | A\mathbf{w} \in S\} \) is a covering of \( V_q^k = \mathbb{F}_q^k \) and \( |W| = |S| \cdot q^{-k+r} \).

**Proof.** Take \( \mathbf{x} \in \mathbb{F}_q^k \), then \( A\mathbf{x} \in \mathbb{F}_q^r \), so we may write \( A\mathbf{x} = s + \alpha a_i \). Let \( e_i = (0,0,\ldots,1,0\ldots,0) \) denote the \( i \)th unit vector in \( \mathbb{F}_q^k \), then \( A(\mathbf{x} - \alpha e_i) = s \in S \) hence \( \mathbf{x} - \alpha e_i \in W \), and \( d(\mathbf{x}, W) \leq 1 \).

**Application**

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 1 & \cdot & \cdot \\
1 & 1 & t & 1 & \ldots t \\
\end{pmatrix}
\]

\[
S = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
0 & 0 & 0 \\
0 & 1 & \cdot & q-t \\
\end{pmatrix}
\]
the columns of $A$ are all projective $(r-1)$-vectors over $\mathbb{F}_q$, repeated $t$ times, with last coordinates $1, 2, \ldots, t$ together with the vector $(0, 0, \ldots, 0, 1)^T$, so $k = 1 + t \frac{q^{r-1} - 1}{q-1}$.

It is easily checked, using the pigeonhole principle, that the pair $A, S$ satisfies the conditions, hence

$$\sigma(k, q) \leq (q - t + 1)q^{k-r}.$$ 

4. A sequence of cases meeting the Rodemich bound

**Theorem.** $\sigma(n, kt) \leq \sigma(n, k)t^{n-1}$.

**Proof.** Let $W$ be a covering of $V^n_k$. Regard $V^n_{kt}$ as obtained from $V^n_k$ by replacing each point by $V^n_{kt}$ and give $V^n_{kt}$ coordinates as follows:

For $a = (a_1, a_2, \ldots, a_n) \in V^n_{kt}$ and $b = (b_1, b_2, \ldots, b_n) \in V^n_t$ the point in position $b$ of the set $V^n_t$ replacing $a$ gets coordinates

$$(a_1 - 1)t + b_1, \ (a_2 - 1)t + b_2, \ldots, (a_n - 1)t + b_n).$$

Now for each rook in $W$ fill the corresponding set $V^n_t$ with $t^{n-1}$ rooks placed at the points $(x_1, x_2, \ldots, x_n)$ satisfying $x_1 + x_2 + \ldots + x_n \equiv 0 \pmod{t}$. It is easy to verify that the set of rooks thus defined covers $V^n_{kt}$.

**Corollary.** If $q$ is a prime power then $\sigma(q+1, qt) = q^{q-1}t^q$.

**Proof.** Since $\sigma(q+1, q) = q^{q-1}$ by the Hamming bound we get $\sigma(q+1, qt) \leq q^{q-1}t^q$. Rodemich's equality however, gives $\sigma(q+1, qt) \geq q^{q-1}t^q$. 


References


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