Strong convergence of the resolvents and semigroups associated with strongly accretive operators in general Banach spaces

Citation for published version (APA):
STRONG CONVERGENCE OF THE RESOLVENTS AND SEMIGROUPS
ASSOCIATED WITH STRONGLY ACCRETIVE OPERATORS IN GENERAL BANACH SPACES

by

Liu Guizhong

University of Technology
Dept. of Mathematics & Computing Science
Den Dolech 2, P.O. box 513
5600 MB EINDHOVEN
The Netherlands
STRONG CONVERGENCE OF THE RESOLVENTS AND SEMIGROPS
ASSOCIATED WITH STRONGLY ACCRETIVE OPERATORS IN GENERAL BANACH SPACES

by

Liu Guizhong

Let $E$ be a Banach Space with norm $\| \cdot \|$. A subset $A \subseteq E \times E$ is said to be an $\omega$-accretive operator in $E$ if

$$
\| (x_1 - x_2) + t(y_1 - y_2) \| \geq (1 - \omega t) \| x_1 - x_2 \| ,
$$

(1)

where $\omega$ is some real number. Usually a $0$-accretive operator is just called an accretive operator. Any $\omega$-accretive operator $A$ with $\omega < 0$ is known as a strongly accretive operator. Note that, a subset $A$ of $E \times E$ is an $\omega$-accretive operator if and only if for each $[x_i, y_i], i = 1, 2$, there exists an $f \in F(x_1 - x_2)$ ($F$ is the normalized (multi-valued) duality mapping) such that

$$
\omega \| x_1 - x_2 \|^2 + (y_1 - y_2, f) \geq 0.
$$

(2)

Here $(\cdot, \cdot)$ denotes the duality pairing of $E$ and $E^*$. The well known Crandall-Liggett Theorem ([1]) asserts that, an accretive operator $A$ in $E$ satisfying the range condition

$$
R(I + tA) \supset D(A), \quad \forall t > 0
$$

(3)

generates via the exponential formula

$$
S(t)x = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^k , \quad S(t)x \in D(A), \quad t \geq 0
$$

(4)

*) Department of Mathematics, Xi'an Jiaotong University,
Xi'an, Shaanxi Province, China.
\[(J_t = (I + tA)^{-1} \text{ is the resolvent}) \text{ an } \omega \text{-contractive semigroup } S \text{ on } D(A), \text{ that is}\]

\[
\| S(t)x - S(t)y \| \leq e^{\omega t} \| x - y \|, \forall x, y \in \overline{D(A)}, \ t \geq 0. \quad (5)
\]

In a Hilbert Space \(H\) it is known that a strongly accretive operator \(A \subset H \times H\) satisfying the range condition (3) has a unique zero point \(x^*\) and for any \(x \in \overline{D(A)}\), \(\lim_{t \to \infty} S(t)x = x^*\), \(S(t)x^* = x^*\), \(\forall t \geq 0\).

See Pazy ([3]). It seems that in General Banach Spaces there are no such results on asymptotic properties, although there do exist results on the limits of \(S(t)x/t\) and \(J_t x/t\) about accretive operators in special Banach Spaces ([2],[4] and [5]). In this short paper we present results on the strong limits of the resolvents and semigroups associated with strongly accretive operators in general Banach Spaces.

**Theorem** Assume that an \(\omega\)-accretive (\(\omega < 0\)) operator \(A\) in a Banach Space \(E\) satisfies the range condition (3). Then there exists a unique zero point \(x^*\) for \(A\) and

\[
\lim_{t \to \infty} J_t x = \lim_{t \to \infty} S(t)x = x^* = S(t)x^* = J_t x^*, \quad \forall x \in \overline{D(A)}, \ t \geq 0. \quad (6)
\]

Moreover, the following estimates hold true

\[
\| J_t x - x^* \| \leq (1 - \omega t)^{-1} \| x - x^* \| \quad (7)
\]

\[
\| J_t x - x^* \| \leq |\omega|^{-1} \max_{0 \leq s \leq t} \| S(s)x - x^* \|. \quad (8)
\]

\[
\| S(t)x - x^* \| \leq e^{\omega t} \| x - x^* \| \quad (9)
\]

\[
\| S(t)x^* \| \leq (2 - e^{-\omega \varepsilon})(1 - e^{\omega \varepsilon})^{-1} \max_{0 \leq s \leq t} \| S(s)x - x \| e^{\omega t}, \forall t \geq 0, \ \varepsilon > 0. \quad (10)
\]

**Proof.** Fix \(t_0 > 0\). From the range condition (3) it is clear that

\[
J_{t_0} : \overline{D(A)} \to D(A) \subset \overline{D(A)}. \quad \text{From (1) it follows that}
\]

\[ \| J_{t_0}x - J_{t_0}y \| \leq (1 - \omega t_0)^{-1} \| x - y \|, \forall x, y \in D(A). \] (11)

So, as \( \omega < 1 \), according to the Banach contraction principle there exists a unique fixed point \( x^* \) of \( J_{t_0} \) in \( D(A) \) (actually in \( D(A) \)).

Since the fixed points of \( J_{t_0} \) in \( D(A) \) are precisely the zero points of \( A \), \( x^* \) must be the unique zero point of \( A \).

For any \( x \in D(A) \), \([J_{t_0}x, t_{t_0}^{-1}(x-J_{t_0}x)] \in A, [x^*,0] \in A\), therefore we obtain from (2) that
\[ \omega \| J_{t_0}x - x^* \|^2 + (t_{t_0}^{-1}(x-J_{t_0}x), f) \geq 0 \] (12)
where \( f \in F(J_{t_0}x - x^*) \). Substituting
\[ (t_{t_0}^{-1}(x-J_{t_0}x), f) = t_{t_0}^{-1}(x-x^*, f) - t_{t_0}^{-1}(J_{t_0}x - x^*, f) \]
\[ = t_{t_0}^{-1}(x-x^*, f) - t_{t_0}^{-1} \| J_{t_0}x - x^* \|^2 \]
in (12) yields
\[ (\omega t_{t_0} - 1) \| J_{t_0}x - x^* \|^2 + (x-x^*, f) \geq 0 \]
from which the estimate (7) is immediately obtained and therefore
\[ \lim_{t \to \infty} J_{t_0}x = x^*. \]

Noting that \([J_{t_0}, t_{t_0}^{-1}(x-J_{t_0}x)] \in A, [J_{s_0}^{-1}(x-J_{s_0}x)] \in A \) \( (t, s > 0) \)

we have from (2) that
\[ \omega \| J_{t_0}x - J_{s_0}x \|^2 + (t_{s_0}^{-1}(x-J_{t_0}x) - s_{s_0}^{-1}(x-J_{s_0}x), f) \geq 0 \] (13)
where \( f \in F(J_{t_0}x - J_{s_0}x) \). Substitution of
\[ (t_{s_0}^{-1}(x-J_{t_0}x) - s_{s_0}^{-1}(x-J_{s_0}x), f) \]
\[ = (t_{s_0}^{-1}(J_{s_0}x - J_{t_0}x) + (t_{s_0}^{-1} - s_{s_0}^{-1}) (x-J_{s_0}x), f) \]
\[ = t_{s_0}^{-1} \| J_{t_0}x - J_{s_0}x \|^2 + (t_{s_0}^{-1} - s_{s_0}^{-1})(x-J_{s_0}x, f) \]
in (13) yields
\[ (\omega t_{s_0} - 1) \| J_{t_0}x - J_{s_0}x \|^2 + (1 - ts_{s_0}^{-1})(x-J_{s_0}x, f) \geq 0 \]
and therefore
\[ \| J_{t_0}x - J_{s_0}x \| \leq (1 - \omega t_{s_0})^{-1} \| 1 - ts_{s_0}^{-1} \| \| x-J_{s_0}x \|. \]
from which the estimate (8) follows by letting \( t \to \infty \).

As \( x^* \) is the zero point of \( A \), it follows that in fact \( J_t x^* = x^* \), \( \forall t > 0 \). So, from the exponential formula (4) we know that

\[
S(t)x^* = x^*, \quad \forall t \geq 0.
\]

Estimate (9) is obtained by setting \( y = x^* \) in (5) and therefore \( \lim_{t \to \infty} S(t)x = x^* \).

Let \( \varepsilon > 0 \) be given. Set \( M = \max_{t \in [0, \varepsilon]} \| S(s)x - x \| \).

Express \( s \geq \varepsilon \) as \( s = n\varepsilon + \delta \), where \( n \) is a positive integer, \( 0 \leq \delta < \varepsilon \), both uniquely determined by \( s \). Then

\[
\| x - S(s)x \| \leq \| x - S(\delta)x \| + \| S(\delta)x - S(\delta)S(n\varepsilon)x \|
\]

\[
\leq M + \| x - S(n\varepsilon)x \|
\]

\[
\leq M + \sum_{k=0}^{n-1} \| S(k\varepsilon)x - S((k+1)\varepsilon)x \|
\]

\[
\leq M + \sum_{k=0}^{n-1} e^{\omega(k+1)\varepsilon} \| x - S(\varepsilon)x \|
\]

\[
\leq (2 - e^{\omega\varepsilon})(1 - e^{\omega\varepsilon})^{-1} M
\]

from which we obtain

\[
\| S(t)x - S(t+s) \| \leq e^{\omega t} \| x - S(s)x \|
\]

\[
\leq (2 - e^{\omega\varepsilon})(1 - e^{\omega\varepsilon})^{-1} M e^{\omega t}, \quad \forall t > 0, s \geq \varepsilon.
\]

Now the estimate (10) follows from this inequality by letting \( s \to \infty \).

The theorem is proved.
References.


