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Let $E$ be a Banach Space with norm $\| \cdot \|$. A subset $A \subseteq E \times E$ is said to be an $\omega$-accretive operator in $E$ if

$$
\| (x_1 - x_2) + t(y_1 - y_2) \| \geq (1 - \omega t) \| x_1 - x_2 \| ,
$$

\[ \forall [x_i, y_i] \in A, \ i = 1, 2, \ \forall t > 0, \ \omega t < 1 \tag{1} \]

where $\omega$ is some real number. Usually a $0$-accretive operator is just called an accretive operator. Any $\omega$-accretive operator $A$ with $\omega < 0$ is known as a strongly accretive operator. Note that, a subset $A$ of $E \times E$ is an $\omega$-accretive operator if and only if for each $[x_i, y_i]$, $i = 1, 2$, there exists an $f \in F(x_1 - x_2)$ (F is the normalized (multi-valued) duality mapping) such that

$$
\omega \| x_1 - x_2 \|^2 + (y_1 - y_2, f) \geq 0. \tag{2}
$$

Here $(\cdot, \cdot)$ denotes the duality pairing of $E$ and $E^*$. The well known Crandall-Liggett Theorem ([1]) asserts that, an accretive operator $A$ in $E$ satisfying the range condition

$$
R(I + tA) \supset D(A), \ \forall t > 0 \tag{3}
$$

generates via the exponential formula

$$
S(t)x = \lim_{n \to \infty} J^{n t/n}x, \ x \in D(A), \ t \geq 0 \tag{4}
$$

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\((J_t = (I + tA)^{-1}\) is the resolvent) an \(\omega\)-contractive semigroup \(S\) on \(D(A)\), that is
\[
\|S(t)x - S(t)y\| \leq e^{\omega t}\|x - y\|, \quad \forall x, y \in D(A), \quad t \geq 0.
\]  

In a Hilbert Space \(H\) it is known that a strongly accretive operator \(A \subset H \times H\) satisfying the range condition (3) has a unique zero point \(x^*\) and for any \(x \in D(A)\), \(\lim_{t \to \infty} S(t)x = x^*\), \(S(t)x^* = x^*\), \(\forall t \geq 0\).  

See Pazy ([3]). It seems that in General Banach Spaces there are no such results on asymptotic properties, although there do exist results on the limits of \(S(t)x/t\) and \(J_t x/t\) about accretive operators in special Banach Spaces ([2],[4] and [5]). In this short paper we present results on the strong limits of the resolvents and semigroups associated with strongly accretive operators in general Banach Spaces.

**Theorem**  Assume that an \(\omega\)-accretive \((\omega < 0)\) operator \(A\) in a Banach Space \(E\) satisfies the range condition (3). Then there exists a unique zero point \(x^*\) for \(A\) and
\[
\lim_{t \to \infty} J_t x = \lim_{t \to \infty} S(t)x = x^* = S(t)x^* = J_t x^*, \quad \forall x \in D(A), \quad t \geq 0.
\]

Moreover, the following estimates hold true
\[
\begin{align*}
\|J_t x - x^*\| & \leq (1 - \omega t)^{-1}\|x - x^*\| \\
\|J_t x - x^*\| & \leq |\omega|^{-1}t^{-1}\|x - J_t x\| \\
\|S(t)x - x^*\| & \leq e^{\omega t}\|x - x^*\| \\
\|S(t)x^*\| & \leq (2 - e^{-\omega})e^{\omega \varepsilon}(1 - e^{-\omega\varepsilon})^{-1}\max_{s \in [0, \varepsilon]} \|S(s)x - x\|e^{\omega t}, \\
\end{align*}
\]
\[
\forall t \geq 0, \quad \varepsilon > 0.
\]

**Proof.** Fix \(t_0 > 0\). From the range condition (3) it is clear that
\[
J_{t_0} : D(A) \to D(A) \subset D(A). \quad \text{From (1) it follows that}
\]
\[ \| J_t x - J_s x \| \leq (1 - \omega t_0)^{-1} \| x - y \|, \forall x, y \in \overline{D(A)}. \]  

(11)

So, as \( \omega < 1 \), according to the Banach contraction principle there exists a unique fixed point \( x^* \) of \( J_t \) in \( \overline{D(A)} \) (actually in \( D(A) \)). Since the fixed points of \( J_t \) in \( \overline{D(A)} \) are precisely the zero points of \( A \), \( x^* \) must be the unique zero point of \( A \).

For any \( x \in \overline{D(A)} \), \([J_t x, t^{-1}(x - J_t x)] \in A, [x^*, 0] \in A\), therefore we obtain from (2) that

\[ \omega \| J_t x - x^* \|^2 + (t^{-1}(x - J_t x), f) \geq 0 \]  

(12)

where \( f \in \mathcal{F}(J_t x - x^*) \). Substituting

\[ \begin{align*}
(t^{-1}(x - J_t x), f) &= t^{-1}(x - x^*, f) - t^{-1}(J_t x - x^*, f) \\
&= t^{-1}(x - x^*, f) - t^{-1} \| J_t x - x^* \|^2
\end{align*} \]

in (12) yields

\[ (\omega t - 1) \| J_t x - x^* \|^2 + (x - x^*, f) \geq 0 \]

from which the estimate (7) is immediately obtained and therefore

\[ \lim_{t \to \infty} J_t x = x^*. \]

Noting that \([J_t t^{-1}(x - J_t x)] \in A, [J_s x, s^{-1}(x - J_s x)] \in A \) \( (t, s > 0) \) we have from (2) that

\[ \omega \| J_t x - J_s x \|^2 + (t^{-1}(x - J_t x) - s^{-1}(x - J_s x), f) \geq 0 \]  

(13)

where \( f \in \mathcal{F}(J_t x - J_s x) \). Substitution of

\[ \begin{align*}
(t^{-1}(x - J_t x) - s^{-1}(x - J_s x), f) &= (t^{-1}(J_s x - J_t x) + (t^{-1} - s^{-1})(x - J_s x, f) \\
&= t^{-1} \| J_t x - J_s x \|^2 + (t^{-1} - s^{-1})(x - J_s x, f)
\end{align*} \]

in (13) yields

\[ (\omega t - 1) \| J_t x - J_s x \|^2 + (1 - ts^{-1})(x - J_s x, f) \geq 0 \]

and therefore

\[ \| J_t x - J_s x \| \leq (1 - \omega t)^{-1} \| 1 - ts^{-1} \|_2 \| x - J_s x \| \]
from which the estimate (8) follows by letting \( t \to \infty \).

As \( x^* \) is the zero point of \( A \), it follows that in fact \( J_t x^* = x^* \), \( \forall t > 0 \). So, from the exponential formula (4) we know that

\[
S(t)x^* = x^*, \quad \forall t \geq 0.
\]

Estimate (9) is obtained by setting \( y = x^* \) in (5) and therefore \( \lim_{t \to \infty} S(t)x = x^* \).

Let \( \varepsilon > 0 \) be given. Set \( M = \max_{t \in [0, \varepsilon]} \| S(s)x - x \| \).

Express \( s \geq \varepsilon \) as \( s = n \varepsilon + \delta \), where \( n \) is a positive integer, \( 0 \leq \delta < \varepsilon \), both uniquely determined by \( s \). Then

\[
\| x - S(s)x \| \leq \| x - S(\delta)x \| + \| S(\delta)x - S(\delta - \varepsilon)x \| + \| S(\varepsilon)x - S(\delta - \varepsilon)x \|.
\]

\[
\leq M + \| x - S(\delta - \varepsilon)x \| + \| S(\delta)x - S(\delta - \varepsilon)x \|
\]

\[
\leq M + \sum_{k=0}^{n-1} \| S(\varepsilon)x - S(k+1)\varepsilon)x \|
\]

\[
\leq M + \sum_{k=0}^{n-1} e^{\omega k} \| x - S(\varepsilon)x \|
\]

\[
\leq (2 - e^{\omega \varepsilon})(1 - e^{\omega \varepsilon})^{-1} M
\]

from which we obtain

\[
\| S(t)x - S(t+s) \| \leq e^{\omega t} \| x - S(s)x \| \leq (2 - e^{\omega \varepsilon})(1 - e^{\omega \varepsilon})^{-1} Me^{\omega t}, \forall t > 0, s \geq \varepsilon.
\]

Now the estimate (10) follows from this inequality by letting \( s \to \infty \).

The theorem is proved.
References.


