A collection of modelling problems carried out in the academic year 1990-1991 by the ECMI-students at the Eindhoven University of Technology

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Published: 01/01/1991

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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STUDENT REPORT 91-04

A COLLECTION OF MODELLING PROBLEMS
1990 - 1991

Marcel van Asperdt
Jacqueline van den Bosch
Axel Bloemen
Frank Holland
Jan Wiepke Knobbe
Richard de Lange
Joost Meuwissen
Sandra Roijakkers

September 1991
A collection of Modelling Problems carried out in the academic year 1990-1991 by the ECMI-students at the Eindhoven University of Technology:

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1 Introduction

Problem description

Various models exist to predict the number of soldiers of the fighting armies in a war. One of these models is the CONCOM combat model. This model depends on two parameters, namely the CLR (Combat Lost Rate) and the OLR (Operational Lost Rate) of the involved armies. Our goal is to estimate these parameters for the Japanese and American troops in the battle of Iwo Jima on the basis of known data of this battle.
2 Model and analysis

2.1 The model

We will use the following notation:

- \( A \) for the number of active (= able to fight) American troops,
- \( J \) for the number of Japanese soldiers.

Then the CONCOM model has the form:

\[
\begin{align*}
\dot{A} &= -aA - bJ + P(t) \\
\dot{J} &= -cA - dJ + Q(t)
\end{align*}
\]  
(1.1)

in which the \( \cdot \) above a variable means its time derivative. \( P(t) \) and \( Q(t) \) are the reinforcements of respectively the Americans and Japanese troops. The parameters \( b \) and \( c \) are respectively the CLR of the Americans and Japanese, \( a \) and \( d \) the OLR of the Americans and Japanese.

There were no reinforcements of the Japanese during the battle of Iwo Jima so we take \( Q(t) = 0 \).

The parameters \( a \) and \( d \), the OLR's, form the mathematical description of phenomena as desertion and sickness of the soldiers. Because of the short duration of the battle (only five weeks) and the fact that the battle took place on an island, we can assume these parameters to be zero. So the resulting model has the form:

\[
\begin{align*}
\dot{A} &= -bJ + P(t) \\
\dot{J} &= -cA
\end{align*}
\]  
(1.2)

We can solve this differential equation, resulting in the following solution:

\[
\begin{align*}
A(t) &= \cosh(\lambda t) \cdot A_0 - \frac{b}{\lambda} \sinh(\lambda t) \cdot J_0 + \int_0^t \cosh(\lambda(t - \tau)) \cdot p(\tau) d\tau \\
J(t) &= -\frac{b}{\lambda} \sinh(\lambda t) \cdot A_0 + \cosh(\lambda t) \cdot J_0 - \int_0^t \frac{b}{\lambda} \sinh(\lambda(t - \tau)) \cdot p(\tau) d\tau
\end{align*}
\]  
(1.3)

in which \( \lambda = \sqrt{bc} \) and \( A_0, J_0 \) represent the initial number of respectively American and Japanese soldiers.

Unfortunately this model is of no use if we do not have good estimations for the parameters \( b \) and \( c \) at our disposal.
2.2 Estimation of the parameters.

We can interpret Equations 1.2 as difference equations by discretization. We will take time steps of one day and define

\[ A_i \] is the number of active American troops at day \( i \),
\[ J_i \] is the number of Japanese troops at day \( i \).

Because the war lasted 36 days, we take \( i \) in the range \( 0, \ldots, 36 \). \( A_0 \) and \( J_0 \) again represent the initial number of respectively American and Japanese soldiers.

Using this notation we can rewrite system 1.2

\[
\begin{align*}
A_{i+1} - A_i &= -bJ_i + P_i \\
J_{i+1} - J_i &= -cA_i \\
\end{align*}
\]

where \( P_i \) denotes the number of reinforcements of the American troops at day \( i \), see Table 1.

The data at our disposal about the battle includes the numbers \( A_0, \ldots, A_{36}, J_0 \) and \( J_{36} \). So we can use the last equation of Equations 2.1. Summing these equation for \( i = 0, \ldots, 35 \) we find

\[ J_{36} - J_0 = -c \cdot \sum_{i=0}^{35} A_i \] (2.2)

Substituting the known values (see Table 1 and 2) gives us that

\[ c = 0.01036 \] (2.3)
Table 2: American troops

<table>
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<tr>
<th>i</th>
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<th>i</th>
<th>( A_i )</th>
<th>i</th>
<th>( A_i )</th>
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Table 3: Estimates for Japanese troops

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<th>i</th>
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<th>( J_i )</th>
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<td>32</td>
<td>2253</td>
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<td>36</td>
<td>0</td>
</tr>
</tbody>
</table>

4
Now we have an estimation for \( c \) we can use the last equation of 2.1 to estimate \( J_1, \ldots, J_{35} \). The results of this calculation are given in Table 3.

A similar process is applied to the first equation of 2.1. If we sum this equation for \( i = 0, \ldots, 35 \) we find:

\[
A_{36} - A_0 = -b \cdot \sum_{i=0}^{35} J_i + \sum_{j=0}^{35} P_j \quad (2.4)
\]

The reinforcements \( P_i \) are given in Table 1. Substituting the values in Equation 2.4 yields:

\[
b = 0.04887 \quad (2.5)
\]

### 2.3 Simulation of the war.

Using the values of \( b \) and \( c \) in Equation 1.3 we can check the given values for \( A_i \). First of all we have to determine the reinforcements \( p(t) \). Rewriting the first equation of 2.1 we find for the calculated reinforcements \( P_i \):

\[
P_i = A_{i+1} - A_i + b J_i, \quad i = 0, \ldots, 35. \quad (3.1)
\]

This yields the values for \( P_i \) which can be found in Table 4.

we deduce from Table 4 that the reinforcements took place on days 3, 4 and the days 6, 7, 8 and 9. We attach the reinforcements the values given in Table 5.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( P_i )</th>
<th>( i )</th>
<th>( P_i )</th>
<th>( i )</th>
<th>( P_i )</th>
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<td>17</td>
<td>214</td>
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<td>-96</td>
</tr>
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</table>

Table 4: Calculated reinforcements
These values are in accordance with the given data that the first reinforce-
ment consisted of 6000 soldiers and the second of 13000.
Calculating the Ai's using the first equation of 1.3 and the values of b and c
from respectively formula 2.3 and 2.5 yields the values which can be found
in Appendix A.

2.4 Improvement of the model.
In Appendix A we see that the calculated values for A10, ..., A15 are too large.
Therefore it seems plausible to make distinction in timeperiods, the first
period containing days 0, ..., 9, the second days 10, ..., 15 and the third and
last period days 16, ..., 36. So the CLR for the Americans, i.e. parameter b
will be different in the different periods. We leave the parameter c unchanged
because we do not have more information about the number of Japanese
troops than J0 and J36. We take for the CLR of the Americans in the first,
second and third period respectively the parameters b1, b2 and b3. A similar
calculation as in Section 2.2 yields the following values for the parameters
b1, b2 and b3:

\[ b_1 = 0.05191 \quad b_2 = 0.05616 \quad b_3 = 0.04156 \] (4.1)

The combat lost rate in the second period is greater than the one in the
first period, this is can be explained by the fact that just after arrival of the
American reinforcements the coordination between the American troops is
more difficult than in the beginning of the battle. After some days they get
used to the tricks and fighting of the Japanese and are able to fight more
effectively, so the CLR decreases in the last period. Again the first equation
of 1.3 is used to calculate the Ai's. The result can be found in Appendix B.

<table>
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<th>i</th>
<th>Pi</th>
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<td>4400</td>
</tr>
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<td>9</td>
<td>2100</td>
</tr>
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<tr>
<td>6</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Chosen reinforcements
3 Conclusion

In the case of the battle of Iwo Jima we find for the combat lost rate of the Americans 0.04887 and for the Japanese troops 0.01036. By taking a time factor into account we can refine the model which yields explicable results, see Section 2.4.

From the above we conclude that it is possible to determine the values of the combat lost rates of the fighting troops in the CONCOM model. Even if we have few information at our disposal.
The value of constant \( b \) is \( 4.88700000000000 \times 10^{-2} \)
The value of constant \( c \) is \( 1.03575170000000 \times 10^{-2} \)

Reinforcement day 3 is 4500
Reinforcement day 4 is 1500
Reinforcement day 6 is 3000
Reinforcement day 7 is 3500
Reinforcement day 8 is 4400
Reinforcement day 9 is 2100

day | real \( A \) | calculated \( A \) | difference
--- | --- | --- | ---
t= 0 | data=54000 | At=54000 | difference= 0

t= 1 | data=53692 | At=52963 | difference= 729

t= 2 | data=53400 | At=51953 | difference= 1447

t= 3 | data=55921 | At=55469 | difference= 452

t= 4 | data=56122 | At=56013 | difference= 109

t= 5 | data=55110 | At=55086 | difference= 24

t= 6 | data=56897 | At=57186 | difference= -289

t= 7 | data=59542 | At=59816 | difference= -274

t= 8 | data=63102 | At=63375 | difference= -273

t= 9 | data=64016 | At=64667 | difference= -651

t=10 | data=63462 | At=64667 | difference= -1205

t=11 | data=62398 | At=63890 | difference= -1492

t=12 | data=62001 | At=63145 | difference= -1144

t=13 | data=60790 | At=62431 | difference= -1641

t=14 | data=59824 | At=61750 | difference= -1926

t=15 | data=60000 | At=61750 | difference= -1750

t=16 | data=60459 | At=61130 | difference= -671

t=17 | data=60392 | At=60541 | difference= -149

t=18 | data=60062 | At=59983 | difference= 79

t=19 | data=59102 | At=59455 | difference= -353

t=20 | data=58429 | At=58957 | difference= -528

t=21 | data=58526 | At=58489 | difference= 37

t=22 | data=58300 | At=58051 | difference= 249

t=23 | data=57684 | At=57642 | difference= 42

t=24 | data=57222 | At=57262 | difference= -40

t=25 | data=56314 | At=56912 | difference= -598

t=26 | data=55872 | At=56590 | difference= -718

t=27 | data=55498 | At=56296 | difference= -798

t=28 | data=55029 | At=56031 | difference= -1002

t=29 | data=55002 | At=55795 | difference= -793

t=30 | data=55016 | At=55587 | difference= -571

t=31 | data=55112 | At=55407 | difference= -295

t=32 | data=55000 | At=55255 | difference= -255

t=33 | data=54726 | At=55131 | difference= -405

t=34 | data=54069 | At=55034 | difference= -965

t=35 | data=53696 | At=54966 | difference= -1270

t=36 | data=53748 | At=54926 | difference= -1178

The standard deviation is 1.43310303126499 \times 10^2
APPENDIX B

The value of constant b1 is 5.19051330000000E-0002
The value of constant b2 is 5.61591530000000E-0002
The value of constant b3 is 4.15620660000000E-0002
The value of constant c is 1.03575170000000E-0002

Reinforcement day 3 is : 4500
Reinforcement day 4 is : 1500
Reinforcement day 6 is : 3000
Reinforcement day 7 is : 3500
Reinforcement day 8 is : 4400
Reinforcement day 9 is : 2100

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<td>data=54069</td>
<td>At=55041</td>
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<td>data=53696</td>
<td>At=54980</td>
</tr>
<tr>
<td>t=36</td>
<td>data=53748</td>
<td>At=54943</td>
</tr>
</tbody>
</table>

The standard deviation is 1.11112549198926E+0002
MAINTENANCE

OF A FLOCK OF SHEEP
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Summary

To maintain a flock of sheep in winter you need hay. The amount of hay you have to lay in depends on the size of the flock. So the size of the flock is dependent on the available pasture. A farmer who wants to buy a flock of sheep wants to have information about these topics.

In Introduction the problem is stated. In the section Assumptions some assumptions are made. The main section of this report consists of the mathematical model, see Model. First we make some conventions about the notation that will be used. Then the model is constructed in Equations, and next the model is analysed in Analysis. In Practical Example the model is applied to a practical case to demonstrate how the calculations are performed. In Conclusions we discuss the mathematical results derived with our model. Finally, in Recommendations for the shepherd, on the basis of data we recommend the shepherd to maintain a flock of at most 200 sheep, and to lay in all plentiful grass as hay during summer and autumn to nourish the sheep in winter and spring(!).
Introduction

A farmer with 10 hectares of pasture wants to keep sheep on his land. He wants to know how many sheep he can keep, given the constraint of the 10 hectares of pasture. In addition he wants to know the breakdown of the flock by age. To maintain the flock during the winter, he needs to know how much grass he should lay in during the summer and autumn.

Assumptions

There is information available on:
• daily grass-consumption of sheep and lambs, see Table 1
• average daily grass-growth in each of the four seasons, see Table 2
• number of lambs produced by a ewe, see Table 3.

<table>
<thead>
<tr>
<th></th>
<th>lamb</th>
<th>sheep</th>
</tr>
</thead>
<tbody>
<tr>
<td>winter</td>
<td>0</td>
<td>2.10</td>
</tr>
<tr>
<td>spring</td>
<td>1.00</td>
<td>2.40</td>
</tr>
<tr>
<td>summer</td>
<td>1.65</td>
<td>1.15</td>
</tr>
<tr>
<td>autumn</td>
<td>0</td>
<td>1.35</td>
</tr>
</tbody>
</table>

Table 1: Daily grass-consumption in kg

<table>
<thead>
<tr>
<th>grass-growth</th>
<th>winter</th>
<th>spring</th>
<th>summer</th>
<th>autumn</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2: Daily growth in gr. per m²

<table>
<thead>
<tr>
<th>age</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>number</td>
<td>0</td>
<td>1.8</td>
<td>2.4</td>
<td>2.0</td>
<td>1.8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Average number of lambs produced by a ewe
Before posing a model, we have to make some assumptions:

- the shepherd strives after a stable flock of sheep, i.e. size of the flock and its breakdown by age are constant after a few years
- the shepherd can sell as many sheep as he wants for slaughtering
- a sheep can be slaughtered if it is younger than 3 years
- maximum age of a sheep is 8 years
- the shepherd doesn't buy sheep, except for starting the flock
- the probability that a lamb is male is \( \frac{1}{2} \).

**Model**

**Notation**

In the model the following notation will be used:

\[ e_i := \text{number of ewes in year } i \text{ of its life, } i=1,\ldots,8 \]
\[ r_i := \text{number of rams in year } i \text{ of its life, } i=1,\ldots,8 \]
\[ s_i := \text{number of sheep in year } i \text{ of its life, } i=1,\ldots,8 \]
\[ \alpha := \text{fraction of ewes producing lambs} \]
\[ \lambda_i := \text{average number of lambs per ewe of age } i \]
\[ o_i := \text{fraction of surviving sheep from age } i \text{ to age } i+1, \text{ the assumption that the maximum age for a sheep is 8 years is expressed by } o_8=0. \text{ The assumption that the shepherd can sell young sheep for slaughtering can be expressed in a} \]
small $o_1$ and $o_2$, so we don't have to introduce a special variable for slaughtering.

$B :=$ matrix of birth, i.e. the matrix with elements $b_{ij}$ defined by the number of ewes in year $i$ of its life born from a sheep in year $j$ of its life. Matrix $B$ has as first row $\left(\frac{\alpha_2}{2}, \frac{\alpha_3}{2}, \frac{\alpha_4}{2}, \frac{\alpha_5}{2}, 0, 0, 0\right)$ and the rest of the matrix consists of noughts.

$S :=$ matrix of survival,

$$
S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0_7 \\
\end{pmatrix}.
$$

For the $e_i$'s, $r_i$'s and $s_i$'s we will also use the vector notation, i.e.

$$
e_k^T := (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$$

and similarly for $r$ and $s$.

Equations

The number of ewes in year $n+1$ is determined by the number of surviving ewes of year $n$ and the number of newborn ewes. In formula:

$$
\mathbf{g}(n+1) = (B+S) \mathbf{g}(n)
$$

in which $\mathbf{g}(n)$ stands for the number of ewes in year $n$.

A similar derivation gives for the number of rams in year $n+1$:

$$
\mathbf{r}(n+1) = B \mathbf{g}(n) + S \mathbf{r}(n)
$$
Combining these two formulas we find for the total number of sheep in year $n+1$:

$$ g(n+1) = g(n+1) + r(n+1) = 2B g(n) + S g(n) \quad (3) $$

where the right-hand side equals the sum of the number of surviving sheep of year $n$ and the number of newborn sheep.

**Analysis**

In a stable flock the number of rams and ewes is constant through time, so we find respectively from (1), (2) and (3):

$$ e = (B+S) e \quad (4) $$

$$ r = B e + S r \quad (5) $$

$$ s = 2B e + S s \quad (6) $$

From (4) we see that the calculation of $e$ is an eigenvalue problem of the matrix $B+S$.

From Equations (5) and (6) we respectively deduce:

$$ r = (I-S)^{-1} B e \quad \text{and} \quad s = 2(I-S)^{-1} B e = 2r $$

Since we now have $s = r + e$ and $s = 2r$, we find that for a stable flock the number of rams of age $i$ equals the number of ewes of age $i$. This might be surprising, but it is a consequence of our assumption that a newborn sheep has an equal probability of being ram or ewe. This reduces the problem of finding the composition of a stable flock to finding the vector $e$ from Equation (4).
Define the matrix $A$ by $A := B + S$, so

$$A = \begin{bmatrix}
0 & \alpha \lambda_2 & \frac{\alpha \lambda_3}{2} & \frac{\alpha \lambda_4}{2} & \frac{\alpha \lambda_5}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The greatest eigenvalue of this matrix is important, which can be understood as follows:

Suppose $g(0) = \sum_{i=1}^{8} \alpha_i \psi_i$, where the $\psi_i$'s are the eigenvectors of the matrix $A$. Using recurrence relation (1) we derive

$$g(n) = A^n g = \sum_{i=1}^{8} \alpha_i A^n \psi_i = \sum_{i=1}^{8} \alpha_i \mu_i^n \psi_i,$$

where $\mu_i$ is the eigenvalue corresponding with eigenvector $\psi_i$.

For large $n$ it is obvious that the eigenvalue $\mu_i$ with $|\mu_i|$ maximal of all eigenvalues is the most important one. We also conclude that to obtain a stable flock, this greatest eigenvalue, say $\mu_1$, has to fulfil $\mu_1 = 1$!

Now we know that the eigenvalues of $A$ are important we will analyse the characteristic polynomial of matrix $A$. This polynomial is given by:
We immediately see that eigenvalue $0$ has multiplicity 3. From the above we know that $\mu = 1$ must also be solution to (7). This yields an equation for $\alpha$:

$$\alpha = \frac{2}{\alpha_{1}\lambda_{2} + \alpha_{1}\lambda_{3} + \alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}}.$$  \hspace{1cm} (8)

It appears that choosing factor $\alpha$ (remind that this factor can be chosen by the shepherd as $\alpha$ is the fraction of ewes producing lambs) is very important to obtain a 'stable' flock. Of course this factor has to fulfil $0 < \alpha < 1$, that $\alpha$ is positive is obvious. The values of the $\lambda_i$'s are given in Table 3 (see Assumptions). Using these values we find that the requirement $\alpha < 1$ is fulfilled if the $\alpha_i$'s are at least 0.55 (for $i = 1, \ldots, 7$). We may assume that this is true in practise (not more than 45% of the sheep die per year normally).

We can divide the characteristic polynomial (7) by $\mu^3(\mu - 1)$. This yields:

$$\mu^4 + \mu^3 + C_2 \mu^2 + C_1 \mu + C_0,$$

where

$$C_2 = \frac{\alpha_{1}\lambda_{3} + \alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}}{\alpha_{1}\lambda_{2} + \alpha_{1}\lambda_{3} + \alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}},$$

$$C_1 = \frac{\alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}}{\alpha_{1}\lambda_{2} + \alpha_{1}\lambda_{3} + \alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}},$$

$$C_0 = \frac{\alpha_{1}\lambda_{5}}{\alpha_{1}\lambda_{2} + \alpha_{1}\lambda_{3} + \alpha_{1}\lambda_{4} + \alpha_{1}\lambda_{5}}.$$

We have not found a general proof, showing that this polynomial in general has roots $\lambda$ with $|\lambda| < 1$, so to be sure a stable flock exists we have to check this polynomial with given $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$. 

\[\mu^8 - \frac{\alpha_{1}\lambda_{2}}{2} \mu^6 - \frac{\alpha_{1}\lambda_{3}}{2} \mu^5 - \frac{\alpha_{1}\lambda_{4}}{2} \mu^4 - \frac{\alpha_{1}\lambda_{5}}{2} \mu^3 \]  \hspace{1cm} (7)
The proportions of the age-groups of the stable flock are then given by the eigenvector corresponding to eigenvalue 1. Suppose the eigenvector is \((e_1,e_2,...,e_8)\) then the structure of the stable flock is given by \(s = C(e_1,e_2,...,e_8)\) for some constant \(C\). This constant \(C\) is determined by the 'grass-condition'. We will now consider how these results are applied in a practical example.

**Practical Example**

Consider a situation where \(a_1 = 0.5\), \(a_2 = 0.6\) and \(a_3 = a_4 = a_5 = a_6 = a_7 = 0.8\). Both \(a_1\) and \(a_2\) are small, which is caused by selling young sheep for slaughtering; the \(a_1\) is even smaller than \(a_2\), expressing the vulnerability of lambs to diseases.

Using the values of Table 3 for the \(\lambda_i\)'s we find \(\alpha = 0.82\) from (8). The roots of polynomial (7) are \(-0.055-0.58i\), \(-0.055+0.58i\), \(-0.56-0.32i\) and \(-0.56+0.32i\). So a stable flock exists. Calculating the eigenvector of \(A\) with eigenvalue 1 yields the following breakdown by age of a stable flock:

0-1 year: 38.4 %
1-2 year: 19.2 %
2-3 year: 11.5 %
3-4 year: 9.2 %
4-5 year: 7.4 %
5-6 year: 5.9 %
6-7 year: 4.7 %
7-8 year: 3.7 %
Thus the stable flock is $s = C(38.4, 19.2, 11.5, 9.2, 7.4, 5.9, 4.7, 3.7)$. To determine constant $C$ we use the grass-condition.

The daily grassneed (in kg) for flock $s$ is:

$$38.4C \times 1.00 + 61.6C \times 2.40 \text{ in spring}$$

$$38.4C \times 1.65 + 61.6C \times 1.15 \text{ in summer}$$

$$100C \times 1.35 \text{ in autumn}$$

$$100C \times 2.10 \text{ in winter.}$$

After summer, the lambs are supposed to be grown up to sheep, so in autumn and winter we don't have lambs.

Daily grass-growth on the 10 available hectares of pasture is 300 kg in spring, 700 kg in summer and 400 kg in autumn.

Summing the grassneed of the flock over a year and comparison with the total grass-growth in a year yields the following condition for constant $C$: $C < 2.10$.

The maximum flock the shepherd can keep is $s = (81, 40, 24, 19, 16, 12, 10, 8)$ and consists of 81 lambs and 129 sheep.

The amount of hay the shepherd should lay in daily for winter and spring(!) is 418 kg in the summer and 116 kg in the autumn. This results in a stock of hay of 48 ton at the beginning of the winter.

How many sheep can the shepherd sell for slaughtering?

Every year in spring 81 lambs are born, and $\alpha_1$ is the fraction of how many of them are still alive one year later. In this example we have $\alpha_1 = 0.5$, so the shepherd has to take care that next spring still 40 of them are alive. If 11 lambs die because of
diseases, the shepherd can sell $81 - 11 - 40 = 30$ of them. A similar calculation holds for sheep of one year old.

Conclusions

Calculation for practical cases was performed for several values of the $o_i$'s. From these examples it becomes clear that in all cases a stable flock exists. The value of $\alpha$ is easy to determine with formula (8). In almost all cases, the eigenvector belonging to eigenvalue 1 was the same. This means that the proportions of the age-groups in the stable flock are almost independent of the parameters $o_i$. If the startflock consists of both rams and ewes, stable proportions of the age groups are reached within 5-8 years.

Recommendations for the shepherd

To obtain a stable flock of sheep, buy a flock not only consisting of rams or ewes but of both races; this results within 5-10 years in a stable flock of sheep. The time it takes to reach the stable flock of about 200 sheep (including 80 lambs) is dependent on the size of the startflock not on the age of the sheep bought to start the flock (although the startflock should consist of sheep that are fertile, i.e. of which the age is at most 5 years). The fraction of fertile ewes producing lambs is 70-80% (for an exact calculation see formula (8)). If this fraction increases, so does the size of the flock, while the size of the flock decreases if this fraction is smaller.

The breakdown by age of the stable flock is:

0-1 year: 38.4% = 77 lambs

1-2 year: 19.2% = 38 sheep
2-3 year: 11.5 % = 23 sheep

3-4 year: 9.2 % = 18 sheep

4-5 year: 7.4 % = 15 sheep

5-6 year: 5.9 % = 12 sheep

6-7 year: 4.7 % = 9 sheep

7-8 year: 3.7 % = 8 sheep.

As only 10 hectares of pasture are available, the size of the flock should not exceed 200 sheep. All plentiful grass in summer and autumn should be layed in as hay for maintenance of the flock in winter and spring(!). This yields an amount of almost 50 ton of hay in the beginning of the winter.
Cost Accounting for a Boiler

Summary

We were asked to devise a minimum-cost strategy for heating the water in an electrically controlled gas boiler overnight. In this report, a mathematical model is used to show that the cheapest strategy is to switch the boiler thermostat off in the period that no warm water is needed. However, comparing the costs of this strategy to the costs of letting the boiler thermostat regulate the water temperature nightly and dayly, we see that the savings thus obtained are relatively small.

Given Problem

When heating the water in a gas boiler overnight, two extreme strategies are possible. The first one is to let the thermostat regulate the temperature just like it does during daytime; the other one is to switch the heating element off at night, and switch it on again in the morning, shortly before hot water is needed. Of course, also all possibilities in between could be considered: one may, for example, use the thermostat to ensure that the water temperature never falls below a certain minimum. We were asked to find out which strategy is the cheapest.
Given Information

The following physical laws are given:

I The energy $h$ needed to raise the temperature of $m$ kilogram of water with $\alpha$ degrees, is proportional to the product of $\alpha$ and $m$;
in a formula:

$$h = C_1 \cdot \alpha \cdot m,$$

where $C_1$ is a constant.

II The difference in temperature $\theta$ between a hot substance and its surroundings decreases proportionally to $\theta$.
In a formula:

$$\frac{d\theta}{dt} = -k \cdot \theta,$$

where $k$ is a constant.

III If a hot substance has a surface area $A$, then its loss of heat per unit time $h$ is proportional to the product of $A$ and the difference in temperature $\theta$ between the hot substance and its surroundings.
In a formula:

$$h = C_2 \cdot A \cdot \theta,$$

where $C_2$ is a constant.
Information Required

In order to solve the problem, we need some information about the boiler. First of all, some general knowledge about the way boilers and their thermostats work has to be obtained. From this we can deduce which characteristics of a boiler (e.g. its heat capacity or size) are relevant.

In the thermostat of the boiler two temperatures have to be set: $T_e$ ($T_{cold}$) and $T_h$ ($T_{hot}$), $T_h > T_e$. As soon as the temperature of the water has fallen to $T_e$, the heating element switches on and heats the water until it has reached the temperature $T_h$. Then it switches off and stays off as long as the water temperature remains higher than $T_e$.

We assume that the heating element of the boiler produces a constant quantity of energy per unit of time. Furthermore, for simplicity, we assume that all the water in the boiler is at the same temperature. From these two assumptions we can draw the following graph, showing the temperature of the water in the boiler as a function of time:

![Figure 1: The temperature of the water in the boiler as a function of time.](image)

Figure 1: The temperature of the water in the boiler as a function of time.
Variables and Constants

Before presenting our solution, we give a list of all variables and constants that will occur.

\( T_s \): the temperature outside the boiler \((T_{\text{surroundings}})\).

\( T_h \): the maximum temperature of the water in the boiler.

\( T_c \): the temperature of the water in the boiler at which the heating element switches on.

\( T(t) \): the temperature of the water in the boiler at time \( t \).

\( \theta(t) \): the difference in temperature between the water in the boiler and the temperature outside the boiler at time \( t \quad (\theta(t) = T(t) - T_s) \).

\( V \): the heat capacity of the boiler.

\( A \): the surface area of the boiler.

\( m \): the mass of the water in the boiler.

\( Q \): the energy costs per second when the heating element of the boiler is active.

\( t_1 \): the time the boiler needs to warm the water from \( T_c \) to \( T_h \).

\( t_2 \): the time it takes for the water temperature to drop from \( T_h \) to \( T_c \).

\( t_n \): the length of the time during the night when no hot water is needed.

\( C_1 \): the constant mentioned in Law I. Since one calorie is defined as the energy needed to warm one gram of water through one degree, and as one calorie equals 4.184 Joule, \( C_1 \) equals 4184.

\( C_2 \): the positive constant mentioned in Law III.

\( k \): the positive constant of Law II, which is boiler-dependent.

The units that will be used are

for temperature : Kelvin,
for length : metres,
for time : seconds,
for mass : kilograms,
for heat or energy : Joules,
for capacity : Watt,
for costs : Dutch guilders.

Solution

For a given boiler, we first determine all constants and variables that are independent of the setting of \( T_h \) and \( T_e \).

The mass \( m \) of the water in the boiler, and the surface area \( A \) can be measured; the same holds for \( T_e \). The constant \( C_1 \) is already known to be equal to 4184.

Usually the constants \( C_2 \), \( k \) and \( V \) will also be known from supplied boiler data, but if they are not, they can be calculated by choosing any values of \( T_h \) and \( T_e \), measuring the corresponding \( t_1 \) and \( t_2 \), and executing the calculations below.

First we consider a cooling-off period \([t_b, t_b + t_2]\) \((t_b = t_{\text{begin}})\) during which the water temperature drops from \( T_h \) to \( T_e \). Law II gives

\[
\frac{d\theta(t)}{dt} = -k \cdot \theta(t), \quad \text{or} \quad \theta(t) = C \cdot e^{-k \cdot t}.
\]

The constant \( C \) can be determined by looking at \( \theta(t) \) at \( t = \tau \). We find

\[
\theta(\tau) = T(\tau) - T_e = C \cdot e^{-k \cdot \tau},
\]
and thus

\[ C = (T(\tau) - T_*) \cdot e^{k\tau}, \]

which yields successively

\[ \theta(t) = (T(\tau) - T_*) \cdot e^{-k(t-\tau)}, \quad t \geq \tau, \]

and

\[ T(t) = T_* + (T(\tau) - T_*) \cdot e^{-k(t-\tau)}, \quad t \geq \tau. \]

By taking \( \tau \) equal to \( t_b \), the last expression reduces to

\[ T(t) = T_* + (T_h - T_*) \cdot e^{-k(t-t_b)}, \quad t \geq \tau. \]

Choosing for \( t \) the value \( t_b + t_2 \) gives

\[ T_c = T_* + (T_h - T_*) \cdot e^{-k t_2}, \]

and solving this equation for \( k \), we find

\[ k = \frac{1}{t_2} \cdot \ln \left( \frac{T_h - T_*}{T_* - T_c} \right). \]  

(1)

Let us now have a look at the loss of energy \( h \) during \([t_b, t_b + t_2]\). According to Law III

\[
h = \int_{t_b}^{t_b+t_2} C_2 \cdot A \cdot \theta(t)dt \\
= C_2 \cdot A \cdot (T_h - T_*) \cdot \int_{t_b}^{t_b+t_2} e^{-k(t-t_b)}dt \\
= C_2 \cdot A \cdot (T_h - T_*) \cdot \frac{1}{k} \cdot (1 - e^{-k t_2}) \\
= \frac{1}{k} \cdot C_2 \cdot A \cdot (T_h - T_c).
\]

Law I states that this equals \( C_1 \cdot (T_h - T_c) \cdot m \), so \( C_2 \) can be determined:

\[ C_2 = \frac{C_1 \cdot m \cdot k}{A}. \]
We then consider a warming-up period \([t_b, t_b + t_1]\) in which the water temperature raises from \(T_c\) to \(T_h\). Each second a quantity \(V\) of energy is supplied, which is, however, not completely used to raise the temperature. In fact, Law III states that at a moment \(t \in [t_b, t_b + t_1]\) the loss of heat per second equals \(C_2 \cdot A \cdot (T(t) - T_s)\). Therefore, each second only a quantity of energy \(V - C_2 \cdot A \cdot (T(t) - T_s)\) is available for warming the water.

Let \(\Delta t\) be a short period of time; from Law I we get

\[
( V - C_2 \cdot A \cdot (T(t) - T_s) ) \cdot \Delta t = C_1 \cdot (T(t + \Delta t) - T(t)) \cdot m,
\]

which is equivalent to

\[
\frac{T(t + \Delta t) - T(t)}{\Delta t} = \frac{V - C_2 \cdot A \cdot (T(t) - T_s)}{C_1 \cdot m}
\]

Taking the limit \(\Delta t \to 0\) converts this to

\[
\frac{d(T(t))}{dt} = \frac{V - C_2 \cdot A \cdot (T(t) - T_s)}{C_1 \cdot m},
\]

or

\[
\frac{d(T(t) - T_s)}{dt} = \frac{V}{C_1 \cdot m} - \frac{C_2 \cdot A}{C_1 \cdot m} \cdot (T(t) - T_s).
\]

The solution to this differential equation is given by

\[
T(t) - T_s = \frac{V}{C_2 \cdot A} + C \cdot e^{-\frac{C_2 \cdot A}{C_1 \cdot m} \cdot t},
\]

where the constant \(C\) can be found by substituting for \(t\) the value \(t_b\):

\[
C = e^{\frac{C_2 \cdot A}{C_1 \cdot m} \cdot t_b} \cdot (T_e - T_s - \frac{V}{C_2 \cdot A}).
\]

So

\[
T(t) - T_s = \frac{V}{C_2 \cdot A} + e^{-\frac{C_2 \cdot A}{C_1 \cdot m} \cdot (t - t_b)} \cdot (T_e - T_s - \frac{V}{C_2 \cdot A}).
\]
If we substitute $t = t_h + t_1$ in this expression, the constant $V$ follows:

$$V = C_2 \cdot A \cdot \frac{(T_h - T_s) - (T_e - T_s) \cdot e^{-\frac{C_1}{m} \cdot t_1}}{1 - e^{-\frac{C_1}{m} \cdot t_1}}$$

$$= C_1 \cdot m \cdot k \cdot \frac{(T_h - T_s) - (T_e - T_s) \cdot e^{-k \cdot t_1}}{1 - e^{-k \cdot t_1}}. \quad (2)$$

Finally, we have to determine $Q$. For this, we note that

$$Q = V_{\text{gross}} \cdot \text{costs gas per Joule},$$

where $V_{\text{gross}}$ equals $V$ divided by the efficiency of the boiler, and where costs gas per Joule denotes the costs of the quantity of gas that yields a Joule of energy when burned.

We take Hfl 0.55 per m$^3$ as the gas price and 60 percent as the efficiency of the boiler; furthermore we know from literature that the heat of combustion for gas is $30 \cdot 10^6$ Joule per m$^3$. All this results in

$$Q = \frac{V}{0.6} \cdot \frac{0.55}{30 \cdot 10^6} = 3.1 \cdot 10^{-6} \cdot V \quad \text{Dutch guilders per second.} \quad (3)$$

We are now able to express $t_1$ and $t_2$ in terms of known constants, $T_h$ and $T_e$, since rewriting (1) and (2) respectively yields:

$$t_2 = \frac{1}{k} \cdot \ln \left( \frac{T_h - T_a}{T_e - T_a} \right), \quad (4)$$

$$t_1 = \frac{1}{k} \cdot \ln \left( \frac{T_e - T_s - \frac{V}{C_1 \cdot m \cdot k}}{T_h - T_s - \frac{V}{C_1 \cdot m \cdot k}} \right). \quad (5)$$

From these formulas it is easy to determine the costs of the heating at night if the temperatures $T_h$ and $T_e$ are set:

$$\left[ \frac{t_n}{t_1 + t_2} \right] \cdot t_1 \cdot Q \leq \text{costs} \leq \left[ \frac{t_n}{t_1 + t_2} \right] \cdot t_1 \cdot Q.$$
As an estimate for the costs we will therefore use \( \frac{t_n}{t_1 + t_2} \cdot t_1 \cdot Q \).

If we switch the thermostat off at night, we first have to determine the temperature \( T_m \) (\( T_{\text{minimum}} \)) of the water at the end of the night, and then the time \( t_m \) it takes to warm the water from \( T_m \) to \( T_h \). Analogous calculations as on pages 5-7 give

\[
T_m = T_s + (T_h - T_s) \cdot e^{-k t_n}
\]

\[
t_m = \frac{1}{k} \cdot \ln \left( \frac{T_m - T_s - \frac{V}{C_1 \cdot m \cdot k}}{T_h - T_s - \frac{V}{C_1 \cdot m \cdot k}} \right),
\]

so we have an expression for the costs \( t_m \cdot Q \) of this strategy.

The attentive reader will have noticed that in both cases the temperature at \( t = t_n \) is usually lower than \( T_h \): in the first case, the temperature will be \( T_h \) at \( t = t_n + t^* \) with \( 0 \leq t^* \leq t_1 \), while in the second case the temperature is \( T_h \) at time \( t = t_n + t_m \). As a consequence, we should not let the heating element switch on at time \( t_n \), but at time \( t_n - t^* \) in the first case, and at time \( t_n - t_m \) in the second case. Since, however, \( t^* \) and \( t_m \) are very small compared to \( t_n \), we can neglect this.

Now we have to find the minimum of the costs of letting the thermostat regulate the temperature of the water in the boiler at night, or switching it off. In practice, this means that we will have to determine the minimum of

\[
\frac{t_n}{t_1 + t_2} \cdot t_1 \cdot Q = \frac{t_n \cdot \ln \left( \frac{T_m - T_s - \frac{V}{C_1 \cdot m \cdot k}}{T_h - T_s - \frac{V}{C_1 \cdot m \cdot k}} \right)}{\ln \left( \frac{(T_h - T_s)(T_m - T_s - \frac{V}{C_1 \cdot m \cdot k})}{(T_h - T_m)(T_m - T_s - \frac{V}{C_1 \cdot m \cdot k})} \right)} \cdot Q
\]

and

\[
t_m \cdot Q = \frac{1}{k} \cdot \ln \left( \frac{(T_h - T_s) \cdot e^{-k t_n} - \frac{V}{C_1 \cdot m \cdot k}}{T_h - T_s - \frac{V}{C_1 \cdot m \cdot k}} \right) \cdot Q.
\]
We do not exaggerate when we say that these two expressions are rather difficult to compare!

Therefore we tried another approach to solve the problem, by drawing the following graph to illustrate what happens inside the boiler:

![Graph](image)

Figure 2: The temperature of the water in the boiler as a function of time.

We define \( u(t) \) as the energy that flows out of the boiler at moment \( t \) due to inefficiency, so that \( u(t) = k \cdot (T(t) - T_s) \). All the energy that flows out, will need to be replaced, since we require both \( T(0) \) and \( T(t_n) \) to be equal to \( T_h \). To minimize the energy that has to be added, we have to minimize the energy loss, i.e. we have to find

\[
\min_{u(t)} \left\{ \int_0^{t_n} u(t) dt \right\} = \min_{T(t)} \left\{ k \cdot \int_0^{t_n} T(t) dt - k \cdot t_n \cdot T_s \right\} = \min_{T(t)} \left\{ k \cdot \int_0^{t_n} T(t) dt \right\} - k \cdot t_n \cdot T_s.
\]
The first expression is precisely equal to the area below the graph multiplied by $k$, while the second expression is a constant. Thus, we find that the area below the graph has to be minimized, which of course happens when the thermostat is switched off during the night.

For a given boiler, with fixed temperature settings, the savings made through switching the thermostat off at night can now readily be determined using formulas (8) and (9): only the values of the occurring variables need to be inserted. In the Appendix, this is done for a specific boiler.

Conclusions

We can summarize the above calculations by saying that the costs are always minimized by switching the thermostat of the boiler off at night. The savings with respect to letting the thermostat regulate the water temperature throughout the night and day can be determined by comparing formulas (8) and (9); such savings, however, appear to be typically in the order of Hfl 0.10 a night. Usually, this would not justify the costs of a timer and increased wear and tear. Besides that, warm water is not immediately available at night in case it would be needed.
Appendix

In the example shown in Table 1, we considered a specific boiler, for which we determined the costs in a number of possible cases. Case 1 is the reference case; Case 2 corresponds to a longer night, while in Case 3 both $T_h$ and $T_c$ have been set to a higher temperature. In Case 4, a situation is considered in which the minimum temperature of the water in the boiler is allowed to be smaller at night.

In the example, we first have computed $Q$ with formula (3), and we determined $t_1$ and $t_2$ using formulas (4) and (5) respectively. To compute $T_m$, we applied formula (6), after which $t_m$ could be found with (7). Finally, using formulas (8) and (9), we determined the costs of the two strategies discussed in this report.

The parameters that are independent of the setting of the boiler, or that are constant in the four considered cases, are:

\[
T_* = 293 K, \quad k = 4 \cdot 10^{-6}, \\
V = 7 \cdot 10^3 W, \quad C_1 = 4184, \\
m = 100 \text{ kg}, \quad Q = 2.1 \cdot 10^{-4} \text{ Hfl/sec},
\]

In the table, costs(1) denotes the costs resulting from the boiler thermostat regulate the water temperature, while costs(2) equals the costs resulting from switching the thermostat off at night. The (relative) savings that can be made by using the second strategy are

\[
\frac{\text{costs}(1) - \text{costs}(2)}{\text{costs}(1)} \cdot 100\%,
\]

and these are displayed in the last row of the table.
### Table 1: Cost accounting for a boiler: four possible cases.

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2 (changes w.r.t Case 1)</th>
<th>Case 3 (changes w.r.t Case 1)</th>
<th>Case 4 (changes w.r.t Case 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_h$</td>
<td>343 K</td>
<td>353 K</td>
<td>323 K</td>
<td></td>
</tr>
<tr>
<td>$T_c$</td>
<td>333 K</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_n$</td>
<td>6 hours = 21600 sec</td>
<td>7 hours = 25200 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_1$</td>
<td>670 sec</td>
<td>688 sec</td>
<td>1322 sec</td>
<td></td>
</tr>
<tr>
<td>$t_2$</td>
<td>5579 sec</td>
<td>4558 sec</td>
<td>12771 sec</td>
<td></td>
</tr>
<tr>
<td>$t_m$</td>
<td>1890 sec</td>
<td>2068 sec</td>
<td>2312 sec</td>
<td></td>
</tr>
<tr>
<td>costs(1)</td>
<td>Hfl 0.50</td>
<td>Hfl 0.58</td>
<td>Hfl 0.61</td>
<td>Hfl 0.43</td>
</tr>
<tr>
<td>costs(2)</td>
<td>Hfl 0.40</td>
<td>Hfl 0.44</td>
<td>Hfl 0.49</td>
<td>Hfl 0.40</td>
</tr>
<tr>
<td>savings</td>
<td>18 %</td>
<td>23 %</td>
<td>18 %</td>
<td>7 %</td>
</tr>
</tbody>
</table>

Comparing Case 1 with the other three considered cases we notice that, as one might have expected:

- a longer night makes both costs(1) and costs(2) increase, but the costs of the first strategy grow relatively more (Case 2).

- whatever strategy we use, higher temperature settings ($T_h$ and $T_c$) result in higher costs, but the (relative) savings of applying the second strategy remain unchanged (Case 3).

- choosing a lower temperature for $T_c$ while still letting the thermostat regulate the water temperature at night, can be considered as a compromise between the two strategies we discussed. This is reflected by the costs of applying this ‘compromise’ strategy, which are lower than costs(1), but higher than costs(2) (Case 4).
The Parachute Jump
Problem Statement

When a parachute jump is made from an aeroplane, quite often the first part of the descent is made in 'free fall' at a high altitude. After a certain time the parachutist then opens his parachute to enable a soft landing to be completed.

The important question is to decide when the parachute should be opened. This must obviously not be too late, otherwise the landing speed will be too great; but on the other hand, a free fall can generate high speeds, which adds to the exhilaration of the event.

Usually a plane drops more than one parachutist, which gives rise to another problem concerning the moment when the parachute should be opened: if the parachutists jump 'too close' after each other, or if they open their parachutes 'too late' or 'too early', they may collide with each other during the jump or when landing.

In this report we will investigate the second problem. Thus, we will determine according to which rules the parachutists should jump out of the plane and open their parachutes in order to avoid collisions.

Given Information

The fall of the parachutist can be split into two parts: the free fall before the parachute is opened, and the parachute-fall afterwards.

I During the free fall, the air resistance \( r_f \) is proportional to the speed of the parachutist. In a formula:

\[
r_f = k_f \cdot v,
\]

where \( k_f \) equals 0.1367 m\(^{-1}\).

II During the parachute-fall, the air resistance \( r_p \) is proportional to the square of the parachutist's speed, so

\[
r_p = k_p \cdot v^2.
\]

The constant \( k_p \) depends on the parachute used. Usually it is required that a parachute landing should be like falling of a twelve-feet wall, which determines \( k_p : k_p = 0.1828 \) s\(^{-1}\).

In both formulas, the mass of the parachutist (including his parachute) is scaled to 1 kg.
Variables

In the model we have the following variables:

\[ v_0 \]: the speed of the plane (m/s),
\[ y_0 \]: the height of the plane (m),
\[ x(t) \]: the position of a parachutist at \( t \) seconds after his jump (m),
\[ v(t) \]: the speed of a parachutist at \( t \) seconds after his jump (m/s),
\[ a(t) \]: the acceleration of a parachutist at \( t \) seconds after his jump (m/s\(^2\)),
\[ \tau_i \]: the time parachutist \( i \) (\( i \geq 1 \)) makes a free fall (s),
\[ \Delta_i \]: the time between the jumps of parachutist \( i - 1 \) and \( i \) (\( i \geq 2 \)) (s),
\[ T_i \]: the moment at which parachutist \( i \) opens his parachute, thus \( T_1 = \tau_1 \) and for \( i \geq 2 \) holds \( T_i = \sum_{j=2}^{i} \Delta_j + \tau_i \) (s),
\[ g \]: the acceleration of gravity; \( g = 9.8065 \text{ m/s}^2 \).

Analysis of the Problem

If we assume that the plane is travelling horizontally in the positive \( x \)-direction, we can draw a graph of a parachute jump, where the parachutist jumps out of the plane at \( t = 0 \), and opens his parachute \( \tau_i \) seconds later; see Figure 1.

First we consider part I of the graph: the free fall.

Since the sum of the powers on the parachutist equals its mass (1 kg) multiplied by its acceleration, we have

\[ a = k_f \cdot v - g, \]

which can be written as the differential equation

\[ \dot{v} = k_f \cdot v - g. \]

At \( t = 0 \) the parachutist has a speed

\[ v = v_0 = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}, \]

so the solution to the differential equation is

\[ v(t) = \begin{pmatrix} 0 \\ -g/k_f \end{pmatrix} + v_0 + \begin{pmatrix} 0 \\ g/k_f \end{pmatrix} e^{-k_f t}. \]
or

\[ \vec{v}(t) = \begin{pmatrix} 0 \\ -g/k_f \end{pmatrix} + \begin{pmatrix} v_0 \\ g/k_f \end{pmatrix} e^{-k_f t}. \]

Since \( \vec{v}(t) \) equals \( \vec{\alpha}(t) \), the derivative of the position of the parachutist at time \( t \), the last formula can be seen as a differential equation for \( \vec{\alpha}_t \). Solving it yields

\[ \vec{\alpha}(t) = -\frac{1}{k_f} e^{-k_f t} \begin{pmatrix} v_0 \\ g/k_f \end{pmatrix} + \begin{pmatrix} 0 \\ -g/k_f \end{pmatrix} t + \vec{\alpha}(0) + \frac{1}{k_f} \begin{pmatrix} v_0 \\ g/k_f \end{pmatrix}. \]  

In part II of the graph we also have that the sum of the powers on the parachutist equals its mass multiplied by its acceleration. In this case we get

\[ k_p \cdot |\vec{v}| \cdot \dot{\vec{v}} - g = \vec{a} \quad (= \ddot{\vec{v}}). \]

This differential equation for \( \vec{v} \) is not as easy to solve as the previous one. We chose a numerical approach (using ‘Euler Forward’) to approximate the parachutist’s fall. To obtain the picture shown in Figure 2, we ran the program with

\[ v_0 = 40 \text{ m/s}, \]
\[ y_0 = 500 \text{ m}, \]
\[ \tau_1 = 10 \text{ s}, \]
\[ \tau_2 = 8 \text{ s}, \]
\[ \tau_3 = 5 \text{ s}, \]
\[ \Delta_2 = 2 \text{ s}, \]
\[ \Delta_3 = 2 \text{ s}. \]
We see that the $y$-component of the position of the parachutist does not matter very much: the $x$-component mainly determines if the paths of two (or more) parachutists will cross each other before or after their landing.

**Solution**

Now we have determined the orbits of the parachutists, we can investigate how soon the parachutists can jump after each other, and when they can open their parachutes. For safety reasons and efficiency reasons respectively we make the following two requirements:

(a) Two parachutists should land at at least $S$ meter distance.

(b) The plane drops $M$ parachutists within a range of $R$ meter (in one straight-line flight).

First we consider requirement (a). From Figure 2 we conclude that after opening his parachute, a parachutist falls (nearly) vertically. If we write $x(t)$ for the $x$-component of $\mathbf{x}(t)$, the requirement reduces to
Substituting the $x$-component of $x(t)$ from formula (1) yields for $2 \leq i \leq M$:

$$-\frac{v_0}{k_f} e^{-k_f \cdot \tau_{i+1}} + v_0 \sum_{j=2}^{i+1} \Delta_j + \frac{v_0}{k_f} + \frac{v_0}{k_f} e^{-k_f \cdot \tau_i} - v_0 \sum_{j=2}^{i} \Delta_j - \frac{v_0}{k_f} - S \geq 0,$$

and thus

$$\Delta_{i+1} \geq \frac{1}{k_f} \left( e^{-k_f \cdot \tau_{i+1}} - e^{-k_f \cdot \tau_i} \right) + \frac{S}{v_0}.$$

For $i = 1$ we get

$$-\frac{v_0}{k_f} e^{-k_f \cdot \tau_2} + v_0 \cdot \Delta_2 + \frac{v_0}{k_f} + \frac{v_0}{k_f} e^{-k_f \cdot \tau_1} - \frac{v_0}{k_f} - S \geq 0,$$

so

$$\Delta_2 \geq \frac{1}{k_f} \left( e^{-k_f \cdot \tau_2} - e^{-k_f \cdot \tau_1} \right) + \frac{S}{v_0}.$$

Combining both expressions, we get the requirement

$$\Delta_{i+1} \geq \frac{1}{k_f} \left( e^{-k_f \cdot \tau_{i+1}} - e^{-k_f \cdot \tau_i} \right) + \frac{S}{v_0} \quad (1 \leq i \leq M). \quad (2)$$

Requirement (b) can be written as

$$(x$-coordinate of parachutist $M$ at $T_M$) -
(x-coordinate of parachutist $1$ at $T_1$) \leq R,$

that is

$$-\frac{v_0}{k_f} e^{-k_f \cdot \tau_M} + v_0 \sum_{j=2}^{M} \Delta_j + \frac{v_0}{k_f} + \frac{v_0}{k_f} e^{-k_f \cdot \tau_1} - \frac{v_0}{k_f} \leq R,$$

or

$$\sum_{j=2}^{M} \Delta_j \leq \frac{1}{k_f} \left( e^{-k_f \cdot \tau_M} - e^{-k_f \cdot \tau_1} \right) + \frac{R}{v_0}.$$

If we assume $\Delta_i \equiv \Delta$ for all $i$, this requirement reduces to

$$\Delta \leq \frac{1}{(M-1)k_f} \left( e^{-k_f \cdot \tau_M} - e^{-k_f \cdot \tau_1} \right) + \frac{R}{(M-1)v_0}, \quad (3)$$
while the first requirement (formula(2)) becomes

$$\Delta \geq \frac{1}{k_f} \left( e^{-k_f \cdot \tau_{i+1}} - e^{-k_f \cdot \tau_i} \right) + \frac{s}{v_0} \quad (1 \leq i < M)$$

(4)

**Practical Examples**

**Example 1**

If we assume that all parachutists enjoy their free fall equally long, i.e.

$$\tau_i \equiv \tau \quad (1 \leq i \leq M),$$

the two requirements for $\Delta$ reduce to

$$\begin{cases} 
\Delta & \leq \frac{R}{(M-1)v_0} \\
\Delta & \geq \frac{S}{v_0}.
\end{cases}$$

The maximum number of parachutists that can jump is

$$M = 1 + \frac{R}{S}.$$

If want this maximum number of parachutists to jump, we have to choose

$$\Delta = \frac{S}{v_0} \left( = \frac{R}{(M-1)v_0} \right).$$

These results are exactly as one might have expected.

**Example 2**

If we assume that each parachutist enjoys his free fall a fixed time $\vartheta$ longer ($\vartheta > 0$) or shorter ($\vartheta < 0$) than his predecessor, we get

$$\tau_{i+1} = \tau_i + \vartheta \quad (1 \leq i < M).$$

Writing $\tau := \tau_1$, this is equivalent to

$$\tau_i = \tau + (i-1)\vartheta \quad (1 \leq i < M).$$
Requirements (3) and (4) for $\Delta$ reduce to

\[
\begin{align*}
\Delta &\leq \frac{1}{(M-1)k_f} \left( e^{-k_f(\tau + (M-1)\vartheta)} - e^{-k_f \cdot \tau} \right) + \frac{R}{(M-1)v_0} \\
\Delta &\geq \frac{1}{k_f} \left( e^{-k_f(\tau + i \cdot \vartheta)} - e^{-k_f(\tau + (i-1)\vartheta)} \right) + \frac{S}{v_0} & (1 \leq i < M)
\end{align*}
\]

Of course, this set can only have a solution if

\[
\frac{1}{k_f} \left( e^{-k_f(\tau+i\cdot\vartheta)} - e^{-k_f(\tau+(i-1)\vartheta)} \right) + \frac{S}{v_0} \leq \frac{1}{(M-1)k_f} \left( e^{-k_f(\tau+(M-1)\vartheta)} - e^{-k_f \cdot \tau} \right) + \frac{R}{(M-1)v_0}
\]

for $1 \leq i < M$.

In case $\vartheta$ is positive, the left hand side is maximal for $i = M - 1$. Therefore the set of requirements can be reduced to their strongest requirement

\[
\frac{1}{k_f} \left( e^{-k_f(\tau+(M-1)\vartheta)} - e^{-k_f(M-2)\vartheta} \right) + \frac{S}{v_0} \leq \frac{1}{(M-1)k_f} \left( e^{-k_f(\tau+(M-1)\vartheta)} - e^{-k_f \cdot \tau} \right) + \frac{R}{(M-1)v_0},
\]

or, equivalently,

\[
(M-2)e^{-k_f(M-1)\vartheta} - (M-1)e^{-k_f(M-2)\vartheta} + 1 \leq \frac{k_f \cdot e^{k_f \cdot \tau}}{v_0} (R - S(M-1)).
\]

In case $\vartheta$ is negative, the left hand side of (6) is maximal for $i = 1$. Now the set of requirements can be reduced to

\[
\frac{1}{k_f} \left( e^{-k_f(\tau \vartheta)} - e^{-k_f \cdot \tau} \right) + \frac{S}{v_0} \leq \frac{1}{(M-1)k_f} \left( e^{-k_f(\tau+(M-1)\vartheta)} - e^{-k_f \cdot \tau} \right) + \frac{R}{(M-1)v_0},
\]

which is equivalent to

\[
(M-1)e^{-k_f \cdot \vartheta} - e^{-k_f(M-1)\vartheta} - M + 2 \leq \frac{k_f \cdot e^{k_f \cdot \tau}}{v_0} (R - S(M-1)).
\]

In case $\vartheta$ is zero, formula (6) reduces to

\[
\frac{1}{k_f} \left( e^{-k_f \cdot \tau} - e^{-k_f \cdot \tau} \right) + \frac{S}{v_0} \leq \frac{1}{(M-1)k_f} \left( e^{-k_f \cdot \tau} - e^{-k_f \cdot \tau} \right) + \frac{R}{(M-1)v_0},
\]

or

\[
S(M-1) \leq R.
\]
For fixed values of $v_0$, $R$, $S$, $M$ and $\tau$ we can calculate if a certain value of $\vartheta$ is possible, using formula (7), (8) or (9), depending on $\vartheta$ being positive, negative or zero. If the chosen value is possible, formula (5) can be used to calculate the lower and upper bound for $\Delta$.

In the numerical example below we have chosen

\begin{align*}
  v_0 &= 40, \\
  R &= 250, \\
  S &= 10.
\end{align*}

The constant $k_f$ is known to be equal to 0.1828.

The first three columns of Table 1 contain respectively $M$, $\tau$ and $\vartheta$. In the fourth column we displayed the difference between the right hand side and the left hand side of (7), (8) or (9), depending on the sign of $\vartheta$. If a solution for the time between the jumps of two consecutive parachutists (i.e. $\Delta$) exists, the entry in the fourth column must be positive. If it is not, we put only a * in the table. In case a solution exists, column five and six contain the lower, respectively upper bound for $\Delta$. When the lower bound was negative, we replaced it by ‘0’ in the table, since $\Delta$ must be positive.

The results show, for example, that is less parachutists have to be dropped under the same requirements, the time between their jumps can be longer, according to what one might have expected.
Conclusions and Recommendations

If the speed and the height of the plane that drops the parachutists are given, and safety and efficiency requirements (i.e. $R$ and $S$) are chosen, the formulas in this report show accordingly to which rules parachutists should jump out of the plane and open their parachutes. It seems that after opening his parachute, a parachutist immediately falls nearly vertically. Since usually no quick changes occur in nature, it may be more realistic to make a model that uses a more gradual opening of the parachutes, resulting in a model with not only a phase I and II, but also a phase in between them.
The snowgang in the snow

1 Problem.

The roads in Steinbach (south-Germany) are covered with snow. The snow­
gang starts normally to work at 8.00 am, but there is a snowshower expected
at 8.30 am. The snowgang cannot work if the snow is too high. The foreman
of the snowgang therefor wonders if it would be wise to start earlier. The
foreman decides to ask Frank Müller for an advice.

2 Suggested solution.

The easiest solution for Frank is to say that the foreman must start at 6.00
am so that he sure will be ready at 8.30 am. This solution is not acceptable
for the foreman because in this way he has to get out of bed early. So Frank
needs to look for a simple model of the problem, as a result he will see that
if the foreman start one quarter of an hour earlier he can clear the road in
most of the cases. Frank’s solution is valid under the following assumptions:

- It is known how much snow will fall at any time.
- It is known how late the snowshower will start.
- The velocity of the snowgang will depend linearly on the height of the
  snow.
3 Data.

Frank got some information to make his model. The foreman gave him the following data:

- The snow shower takes about one hour.
- The snow shower will vary but will be less than 0.1 cm/s.
- The velocity of the snow gang on a clear road is 10 m/s.
- The snow gang cannot work if the snow height is above 1.5 m.
- Average height of the snow before the snow shower is 0.5 m.
- The snow gang has to clear a road with a length of 12.5 km.

4 Model.

Frank made the following mathematical model:

- $S(t)$: Snowfall on timepoint $t$.
- $H(t)$: Height of the snow just before the snow gang at timepoint $t$.
- $V(H)$: Velocity of the snow gang at a snow height $H$.
- $X(t)$: The cleared distance at timepoint $t$.

In this model all the quantities which you need are defined. If the snowfall $S(t)$ is known then all the other quantities are known. The assumptions already made, define $S(t)$ and $V(H)$:

$$V(H) = \begin{cases} 
10 - \frac{(10/H_{\text{max}})H}{0 < H \leq H_{\text{max}}} \\
0 & H > H_{\text{max}}
\end{cases}$$
The assumptions lead to easy computations:

\[
S(t) = \begin{cases} 
  \frac{c}{a}t & 0 < t < a \\
  a & a \leq t \leq b \\
  c - \frac{c}{a} (t - b) & b < t < s_1 \\
  0 & t > s_1 
\end{cases}
\]

\[
H(t) = \int_0^t S(\tau) d\tau + H_0 \\
X(t) = \int_0^t V(H(\tau)) d\tau + X_0
\]

\(X(t)\) depends linear on \(t\) on each interval \((0, a), (a, b)\) and \((b, s_1)\).

\(H_0 = 0.5m\), the assumption was that the average height before the snowshower was 0.5m.

\(X_0 = 3.5km\), this is the distance which can be cleared before the snowshower starts.

The function \(H(t)\) can be calculated exactly:

\[
H(t) = \begin{cases} 
  \frac{(c/2)a^2}{2} + H_0 & 0 < t < a \\
  \frac{(c/2)a + c(t - a)}{2} + H_0 & a \leq t \leq b \\
  \frac{(c/2)a + c(b - a) + c(t - b) - c/(2(s_1 - b))(t - b)^2}{2} + H_0 & b < t < s_1 \\
  \frac{(c/2)a + c(b - a) + (c/2)(s_1 - b) + H_0}{2} & t \geq s_1 
\end{cases}
\]

Now \(H(t)\) is known and \(X(t)\) can also be calculated.
4.1 Solution procedure.

Frank made the following solution procedure:

\[
\begin{align*}
H(s_l) &\leq H_{\max} \\
\Rightarrow &\text{yes no problem} \\
\downarrow \text{no} \\
\text{determine } t^* \text{ such that } H(t^*) = H_{\max} \\
\downarrow \\
x(t^*) &\geq 9 \text{ km} \\
\Rightarrow &\text{yes no problem} \\
\downarrow \text{no} \\
\text{snowgang should start at 8.00 am } - \left( \frac{9000 - x(t^*)}{v_0} \right)
\end{align*}
\]

with \(v_0\): the velocity of the snowgang at \(t = 0\).

5 Numerical results.

Frank decides to look at the following two cases. Case 1 is a worst case situation and case 2 is an average case situation.

1) \( s_l = 3600 \text{ sec} \quad H_0 = 50 \text{ cm} \)
   \( a = 1200 \text{ sec} \quad H_{\max} = 1.5 \text{ m} \)
   \( b = 2400 \text{ sec} \quad c = 0.1 \text{ cm/s} \)
   
   \( H(3600) = 2.9 \text{ m} > 1.5 \text{ m} \)
   
   \( t^* = 1600 \text{ sec because } H(1600) = 1.5 \text{ m} \)
   
   \( x(t^*) = 6933 \text{ m} \)
   
   the snowgang should start at 8.00 am \(- 9000 - 6933)/v_0 \approx 7.55 \text{ am} \)

2) \( s_l = 3600 \text{ sec} \quad H_0 = 50 \text{ cm} \)
   \( a = 1200 \text{ sec} \quad H_{\max} = 1.5 \text{ m} \quad H(3600) = 1.46 \text{ m} < 1.5 \text{ m}: \text{ no problem} \)
   \( b = 2400 \text{ sec} \quad c = 0.04 \text{ cm/s} \)

for the snowgang.
6 Discussion.

In Franks’s model no stochastic variables are used. It would be more natural to make snowfall stochastic. This can be done by changing one of the variables $a, b, c, s_t, H_0$ into a stochastic variable. Another possibility is to make the "form" of the snowfall stochastic. All these changes make the model difficult to use. Because if you change one of these variables into a stochastic variable you get an integral over a stochastic variable. For example $c$ is made stochastic: $c \sim N(\mu, \sigma^2)$. Then $S(t) \sim N((\mu/a)t, ((\sigma/a)t)^2)$ for $0 < t < a$. But it becomes difficult to say something of the distribution, expectation and variance of $H(t) = \int_0^t S(\tau)d\tau + H_0$. In the model the velocity of the snowgang depends linearly on the height of the snow. This can be made more realistic by using an exponential dependence. For example $V(H) = 10\exp(-H)$.

Another possibility for an extension is to change the solution procedure in a way so that the height of the snow stays below $H_{\text{max}}$ in all places. In this way you prevent that the snowgang gets into trouble the next day. But the difficulty with this is that the snowgang cannot work twice a day.

7 Conclusion.

The model of Frank which is presented here gives the result that if the foreman of the snowgang start one quarter of an hour earlier he can clear the road in most of the cases. The model is not very realistic because snowfall is stochastic and in the model no stochastic variables are used. If stochastic variables are used in the model the mathematical problems to solve will become difficult to solve.
The Optimal Width of Hospital Corridors

1 Description of the Problem

When building a new hospital it might be important to consider the optimal width of the corridors. This might save a lot of money. Not only because less brick etc. will be required but also the heating costs will be less. On the other hand the corridors should be wide enough for hospital beds to pass. This is especially not trivial when one considers the situation that it ought to be possible to go around a corner with a bed.

The hospital direction turned to a mathematical centre with the problem. They asked the mathematicians also to take into account that in the future long pipes have to be transported through the corridors for improving the infra-structure. The direction wants to know what the maximal length of the pipes can be given the width of the corridors.

2 Results

After some days of calculation the mathematical centre came with the following results:

- build corridors of width 1.42 mt. This is the minimal width so that it is for hospital beds (0.90 x 1.90 mt.) still possible to pass and go around a corner;

- the maximal length of the pipes is 4.72 mt. Then it is still possible to go through the corridors.

These results were based on the following assumptions:

- there are only perpendicular corners, so no T-forks or crossings are taken into consideration; (note that if one is possible to go round perpendicular corners it is also possible to go round T-forks and crossings)

- in all corridors there is one-way traffic. The problem is trivial if one considers the situation that beds can go both ways;

- the height of the corridors is 2.50 mt. ;

- all beds are 0.90 x 1.90 mt. ;
3 Mathematical Modelling

The mathematical centre did the following modelling:

While going around a corner there is 5 cm. left at each side of the bed, so the bed can be considered to be 1 x 2 mt. At any time while going around the corner the situation might be as follows:

Figure 1 "The bed while going around the corner"

\[
\begin{align*}
\ell &= \text{length of the bed} \\
[w] &= \text{width of the bed} \\
u_1, u_2 &= \text{the protrusion in respectively vertical and horizontal direction} \\
x &= \text{distance AB} \\
\theta &= \text{angle of the length axis of the bed with the vertical axis} \\
0 \leq x \leq \ell \\
0 \leq \theta \leq \frac{\pi}{2}
\end{align*}
\]

\(x\) en \(\theta\) define the position of the bed. It might be clear that an optimal strategy exists in pushing the bed around the corner always touching point B and minimizing the maximum of \(u_1\) and \(u_2\).

4 Solution

Given \(x\) and \(\theta\) with some simple geometry it is easy to calculate \(u_1(x, \theta)\) and \(u_2(x, \theta)\):
\begin{align}
    u_1(x, \theta) &= x \cos \theta + \omega \sin \theta \\
    u_2(x, \theta) &= (l - x) \sin \theta + \omega \cos \theta
\end{align} \hspace{1cm} (4.1)

As explained in the previous section we are curious of the maximum of \( u_1(x, \theta) \) and \( u_2(x, \theta) \). Define

\[
u_3(x, \theta) = \max[u_1(x, \theta), u_2(x, \theta)] \hspace{1cm} (4.3)\]

Given an angle \( \theta \) we are interested in minimizing \( u_3(x, \theta) \). From figure 2 it follows that given an angle \( \theta \) we should choose \( x(\theta) \) such that \( u_1(x, \theta) = u_2(x, \theta) \) as to minimize \( u_3(x, \theta) \).

Figure 2 ”The protrusion as function of \( x \)”

\[\begin{array}{c}
\text{From } u_1(x, \theta) = u_2(x, \theta) \text{ it follows that } \\
\theta(x) = \frac{\omega (\cos \theta - \sin \theta) + \omega \sin \theta}{\sin \theta + \omega \cos \theta} \hspace{1cm} (4.4)
\end{array}\]

When we define \( u(\theta) \) as the minimal width of the corridor we need, it is clear that \( u(\theta) = u_3(x(\theta), \theta) = u_1(x, \theta) \) and that we are looking for the maximum of \( u(\theta) \) for \( 0 \leq \theta \leq \frac{\pi}{2} \)

\[
u(\theta) = \frac{(\omega (\cos \theta - \sin \theta) + \omega \sin \theta) + \omega \sin^2 \theta + \omega \sin \theta \cos \theta}{\sin \theta + \omega \cos \theta}
\]

\[\leftrightarrow\]
To find optimal values we calculate the derivative:

\[
u'(\theta) = \frac{(\cos^2\theta - \sin^2\theta)(\sin\theta + \cos\theta) + (\sin^2\theta + \cos^2\theta)(\cos\theta - \sin\theta)}{(\sin\theta + \cos\theta)^2}\]

(4.6)

Inspection learns that this function \( u'(\theta) \) is positive on the interval from 0 to \( \frac{\pi}{4} \), zero at \( \frac{\pi}{4} \), and negative on the interval from \( \frac{\pi}{4} \) to \( \frac{\pi}{2} \). So the maximum is found at \( \theta = \frac{\pi}{4} \) and substitution learns that the corridors should be at least of width:

\[
g = u\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}(w + \frac{1}{2})\]

(4.7)

The maximal length \( l_{\text{max}} \) of the pipes that will have to be transported through the corridors (width of pipes \( \approx 0 \)) is:

\[
l_{\text{max}} = \frac{4g}{\sqrt{2}}\]

(4.8)

if one doesn't take into account the possibility to move the pipes up- and downward. If one uses this possibility and one takes a perpendicular viewpoint at the pipe the new situation is as sketched in figure 3.

Figure 3 "A perpendicular viewpoint at the pipe"

\[h = \text{height of the corridors}\]

\[l_{\text{max}} = \text{maximal length of pipe that can be transported through corridors of width } g \text{ in } "\text{old}" \text{ situation}\]

If we define \( l'_{\text{max}} \) as the maximal length of pipes that can be transported through corridors of width \( g \) in the "new" situation \( l'_{\text{max}} \) is:
\[
  l_{\text{max}}^* = \frac{l_{\text{max}}}{\cos(\arctan(l_{\text{max}}/l))} \tag{4.9}
\]

5 Conclusions and Remarks

- Substitution of \( w = 1 \) mt. and \( l = 2 \) mt. into (4.7) learns that the necessary width of the corridors must be \( \sqrt{2} \approx 1.42 \) mt;

- Substitution of \( g = \sqrt{2} \) into (4.8) learns that pipes of length \( l_{\text{max}} = 4 \) mt. can be transported through corridors in the "old" situation;

- If one uses the height of the corridors substitution of \( h = 2.5 \) mt and \( l_{\text{max}} = 4 \) mt. learns that pipes of length \( l_{\text{max}}^* = 4.71 \) mt can be transported through corridors in the "new" situation.

The mathematical centre remarked that the problem is trivial if one allows two-way traffic. Make the corridors then as wide as twice the width of a bed and let only one bed at the time go around a corner.
A Punch Problem

1 Introduction

1.1 The present punch problem
Disks are punched from a steel plate in a production process. Disks with diameter of 25 cm and 10 cm are needed. At this moment 16 disks with a diameter of 25 cm and 9 disks with a diameter of 10 cm are punched from a plate of 1 m length and width.
The following problem arises: is it possible to reduce the amount of waste material by punching these disks in another way from a rectangular plate which length and width are approximately 1 m.

1.2 General punch problems
We formulate two general punch problems.

1. Given the shape or shapes that have to be punched from a steel plate with given length and width, determine a pattern such that the used fraction of area is maximal. A pattern represents the positions of the shapes that have to be punched, on the plate.

2. Given the shape or shapes that have to be punched, determine the length and width of the plate and a pattern such that the used fraction of area is maximal.

A certain ratio of the numbers of different shapes that have to be produced, can be imposed on these problems. We only require that disks of both types are punched from a plate in our punch problem.

1.3 Summary
We have considered different patterns, starting from a plate with variable length and width and a production of only one type of disk. We have analysed these patterns as follows: the pattern that prescribes how disks with a diameter of 25 cm are punched from a plate, is extended by stating how the remaining area of the plate can be used optimally by punching disks with a diameter of 10 cm.
The presently used pattern is the best pattern among the considered ones if
we can choose the length and the width of the plate within certain bounds. It is not certain that no other patterns exist which yield a higher used fraction of area than the patterns we have considered.

We assume that no loss of material occurs in the punching process. This is not the case in practise, so disks touching each other can not be punched from a plate. This implies that in such situations the diameter of the punched disks is smaller, or that the plate has to be larger to punch disks of the required diameters.
2 One type of disk

2.1 Mathematical model

We model the first general problem, assuming that only one type of disk is punched from a rectangular plate, as follows.

Define

\[ L := \text{length of the plate} \]
\[ B := \text{width of the plate} \]
\[ r := \text{radius of a disk} \]
\[ n := \text{the number of disks} \]
\[ (x_i, y_i) := \text{position of the centre of disk } i \text{ on the plate} \]

Problem:

\[
\begin{align*}
\max \quad & \frac{\pi^2 n}{LB} \\
\text{s.t.} \quad & n \in \mathbb{N} \\
& r \leq x_i \leq B - r \quad i = 1, \ldots, n \\
& r \leq y_i \leq L - r \quad i = 1, \ldots, n \\
& (x_i - x_j)^2 + (y_i - y_j)^2 \geq (2r)^2 \quad i = 1, \ldots, n \wedge \\
& j = 1, \ldots, i - 1, i + 1, \ldots, n
\end{align*}
\]

This model can easily be extended to a model that describes the production of different shapes, or a model that describes the second general problem.

2.2 Complexity of the problem

It is probably not possible to develop an algorithm that generates an optimal solution of an instance of this problem within a computing time that is bounded by a polynomial of the size of the instance. For this reason we will consider patterns in the next section, which hopefully are good approximations of an optimal solution.
2.3 Structured patterns

2.3.1 Plate with infinite length and width

We consider the following pattern on a plate with infinite length and width.

![Diagram of a pattern on a plate with infinite length and width.]

The distance \( d \) between two successive disks in a row is subject to the following inequalities.

\[
0 \leq d \leq 2(\sqrt{3} - 1)r
\]

If \( d = 0 \) or \( d = 2(\sqrt{3} - 1)r \), two successive disks in a row or column touch, respectively.

We can divide the plate into parallelograms, as drawn in Figure 1, independently of \( r \) and \( d \). Note that the area of the four parts of disks in a parallelogram equals the area of one disk. The area \( O_p \) of a parallelogram is computed with the help of Figure 2.
The fraction $U$ of disk area per parallelogram is

$U = \frac{4r^2 \sin(2\varphi)}{4 \sin(2\varphi)} \quad \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$

$U$ is maximal ($U = \frac{\pi}{2\sqrt{3}} \approx 0.907$) if $\varphi = \frac{\pi}{6} \lor \varphi = \frac{\pi}{3}$, or $d = 2(\sqrt{3} - 1)r \lor d = 0$.

Both values of $d$ result in the same pattern. $U = \frac{\pi}{2\sqrt{3}}$ is only attained for infinite length and width of the plate. If the plate has a finite length or width, $U$ is smaller because of irregularities in the pattern at the border of the plate.

It is known from literature that $\frac{\pi}{2\sqrt{3}}$ is the largest used fraction of area that can be attained with patterns consisting only of disks of one type.

For this reason we will consider in the next section, among others, patterns for plates with finite length and width, which are similar to the pattern which we have considered in this section.
2.3.2 Plate with finite length and width

We will analyse three different patterns, one similar to the pattern which is presently used, and two patterns similar to the pattern discussed in the previous section.

1. "Straight" pattern

![Diagram of a "straight" pattern on a plate with length \( L \) and width \( B \).]

Define

\[
\begin{align*}
    a_B &:= \text{number of disks per row} \\
    a_L &:= \text{number of disks per column} \\
    U &= \text{used fraction of area}
\end{align*}
\]

The following equations hold:

\[
\begin{align*}
    a_B &= \left\lfloor \frac{B}{2\pi} \right\rfloor, \quad a_L = \left\lfloor \frac{L}{2\pi} \right\rfloor \\
    U &= \frac{a_B a_L \pi^2}{BL} = \frac{\left\lfloor \frac{B}{2\pi} \right\rfloor \left\lfloor \frac{L}{2\pi} \right\rfloor \pi^2}{BL}
\end{align*}
\]
Mark that $U$ is symmetrically in $B$ and $L$.

$U$ is maximal if $L = p2r$, $B = q2r$, $p, q \in \mathbb{N}$:

$$U = \frac{\pi r^2}{pqr} = \frac{\pi}{4} \approx 0.785$$

If we consider a plate with infinite length and width, this pattern results in a significantly smaller used fraction of area than the pattern we have analysed in the previous section.

2. "Oblique" patterns

Figure 4: An "oblique" pattern 1.

Figure 5: An "oblique" pattern 2.
The number of disks $a_B$ in row 1 for pattern 1 and 2 is maximal:

$$a_B = \left\lfloor \frac{B}{r} \right\rfloor$$

The following equations hold:

$$d_1 = \frac{B - 2a_B r}{a_B - 1}, \quad h_1 = \sqrt{(2r)^2 - (r + \frac{1}{2}d_1)^2}$$

$$h_2 = \sqrt{(2r)^2 - (r + \frac{1}{2}d_2)^2} = \sqrt{(2r)^2 - (r + e_2)^2}$$

This last equation implies $e_2 = r + \frac{1}{2}d_2$.

So, a condition to be able to make pattern 2 is $B - 2a_B r \geq r$.

The following equation holds in this case:

$$d_2 = \frac{B - (2a_B + 1)r}{a_B - \frac{1}{2}}$$

Define $g_i$ as the number of rows of disks for pattern $i$, $i \in \{1, 2\}$.

The height of $g_i$ rows of disks is $2r + (g_i - 1)h_i$, so the number of rows of disks $g_i$ is

$$g_i = \left\lfloor \frac{B - 2r}{h_i} \right\rfloor + 1, \quad i \in \{1, 2\}$$

Define

$$T_i := \text{the total number of disks for pattern } i, \quad i \in \{1, 2\}$$

$$U_i := \text{used fraction of area for pattern } i, \quad i \in \{1, 2\}$$

The following equations hold:

$$T_1 = \begin{cases} na_B + n(a_B - 1) = 2na_B - n & \text{if } g_1 = 2n \\ (n + 1)a_B + n(a_B - 1) = 2na_B - n + a_B & \text{if } g_1 = 2n + 1 \end{cases}$$

$$T_2 = g_2a_B$$
\[ U_1 = \begin{cases} 
\frac{(2na_B-n)\pi r^2}{LB} & \text{if } g_1 = 2n \\
\frac{(2na_B-n+a_B)\pi r^2}{LB} & \text{if } g_1 = 2n + 1 
\end{cases} \]

\[ U_2 = \frac{g_2 a_B \pi r^2}{LB} \]

Mark that \( U_i, \ i \in \{1, 2\}, \) is not symmetrically in B and L. If the rows of disks are formed lengthwise and not widthwise, formulas can be derived by exchanging the role of B and L.
3 Examples

3.1 Plate with 1 m length and width

Starting from a plate with a length and width of 1 m, we compare the presently used pattern, which is drawn in Figure 6, and "oblique" pattern 1, which is extended by stating how the remaining area of the plate is optimally used by punching disks with a diameter of 10 cm. This last pattern is drawn in Figure 7.

The corresponding used fractions of area are listed in Table 1, for the case that disks with a diameter of 10 cm are not taken into account, as well as the case that they are taken into account.

![Figure 6: "Straight" pattern on a plate with a length and width of 1.000 m.](image)
Table 1: Used fractions of area for the "straight" pattern.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Only r=0.125</th>
<th>Also r=0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;straight&quot;</td>
<td>0.785</td>
<td>0.856</td>
</tr>
<tr>
<td>&quot;oblique&quot; 1</td>
<td>0.687</td>
<td>0.829</td>
</tr>
</tbody>
</table>

Figure 7: "Oblique" pattern 1 on a plate with a length and width of 1.000 m.

3.2 Plate with variable length and width

Used fractions of area for certain length-width combinations are listed in Table 2, 3 and 4, for "straight" pattern, "oblique" pattern 1 and 2 respectively. Per length-width combination, the fraction is listed in which the disks with a diameter of 10 cm are not taken into account, as well as the case that they are taken into account. The number of disks with a diameter of 25 cm and 10 cm are stated between brackets after these two fractions. The length-width combinations are chosen such that the distance $d$ between two successive disks in a row is zero, and that the length is completely used by rows of disks. Drawings of an "oblique" pattern 1 and 2 are given in Figure 8 and 9, respectively.
<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>1.000</th>
<th>1.250</th>
<th>1.500</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.000</td>
<td></td>
<td>0.785 (16)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.125</td>
<td></td>
<td>0.785 (20)</td>
<td>0.785 (25)</td>
<td></td>
</tr>
<tr>
<td>1.500</td>
<td></td>
<td>0.785 (24)</td>
<td>0.785 (30)</td>
<td>0.785 (36)</td>
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<tr>
<td></td>
<td>1.000</td>
<td>0.861 (12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.116</td>
<td>0.841 (8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.333</td>
<td>0.854 (8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.549</td>
<td>0.852 (12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.000</td>
<td>0.853 (12)</td>
<td></td>
<td></td>
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<tr>
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<tr>
<td>1.116</td>
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<td>0.846 (8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.333</td>
<td></td>
<td>0.857 (12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.549</td>
<td></td>
<td>0.864 (12)</td>
<td></td>
<td></td>
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</table>

Table 2: Used fractions of area for the "straight" pattern.

<table>
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<th>1.250</th>
<th>1.500</th>
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<tbody>
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<td>L</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.900</td>
<td></td>
<td>0.764 (14)</td>
<td>0.785 (18)</td>
<td>0.800 (22)</td>
</tr>
<tr>
<td>1.116</td>
<td></td>
<td>0.792 (18)</td>
<td>0.809 (23)</td>
<td>0.821 (28)</td>
</tr>
<tr>
<td>1.333</td>
<td></td>
<td>0.773 (21)</td>
<td>0.795 (27)</td>
<td>0.810 (33)</td>
</tr>
<tr>
<td>1.549</td>
<td></td>
<td>0.792 (25)</td>
<td>0.811 (32)</td>
<td>0.824 (39)</td>
</tr>
<tr>
<td></td>
<td>1.000</td>
<td>0.833 (8)</td>
<td>0.841 (8)</td>
<td>0.846 (8)</td>
</tr>
<tr>
<td>1.250</td>
<td></td>
<td>0.848 (8)</td>
<td>0.854 (8)</td>
<td>0.859 (8)</td>
</tr>
<tr>
<td>1.500</td>
<td></td>
<td>0.844 (12)</td>
<td>0.852 (12)</td>
<td>0.857 (12)</td>
</tr>
<tr>
<td>0.900</td>
<td></td>
<td>0.853 (12)</td>
<td>0.860 (12)</td>
<td>0.864 (12)</td>
</tr>
<tr>
<td>1.116</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.333</td>
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<td></td>
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</tr>
<tr>
<td>1.549</td>
<td></td>
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Table 3: Used fractions of area for "oblique" pattern 1.
Table 4: Used fractions of area for "oblique" pattern 2.

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<td>0.852 (14)</td>
<td>0.858 (14)</td>
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Figure 8: "Oblique" pattern 1 on a plate with a width of 1.000 m and a length of 0.900 m.
The used fraction of area is 0.856 in the present situation. The smallest plate which yields a higher used fraction of area with an "oblique" pattern is a plate with a width of 1.500 m, a length of 1.116 m and "oblique" pattern 1. The used fraction of area is 0.859 in this case. On the other hand, if we apply the presently used pattern (the "straight" pattern) to a plate with a width of 1.000 m and a length of 1.250 m (20 disks with a diameter of 25 cm, 12 disks with a diameter of 10 cm), the used fraction of area is 0.861. Thus, a smaller plate with the "straight" pattern yields a better result than the larger plate with the "oblique" pattern. This can be observed in general.

We can conclude that the presently used pattern yields a higher used fraction of area than an "oblique" pattern if we can choose the length and width of the plate within certain bounds.

Used fractions of area for certain length-width combinations are listed in Appendix 1, for the "straight" pattern and "oblique" pattern 1 and 2. Disks with a diameter of 10 cm are not taken into account.

We can conclude that no pattern in general (independent of the length and width of the plate) yields the highest used fraction of area.
4 Conclusions

1. We have analysed three different patterns for variable length and width of a plate, assuming that only one type of disks has to be produced. None of these patterns yields, independent of the length and width of the plate and the diameter of the disk, the highest used fraction of area.

2. The presently used pattern is the best of these three patterns if we allow a restricted variation of the length and width of the plate. These patterns for disks with a diameter of 25 cm are extended by stating how the remaining area of the plate is optimally used by punching disks with a diameter of 10 cm.

3. We have not proven that no other patterns exist which yield a higher used fraction of area.

4. We have not considered patterns in which the production of multiple types of disks is taken into account directly.
Aeroplanes and Missiles

1 Introduction

1.1 Problem description

If we detect an aeroplane on a radar screen, we do not yet know if it is hostile or friendly. We expect to obtain an identification on the aeroplane within \( t_{id} \) seconds. If we immediately launch a missile to destroy the aeroplane, and the aeroplane turns out to be friendly, we have to destroy the missile. This results in the following questions: is the time \( t_{hit} \) which the missile needs to hit the aeroplane, larger than the time \( t_{id} \) needed to identify the aeroplane, and is the distance of the missile to the aeroplane large enough at the moment that the missile is destroyed.

To answer these questions, we want to know the trajectory of the missile and the time \( t_{hit} \) which the missile needs to hit the aeroplane.

Our missile has the following characteristics: the velocity \( v_m \) of the missile is constant and always directed towards the aeroplane.

1.2 Results

If we assume that the trajectory of the missile is a straight horizontal line, and that the velocity \( v_a \) of the aeroplane is constant, we give exact answers to our questions: we give an expression which describes the trajectory of the missile, and proof that the missile only hits the aeroplane if \( v_m > v_a \), in which case we give an expression for \( t_{hit} \).

We can still give exact answers to our questions if the trajectory of the aeroplane can be divided into parts such that each part is a straight horizontal line, and that the velocity of the aeroplane during each part is constant.

If we do not start from these assumptions, the analysis becomes more difficult, probably too difficult to give exact answers to our questions.
2 Model construction

In this section we give a model that describes the trajectories of the missile and aeroplane.

We define $t$ as the time elapsed from the launching of the missile.

We describe the positions of the aeroplane and the missile at time $t$ by Euclidian coordinates: $(x_a(t), y_a(t), z_a(t))$ and $(x_m(t), y_m(t), z_m(t))$ respectively. Without loss of generality we assume $x_m(0) = y_m(0) = z_m(0) = 0$.

The following equations hold:

$$\dot{x}_m = \alpha(t)(x_a(t) - x_m(t)) \quad (1)$$

$$\dot{y}_m = \alpha(t)(y_a(t) - y_m(t)) \quad (2)$$

$$\dot{z}_m = \alpha(t)(z_a(t) - z_m(t)) \quad (3)$$

$$\dot{x}_m^2 + \dot{y}_m^2 + \dot{z}_m^2 = v_m^2$$

$$\alpha(t) = \frac{v_m}{\sqrt{(x_a(t) - x_m(t))^2 + (y_a(t) - y_m(t))^2 + (z_a(t) - z_m(t))^2}}$$

By using sphere coordinates with origin $(x_m(t), y_m(t), z_m(t))$, we obtain

$$x_a(t) - x_m(t) = r(t) \sin \theta(t) \cos \varphi(t) \quad (4)$$

$$y_a(t) - y_m(t) = r(t) \sin \theta(t) \sin \varphi(t) \quad (5)$$

$$z_a(t) - z_m(t) = r(t) \cos \theta(t) \quad (6)$$

with

$$r(t) = \sqrt{(x_a(t) - x_m(t))^2 + (y_a(t) - y_m(t))^2 + (z_a(t) - z_m(t))^2}$$

Obviously, the following equation holds:

$$\alpha(t) = \frac{v_m}{r(t)}$$
3 Analysis

3.1 General analysis

From (1) and (4) we obtain

\[ \dot{x}_a - \dot{x}_m = \dot{x}_a - v_m \sin \theta \cos \varphi \]

From (4) we obtain

\[ \dot{x}_a - \dot{x}_m = \dot{r} \sin \theta \cos \varphi + r \dot{\theta} \cos \theta \cos \varphi - r \dot{\varphi} \sin \theta \sin \varphi \]

Combining the last two equations yields

\[ \dot{r} \sin \theta \cos \varphi + r \dot{\theta} \cos \theta \cos \varphi - r \dot{\varphi} \sin \theta \sin \varphi = \dot{x}_a - v_m \sin \theta \cos \varphi \quad (7) \]

From (2) and (5) we obtain

\[ \dot{y}_a - \dot{y}_m = \dot{y}_a - v_m \sin \theta \sin \varphi \]

From (5) we obtain

\[ \dot{y}_a - \dot{y}_m = \dot{r} \sin \theta \sin \varphi + r \dot{\theta} \cos \theta \sin \varphi + r \dot{\varphi} \sin \theta \cos \varphi \]

Combining the last two equations yields

\[ \dot{r} \sin \theta \sin \varphi + r \dot{\theta} \cos \theta \sin \varphi + r \dot{\varphi} \sin \theta \cos \varphi = \dot{y}_a - v_m \sin \theta \sin \varphi \quad (8) \]

From (3) and (6) we obtain

\[ \ddot{z}_a - \ddot{z}_m = \ddot{z}_a - v_m \cos \theta \]

From (6) we obtain

\[ \ddot{z}_a - \ddot{z}_m = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \]

Combining the last two equations yields

\[ \dot{r} \cos \theta - r \dot{\theta} \sin \theta = \ddot{z}_a - v_m \cos \theta \quad (9) \]
3.2 Specific analysis

3.2.1 The equations after assumptions

We cannot continue our analysis easily at this stage without making assumptions on the movement of the aeroplane. We assume that the trajectory of the aeroplane is a straight horizontal line, and that the velocity of the aeroplane \( v_a \) is constant. The trajectories of the aeroplane and the missile are in the same plane (not necessarily vertical) in this case, so we can describe the positions of the aeroplane and the missile with two coordinates, say \( x(t) \) and \( y(t) \):

\[
x_a(t) = x_0 + v_a t
\]

\[
y_a(t) = y_0
\]

Without loss of generality we assume \( v_a > 0 \).

In this case \( \theta(t) = 0 \), equations (7) and (8) are simplified:

\[
\dot{r} \cos \phi - r \dot{\phi} \sin \phi = v_a - v_m \cos \phi
\]

\[
\dot{r} \sin \phi + r \dot{\phi} \cos \phi = -v_m \sin \phi
\]

From (10) and (11) we obtain

\[
\dot{r} = v_a \cos \phi - v_m
\]

\[
\dot{\phi} = -\frac{v_m \sin \phi}{r}
\]

3.2.2 The radius \( r \) as function of the angle \( \phi \)

We can consider \( r \) as a function of \( \phi \) as long as \( \dot{\phi} \neq 0 \), or \( \phi \neq 0 \) \& \( \phi \neq \pi \).

After dividing (12) by (13) we obtain a differential equation which has the following solution:

\[
r(\phi) = C \frac{1}{\sin \phi} \left( \tan \frac{\phi}{2} \right)^{\frac{v_m}{v_a}}
\]

with
\[ C = \frac{r(0) \sin \varphi(0)}{(\tan \frac{\varphi(0)}{2}) \frac{r_m}{v_a}} \]

We observe the following:

\[
\lim_{\varphi \to 0} r(\varphi) = \lim_{\varphi \to 0} \frac{C}{2} \frac{(\sin \frac{\varphi}{2}) \frac{r_m-1}{v_a}}{(\cos \frac{\varphi}{2}) \frac{r_m+1}{v_a}}
\]

\[
= \begin{cases} 
0 & \text{if } v_m > v_a \\
C & \text{if } v_m = v_a \\
\infty & \text{if } v_m < v_a
\end{cases}
\]

We conclude that the missile hits the aeroplane only if \( v_m > v_a \). Intuitively it is clear that in this case the missile will indeed hit the aeroplane.

### 3.2.3 The time \( t \) as function of the angle \( \varphi \)

By substituting (14) in (13) we obtain a differential equation which has the following solution:

\[
t(\varphi) = \begin{cases} 
C \left( \frac{C'}{2v_a} \left( \frac{v_m-v_a}{v_m-v_a} \left( \tan \frac{\varphi}{2} \right) \frac{r_m-v_a}{v_m} - \frac{v_m-v_a}{v_m+v_a} \left( \tan \frac{\varphi}{2} \right) \frac{r_m+v_a}{v_m} \right) \right) & \text{if } v_m \neq v_a \\
\frac{C}{2v_a} \left( C' - \ln(\tan \frac{\varphi}{2}) - \frac{1}{2} (\tan \frac{\varphi}{2})^2 \right) & \text{if } v_m = v_a
\end{cases}
\]

(15)

with

\[ C' = \begin{cases} 
\frac{v_m-v_a}{v_m-v_a} \left( \tan \frac{\varphi(0)}{2} \right) \frac{r_m-v_a}{v_m} + \frac{v_m-v_a}{v_m+v_a} \left( \tan \frac{\varphi(0)}{2} \right) \frac{r_m+v_a}{v_m} & \text{if } v_m \neq v_a \\
\ln(\tan \frac{\varphi(0)}{2}) + \frac{1}{2} (\tan \frac{\varphi(0)}{2})^2 & \text{if } v_m = v_a
\end{cases} \]

### 3.2.4 Calculation of \( t_{hit} \)

The missile hits the aeroplane if and only if \( \varphi \downarrow 0 \) under the condition that \( v_m > v_a \). From (15) we obtain

\[ t_{hit} = \lim_{\varphi \to 0} t(\varphi) = \frac{CC'}{2v_a} \]

Note that \( \lim_{\varphi \to 0} t(\varphi) = \infty \) if \( v_m \leq v_a \).
By substituting $C$ and $C'$ and simplifying the resulting expression we obtain:

$$t_{hit} = \frac{r(0)}{v_m^2 - v_a^2} (v_m + v_a \cos \varphi(0))$$

(16)

3.2.5 The trajectory of the missile

From (5) (with $\theta(t) = 0$) and (14) we derive

$$\tan \frac{\varphi}{2} = \left( \frac{1}{C} (y_0 - y_m) \right) \frac{v_m}{v_a}$$

By using this expression and substituting (15) in

$$x_m(\varphi) = x_a(t(\varphi)) - r(\varphi) \cos \varphi$$

we obtain the following expression for the trajectory of the missile:

$$x_0 - x_m = \left\{ \begin{array}{ll}
- \frac{C C'}{2} + \frac{x_m}{2} (y_0 - y_m) \frac{\left( \frac{1}{C} (y_0 - y_m) \right) \frac{x_m}{v_a}}{v_m - v_a} - \frac{\left( \frac{1}{C} (y_0 - y_m) \right) \frac{x_m}{v_m + v_a}}{v_m + v_a} & \text{if } v_m \neq v_a \\
\frac{C }{2} \left( 1 - C' - \ln C' \right) + \frac{C }{2} \ln(y_0 - y_m) - \frac{1}{4C} (y_0 - y_m)^2 & \text{if } v_m = v_a
\end{array} \right.$$

(17)
4 Conclusions

Firstly we itemize the assumptions we have made, secondly we itemize the results we have obtained under these assumptions and lastly we give some remarks concerning our assumptions.

4.1 Assumptions

- The trajectory of the missile is a straight horizontal line.
- The velocity $v_a$ of the aeroplane is constant.

4.2 Results

- The trajectory of the missile is given by (17).
- The missile will hit the aeroplane if and only if $v_m > v_a$, in which case the time $t_{hit}$ needed to hit the aeroplane is given by (16).

4.3 Remarks on our assumptions

We have assumed that the trajectory of the aeroplane is a straight horizontal line and that the velocity of the aeroplane is constant. Our analysis still holds if the trajectory of the aeroplane can be divided into parts such that each part is a straight horizontal line, and that the velocity of the aeroplane during each part is constant. In this case we can apply our results to each of these parts of the trajectory, resulting in a trajectory of the missile which is a concatenation of trajectories. Note that the total trajectory of the aeroplane is not necessarily a straight horizontal line.

If we do not start from these assumptions, the analysis becomes more difficult, probably too difficult to give exact answers to our questions.
Travelling Allowance.

1 Introduction.

Mr. Smith, a commercial traveller who works at a trading-firm, has to travel a lot for his work. His boss Mr. Collins gives him a travelling allowance for every kilometre that he has travelled for his work. So every friday afternoon, Mr. Smith goes to Mr. Collins' secretary, Mrs. Jones, with his declaration to get his travelling allowance. Of course Mrs. Jones has to control the declaration from Mr Smith first with the help of a map. Since Mrs. Jones is always very busy on friday afternoon and wants to leave early from work for the weekend she has little time to control the declaration. So she measures on the map the distance as the crow flies. Of course Mr. Smith does not agree with this method because in this case his travelling allowance is smaller than he deserves. So they went to their boss Mr. Collins to ask him whether he could find a solution for their problem so that Mrs. Jones only has to measure the distance as the crow flies and Mr. Smith gets a travelling allowance that he deserves.

2 Suggested solution.

After Mrs. Jones and Mr. Smith left the office of Mr. Collins, Mr. Collins tried to figure out a solution for the problem. He came to the following solution:
1. First he draw the following picture:

\[ \begin{array}{c}
\text{Figure 1.}
\end{array} \]

So in this case Mr Smith has to travel from city A to city B and he takes route 1, and Mrs. Jones measures the distance of route 2.

2. After that Mr. Collins assumed that:

(a) A route doesn't lie outside the square.
(b) A route doesn't oscillate much.
(c) One travels in the right direction.

With this picture it was obvious to Mr. Collins that:

\[ \text{dist. route } 2 \leq \text{dist. route } 1 \leq \text{dist. route } 3 \leq \sqrt{2} \cdot \text{dist. route } 2 \]

So the next day, Mr. Collins gives the following advice to Mrs Jones: Multiply the distance as the crow flies by $\sqrt{2}$, and then Mr Smith gets the travelling allowance that he deserves. However, Mr. Smith was not convinced that with this method he gets the travelling allowance that he deserves so he asked a mathematical expert to find a solution for the problem.
3 Model construction.

The expert assumes that the road $\gamma$, that Mr. Smith drives, can be written as $(t, y(t))$ with $a \leq t \leq b$ and $y(a) = y(b) = 0$ and $y$ a continuous differentiable function of $t$.

![Graph of y(t) from A to B]

The length of the road $\gamma$ is now equal to

$$s(b) = \int_a^b \sqrt{1 + (\dot{y}(\tau))^2} d\tau$$
$$\leq \sqrt{1 + m^2(b - a)}$$
with $m = \max_{\tau \in [a,b]} |\dot{y}(\tau)|$

In this way the expert finds that there is a constant $\alpha$ so that $s(b) \leq \alpha(b - a)$. The expert will look to $\alpha$ instead of looking to $m$ because the value of $m$ can be very large in small intervals. And if the value of $m$ is very large on only a small interval the value of $s(b)$ won't change very much. So the expert will give an estimation of the value of $\alpha$. The expert also wants to know something about the reliability of this estimation. This will be done with the data in the next table. The data are produced with the assumption that Mr. Smith drives the fastest way.
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km in a crow line (α)

* : The fastest way lies outside the Netherlands, therefore we did not get enough data to compute α
## SMALL CITIES

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average of $\alpha$ in non-disjunct intervals with length 25 km for small cities

average of $\alpha$ in non-disjunct intervals with length 25 km for big cities

6.
3.1 Conclusions from graphs.

The expert gets the following conclusions from the graphs:

- The value of $Q$ at small distances is smaller than the value of $Q$ at larger distances.
- There is an upper limit for the value of $Q$.

4 Reliability of the estimation.

The expert wants to know an estimation of $Q$ and the reliability of that estimation and therefore he needs to know the distribution of $Q$. He assumes that $Q$ has a normal distribution $N(\mu, \sigma^2)$ with $\mu$ the expectation of $Q$ and $\sigma$ the variance of $Q$. To check this assumption the expert has made the following graph. In the graph the frequency of $Q$ in an interval is plotted.
number of measurements of $X$ in disjunct intervals with length 0.04 for small cities

number of measurements of $\alpha$ in disjunct intervals with length 0.04 for big cities
5 Numerical results.

The expert concludes from the graphs that $\alpha$ may have a normal distribution. The number of measurements is too small to be sure of that. For small cities he gets the following results:

$\alpha$ has a $N(\mu, \sigma^2)$ distribution with $\mu = 1.28$ and $\sigma = 0.06833$. He wants to have a $\beta$ so that $P(X > \beta) \leq 0.01$ with $X$ the value of $\alpha$. He finds that this is true for $\beta = 1.439$.

For big cities he gets $\mu = 1.27, \sigma = 0.09875$ and $\beta = 1.500$.

The suggested solution was $\alpha = \sqrt{2}$, the expert checks this solution and finds $P(X > \sqrt{2}) = 0.025$ for small cities and $P(X > \sqrt{2}) = 0.0721$ for big cities. So the value of $\sqrt{2}$ was not so bad at all. The suggestion of the expert for the secretary will be: use $\alpha = 1.5$.

6 Conclusion.

With the assumption that the factor $\alpha$, with which you have to multiply the distance as the crow flies to get the real distance, has a normal distribution, the expert advises to use $\alpha = 1.5$. So the secretary has to multiply the distance as the crow flies with 1.5 to get an estimation of the distance which Mr. Smith has driven.
1 Introduction

A barrister has a client who is suspected of murdering somebody. It is said that his client jumped out from a high window to avoid his arrest. The client claims that, had he been the person jumping, he would have injured himself owing to his weak kneejoints. So the problem is to see whether it is likely that the person was able to run away after such a jump.

2 Modelconstruction

In order to study the motion of a person falling out of a window we make the following assumptions:

1. The person falls down in a straight line with zero initial velocity

2. When falling the human body is represented by a homogeneous cylinder as shown in figure 1a. After reaching the ground we represent the body as shown in figure 1b. Here both legs are represented by two collared springs and the mass is thought to be packed in two pointmasses $m_1$ and $m_2$.

3. The person is not able to run away after jumping out of a window if the energy supply to the collared springs exceeds a known critical amount of energy, say $E_{\text{crit}}$. Weak kneejoints are modelled by relatively small values of $E_{\text{crit}}$. 
When falling down only two forces affect the cylinder, namely:
1. Gravitation, noted by \( F_g \)
2. Friction, noted by \( F_{wr} \)

For these forces the following formulas hold:

\[
F_g = mg \quad [N] \tag{1}
\]

and

\[
F_{wr} = \frac{1}{2} \rho \ c_d \ v^2 \ \pi \ r^2 \quad [N], \tag{2}
\]

where

- \( m \quad [\text{kg}] \) is the total mass of the cylinder,
- \( \rho \quad [\text{kg} \ \text{m}^{-3}] \) is the mass-density of air,
\[ v \text{ [m s}^{-1}\text{]} \text{ is the velocity of the cylinder,} \]
\[ l \text{ [m]} \text{ is the length of the cylinder,} \]
\[ r \text{ [m]} \text{ is the radius of the cylinders bottom and} \]
\[ c_d \text{ is the friction constant for air.} \]

Applying Newton's law our model for the motion of a falling body is represented by the first-order differential equation
\[
\frac{dv}{dt} = F_g - F_{\text{friction}} = g - \alpha v^2 ,
\]
where
\[
\alpha = \frac{\rho c_d \pi r^2}{2m}.
\]

After some basic calculation we get the solution
\[
v(t) = \beta \tanh \left( \frac{g}{\beta} t \right) ,
\]
where
\[
\beta = \sqrt{\frac{2g}{\alpha c_d}}.
\]

Next we determine the covered distance \( S \) of the cylinder
\[
S(t) = \int_0^t v(\tau) d\tau = \frac{1}{\alpha} \log [\cosh (\alpha \beta t)].
\]

Now suppose that the cylinder hits the ground at time \( t_1 \), thus:
\[
S(t_1) = h \Rightarrow \cosh (\alpha \beta t_1) = \exp(h \alpha).
\]

So the kinetic energy just before hitting the ground is (use equation (8)):
\[
E_{\text{kin}}(t_1) = \frac{1}{2} m (v(t_1))^2 = \frac{m g}{2 \alpha} [1 - \exp(-2 h \alpha)] \text{ [J]}.
\]
When the ground is reached the knees and ankles bend in order to meet the smack. This causes a decrease in potential energy, noted by $E_{zak}$.

\[ E_{zak} = (m_1 + m_2)gl_1(1 - \cos \varphi_1) + m_2gl_2(1 - \cos \varphi_2)[J]. \tag{10} \]

Note that $E_{zak}$ is maximal when the angles $\varphi_1$ and $\varphi_2$ are maximal. However these maximal angles depend on the physical state of the client. Together with the kinetic energy this energy is supplied to the colled springs. According to our assumptions we conclude that the client would not have been able to run away whenever

\[ E_{zak} + E_{\text{kin}}(t_1) > E_{\text{crit}}. \tag{11} \]

Evidently this would mean that the client is not guilty. Using equation (9) we rewrite equation (11) in a condition for the height of the window $h$.

\[ h > h_{\text{crit}}(\alpha) = \frac{-1}{2\alpha} \log \left[ 1 - \frac{2\alpha}{mg} (E_{\text{crit}} - E_{zak}) \right]. \tag{12} \]
In case of absence of the friction force \((\alpha = 0)\) we get

\[
h > h_{\text{crit}}(0) = \frac{E_{\text{crit}} - E_{\text{zub}}}{m g}.
\]

(13)

### 3 Results

In order to check our model, we have calculated the critical height \(h_{\text{crit}}\) for the following typical values of parameters. However the choice of \(E_{\text{crit}}\) is rather arbitrary.

- \(g = 10 \, [m \, s^2] \); \(h = 10 \, [m] \);
- \(m = 80 \, [kg] ; m_1 = 30 \, [kg] ; m_2 = 50 \, [kg] \);
- \(l = 1.90 \, [m] ; l_1 = 0.50 \, [m] ; l_2 = 0.50 \, [m] ; r = 0,30 \, [m] \);
- \(\varphi_1 = \frac{\pi}{6} \, [\text{rad}] ; \varphi_2 = \frac{2\pi}{3} \, [\text{rad}] \);
- \(c_d = 1 ; \rho = 1 \, [kg \, m^{-3}] \) and
- \(E_{\text{crit}} = 5000 \, [J]\).

This yields \(h_{\text{crit}}(\alpha) = 5.6 \, [m]\), which means that the person would not have been able to run away after jumping out of the window. So we may assume that the client is not guilty.

When neglecting the air resistance we get \(h_{\text{crit}}(0) = 5.5 \, [m]\), so the friction has only small influence on the critical height.

### 4 Conclusions

Whenever the height of the window exceeds the critical height \(h_{\text{crit}}\) the barrister can argue that his client is not guilty as he probably would have injured himself by jumping out of the window.

In any other case the barrister should look for other evidence to prove that his client is not guilty.
5 Some remarks

1. In practice the critical energy $E_{\text{crit}}$ is not known, but should be determined by experiment, which seems rather complicated.

2. We have not taken into account the nature of the ground where the landing took place. It may well be that including this effect significantly changes the critical height, as the murderer has more chance to escape when landing on a soft soil.
WAR!

1 Description of the Problem

With the Golf-crisis in our mind it might be interesting to investigate whether it is possible to predict the winner of a war when we know something about the amount of soldiers at both sides and about their ability to fight a war. If we succeed in this, two other interesting questions arise:
- how many soldiers will be left of the winning army at the end of the war?
- how many soldiers will the loosing army need to win the war?

Assume that there are two armies X and Y. X consists of 10,000 soldiers, whereas Y consists of 5,000 soldiers. Because of superior military means it is for soldier y possible to fight the war 1.5 times as effective as soldier x. Army X wants to attack army Y. Who will win the war?

2 Results

After some modelling we obtain the following results:

- After some time hard fighting army X will eventually win the war;
- Although X will beat army Y their losses will be considerably: at the end there will only be 2500 soldiers left;
- To defend themselves succesfully army Y will need at least 6667 soldiers, when army X starts with 10,000 soldiers.

These results were based on the following assumptions:

- An army will only attack an other army with more soldiers when the effectiveness of its soldiers is higher;
- It is possible to express the effectiveness in a mathematical formula. In words this formula means that in the same period of time three soldiers of army X will die whenever two soldiers of army Y die;
• The war is over whenever one army has been exterminated;

• The amount of soldiers X and the amount of soldiers Y are real numbers;

• The interaction between army X and army Y is modelled by a function of X and Y.

3 Model Construction

Define the following two variables:

\[ X(t) := \text{number of soldiers of army X at time } t \]
\[ Y(t) := \text{number of soldiers of army Y at time } t \]

The change in X and Y from a time \( t \) to a time \( t + \Delta t \) equals:

\[
X(t + \Delta t) - X(t) = -\alpha \cdot \Delta t \cdot a \cdot f(X, Y)
\]
\[
Y(t + \Delta t) - Y(t) = -\Delta t \cdot a \cdot f(X, Y)
\]

For example when we choose \( f(X, Y) = X \cdot Y \) this means that in a time \( \Delta t \) a fraction \( -\alpha \cdot a \cdot Y \) of X will be killed. Something similar holds for Y. \( \alpha \) is the effectiveness parameter.

When we divide both sides by \( \Delta t \) and let \( \Delta t \to 0 \) we obtain the following two differential equations:

\[
\dot{X}(t) = -\alpha \cdot f(X, Y)
\]
\[
\dot{Y}(t) = -f(X, Y)
\]

The variable \( a \) has disappeared because the two equations were made dimensionless by setting \( t^* = a \cdot t \). Elimination of \( dt^* \) yields:

\[
\frac{\dot{X}}{Y} = \alpha
\]
This formula expresses the effectiveness as mentioned in the previous section. We can solve this differential equation easily:

\[ X(t) = \alpha Y(t) + C \]

with \( C \) a constant that follows from the initial conditions. If we set \( X_0 \) the number of soldiers of army \( X \) at the beginning of the war (at time \( t = 0 \)) and \( Y_0 \) the number of soldiers of army \( Y \) at the beginning of the war, then

\[ C = X_0 - \alpha Y_0 \]

The war is over when either \( X(t_1) = 0 \) or \( Y(t_2) = 0 \), for some \( t_1, t_2 \geq 0 \). So

\[ X(t_1) = 0 \Leftrightarrow Y(t_1) = \frac{-C}{\alpha} \]

\[ Y(t_2) = 0 \Leftrightarrow X(t_2) = C \]

This implies that

- \( C < 0 \) : \( Y(t_1) > 0 \) \( \Rightarrow \) army \( Y \) wins the war;
- \( C = 0 \) : \( Y(t_1) = 0 \) \( \Rightarrow \) outstanding;
- \( C > 0 \) : \( X(t_2) > 0 \) \( \Rightarrow \) army \( X \) wins the war.

4 Conclusions and Remarks

For the three cases mentioned at the end of the previous section we will translate the conditions on \( C \) to conditions on the initial values of \( X \) and \( Y \). Moreover we will discuss the three questions we posed in section 1 in each case. (Remark that \( \alpha > 1 \) because the effectiveness of the defending army must be higher than the effectiveness of the attacking army).
\[ C < 0 \Leftrightarrow 1 < \alpha \leq \frac{X_0}{Y_0} \]

1) X shall win the war.
2) At the end of the war there will be \( C = X_0 - \alpha Y_0 \) soldiers of army X left.
3) Y can only win the war when it raises its army up to at least \( \frac{X_0}{\alpha} \) soldiers.

\[ C = 0 \Leftrightarrow \alpha = \frac{X_0}{Y_0} \]

1) No army shall win the war. Both armies will be exterminated.
2) No soldiers from army X or Y will be left at the end.
3) When either army X or army Y raises its army with at least one soldier it will win the war.

\[ C > 0 \Leftrightarrow \alpha > \frac{X_0}{Y_0} \]

1) Y shall win the war.
2) At the end of the war there will be \( C = Y_0 - \frac{X_0}{\alpha} \) soldiers of army Y left.
3) X can only win the war when it raises its army up to at least \( \alpha Y \) soldiers.
In our specific case

\[ \alpha = \frac{3}{2} < \frac{X_0}{Y_0} = \frac{10000}{5000} \]

So X will win the war. At the end there will be \( X_0 - \alpha Y_0 = 10000 - \frac{3}{2} \cdot 5000 = 2500 \) soldiers of army X left. To win the war Y will have to raise its army up to at least

\[ \frac{X_0}{\alpha} = \frac{10000}{\frac{3}{2}} = 6667 \]

soldiers.

It is possible to derive explicit functions of t for X and Y by substituting the expression for Y back into the differential equation. This makes only sense when one wants to know how long it will take before the war comes to an end.

In our modelling it makes no difference what kind of interaction we assume between X and Y. Everything is determined by the equation that expresses the effectiveness:

\[ \frac{\dot{X}}{Y} = \alpha \]

Of course when explicitly solving the two differential equations for \( X(t) \) and \( Y(t) \) it is important to know what the interaction between X and Y will be.
A Population Size Problem

1 Problem description

Our problem is to develop a model that predicts the size of the population of the U.S. in the year 2000, given the sizes of the population of the U.S. from 1820 until 1930.

The size $N_k(t)$ of the population of the U.S for $t = 1820, 1830, \ldots, 1970$, and the factor of growth $\lambda_k(t) = \frac{N_k(t) - N_k(t-10)}{N_k(t-10)}$ for $t = 1830, 1840, \ldots, 1970$ are listed in Table 1.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N_k(t)$</th>
<th>$\lambda_k(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1820</td>
<td>9.6</td>
<td></td>
</tr>
<tr>
<td>1830</td>
<td>12.9</td>
<td>0.344</td>
</tr>
<tr>
<td>1840</td>
<td>17.1</td>
<td>0.326</td>
</tr>
<tr>
<td>1850</td>
<td>23.2</td>
<td>0.357</td>
</tr>
<tr>
<td>1860</td>
<td>31.4</td>
<td>0.353</td>
</tr>
<tr>
<td>1870</td>
<td>39.8</td>
<td>0.268</td>
</tr>
<tr>
<td>1880</td>
<td>50.2</td>
<td>0.261</td>
</tr>
<tr>
<td>1890</td>
<td>62.9</td>
<td>0.253</td>
</tr>
<tr>
<td>1900</td>
<td>76.0</td>
<td>0.208</td>
</tr>
<tr>
<td>1910</td>
<td>92.0</td>
<td>0.211</td>
</tr>
<tr>
<td>1920</td>
<td>106.5</td>
<td>0.158</td>
</tr>
<tr>
<td>1930</td>
<td>123.2</td>
<td>0.157</td>
</tr>
<tr>
<td>1940</td>
<td>132.2</td>
<td>0.073</td>
</tr>
<tr>
<td>1950</td>
<td>151.3</td>
<td>0.144</td>
</tr>
<tr>
<td>1960</td>
<td>179.3</td>
<td>0.185</td>
</tr>
<tr>
<td>1970</td>
<td>207.0</td>
<td>0.154</td>
</tr>
</tbody>
</table>

Table 1: Population size and factor of growth.

$N_k(t)$ and $\lambda_k(t)$ are plotted in Figure 1 and 2 respectively.
Figure 1: Population size.

Figure 2: Factor of growth.
2 Model construction

2.1 Model

Define $N(t)$ as the size of the population of the U.S. in the year $t$. We assume that $N(t)$ converges to a certain value $N(\infty)$ as $t \to \infty$. Furthermore we assume that this behaviour is caused by a factor of growth which decreases to zero in time. A way to model this behaviour of the factor of growth is:

$$\lambda(t) = \lambda \frac{N(\infty) - N(t)}{N(\infty)} \quad (1)$$

with $\lambda$ the initial factor of growth.

This results in the following equation:

$$N(t + \Delta t) - N(t) = N(t)(\lambda \Delta t) \frac{N(\infty) - N(t)}{N(\infty)} \quad (2)$$

Equation (2) can be written as:

$$\frac{dN(t)}{dt} = \lambda N(t) - \frac{\lambda}{N(\infty)} [N(t)]^2 \quad (3)$$

After some calculations follows:

$$\ln \left( \frac{N(t)}{1 - \frac{N(t)}{N(\infty)}} \right) = \lambda t + c \quad (4)$$

with $c$ an integration constant.

The solution of the differential equation (3) is

$$N(t) = \frac{e^c e^{\lambda t}}{1 + \frac{e^c}{N(\infty)} e^{\lambda t}} \quad (5)$$

2.2 Sensitivity analysis

In this section we will look at the sensitivity analysis of $N(t)$ in equation (5). In this analysis we give each parameter a small perturbation and look at the influence of this perturbation on the value of $N(t)$. First we replace $N(\infty)$ by $N(\infty)(1 + \epsilon)$ and calculate $\tilde{N}(t)$. It follows that

$$N(t)(1 - 2\epsilon) \leq \tilde{N}(t) \leq N(t)(1 + 2\epsilon)$$
Next we replace $c$ by $c(1 + \epsilon)$ and calculate $\tilde{N}(t)$. It follows that

$$N(t)(1 - 2c\epsilon) \leq \tilde{N}(t) \leq N(t)(1 + 2c\epsilon)$$

Next we replace $\lambda$ by $\lambda(1 + \epsilon)$ and calculate $\tilde{N}(t)$. It follows that

$$N(t)(1 - 2\lambda\epsilon t) \leq \tilde{N}(t) \leq N(t)(1 + 2\lambda\epsilon t)$$

This means that $N(t)$ is most sensitive for a perturbation of $\lambda$. From the calculations of the parameters $N(\infty)$, $\lambda$ and $c$ in the next section follows that the values of $\lambda$ and $c$ are almost the same for different values of $N(\infty)$ close to each other. So $N(t)$ can be calculated well.
3 Solution method

3.1 Introduction

In this section we will explain how we determine values of the unknown parameters \( \lambda, c \) and \( N(\infty) \) of equation (5). Given the values of \( N(t) \) for \( t = 1860, 1870, \ldots, 1930 \), values of \( \lambda, c \) and \( N(\infty) \) will be determined such that equation (5) gives good estimations of \( N(t) \) for \( t = 1860, 1870, \ldots, 1930 \).

We hope that equation (5), given these values of \( \lambda, c \) and \( N(\infty) \), gives good estimations of \( N(t) \) for \( t = 1940, 1950, \ldots, 2000 \). We partly control the quality of these estimations by comparing them with known values for \( t = 1940, 1950, \ldots, 1970 \).

We have not used the known values of \( N(t) \) for \( t = 1820, 1830, \ldots, 1850 \), but also values of \( \lambda, c \) and \( N(\infty) \) will be listed starting from data including these.

3.2 Method

Note that the righthand side of equation (4) is linear in \( t \). Define

\[
y(t, N(\infty)) := \ln \left( \frac{N(t)}{1 - \frac{N(t)}{N(\infty)}} \right)
\]

We obtain

\[
y(t, N(\infty)) = \lambda t + c
\]

Given a value of \( N(\infty) \), we determine, by using the least squares method, values \( \hat{\lambda}(N(\infty)) \) and \( \hat{c}(N(\infty)) \) of \( \lambda \) and \( c \), so that

\[
\sum_{t \in T} (e(t, N(\infty)))^2
\]

is minimal, where

\[
e(t, N(\infty)) = \hat{y}(t, N(\infty)) - y_k(t, N(\infty))
\]

\[
\hat{y}(t, N(\infty)) = \hat{\lambda}(N(\infty))t + \hat{c}(N(\infty))
\]

\[
y_k(t, N(\infty)) : \text{the known value of } y(t, N(\infty))
\]
\[ T = \{ t_f, t_f + 10, \ldots, t_i \} : \text{the set of years } t \text{ for which } N_k(t) \text{ is used to determine } \hat{\lambda}(N(\infty)) \text{ and } \hat{c}(N(\infty)). \]

We hope that these values \( \hat{\lambda}(N(\infty)) \) and \( \hat{c}(N(\infty)) \) are so, that

\[
\sum_{t \in T} (f(t, N(\infty)))^2
\]

is minimal, where

\[
e(t, N(\infty))) = \hat{N}(t, N(\infty) - N_k(t))
\]

\[
\hat{N}(t, N(\infty)) = \frac{e^{\hat{\lambda}(N(\infty))t} e^{\hat{\lambda}(N(\infty))t}}{1 + e^{\hat{\lambda}(N(\infty))t} e^{\hat{\lambda}(N(\infty))t}}
\]

We apply this least squares method for different values of \( N(\infty) \), and choose that value of \( N(\infty) \), say \( N_{\infty} \), so that

\[
\sum_{t \in T} (f(t, N_{\infty}))^2 = \min \{ \sum_{t \in T} (f(t, N(\infty)))^2 | N(\infty) \in N \}
\]

Define

\[ \hat{\lambda} := \hat{\lambda}(N_{\infty}), \ \hat{c} := \hat{c}(N_{\infty}) \]

In practice it showed that both \( \sum_{t \in T} (e(t, N(\infty)))^2 \) and \( \sum_{t \in T} (f(t, N(\infty)))^2 \) are convex functions of \( N(\infty) \), and that the values of \( N(\infty) \) at which these functions have minimal value, are relatively close. This last observation indicates that \( \hat{\lambda}(N(\infty)) \) and \( \hat{c}(N(\infty)) \) are so, that \( \sum_{t \in T} (f(t, N(\infty)))^2 \) is relatively close to its minimal value for fixed \( N(\infty) \), and that \( N_{\infty}, \hat{\lambda} \) and \( \hat{c} \) are so, that \( \sum_{t \in T} (f(t, N_{\infty}))^2 \) is relatively close to the minimal value of \( \sum_{t \in T} (f(t, N(\infty)))^2 \) for variable \( N(\infty) \).

### 3.3 Numerical results

We have performed the method starting from the values of \( N_k(t) \) for \( t = 1860, 1870, \ldots, 1930 \). We do not use the values of \( N_k(t) \) for
\( t = 1820, 1830, \ldots, 1850 \) because there are reasons to believe that these values are not reliable. From 1860, the growth factor shows a behaviour that roughly agrees with our assumptions.

We have computed the following values:

\[
N_\infty = 218 \text{ million}, \quad \hat{\lambda} = 0.2920, \quad \hat{c} = 3.3062.
\]

These values are computed so, that substitution of \( t = 1, 2, \ldots \) in equation (5) gives an estimation of \( N(t) \) for \( t = 1860, 1870, \ldots \).

The estimated values of \( N(t) \), the known values \( N_k(t) \) and the absolute and relative differences of these values are listed in Table 2.

<table>
<thead>
<tr>
<th>Year</th>
<th>( t ) = 1860</th>
<th>( N(t) ) = 31.3</th>
<th>( N_k(t) ) = 31.4</th>
<th>( N(t) - N_k(t) ) = -0.1</th>
<th>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
<td>( t ) = 1870</td>
<td>( N(t) ) = 40.0</td>
<td>( N_k(t) ) = 39.8</td>
<td>( N(t) - N_k(t) ) = 0.2</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 0.5</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1880</td>
<td>( N(t) ) = 50.4</td>
<td>( N_k(t) ) = 50.2</td>
<td>( N(t) - N_k(t) ) = 0.2</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 0.4</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1890</td>
<td>( N(t) ) = 62.6</td>
<td>( N_k(t) ) = 62.9</td>
<td>( N(t) - N_k(t) ) = -0.3</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -0.5</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1900</td>
<td>( N(t) ) = 76.3</td>
<td>( N_k(t) ) = 76.0</td>
<td>( N(t) - N_k(t) ) = 0.3</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 0.4</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1910</td>
<td>( N(t) ) = 91.4</td>
<td>( N_k(t) ) = 92.0</td>
<td>( N(t) - N_k(t) ) = -0.6</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -0.7</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1920</td>
<td>( N(t) ) = 107.1</td>
<td>( N_k(t) ) = 106.5</td>
<td>( N(t) - N_k(t) ) = 0.6</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 0.6</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1930</td>
<td>( N(t) ) = 123.0</td>
<td>( N_k(t) ) = 123.2</td>
<td>( N(t) - N_k(t) ) = -0.2</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -0.2</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1940</td>
<td>( N(t) ) = 138.2</td>
<td>( N_k(t) ) = 132.2</td>
<td>( N(t) - N_k(t) ) = 6.0</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 4.5</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1950</td>
<td>( N(t) ) = 152.4</td>
<td>( N_k(t) ) = 151.3</td>
<td>( N(t) - N_k(t) ) = 1.1</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = 0.7</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1960</td>
<td>( N(t) ) = 164.9</td>
<td>( N_k(t) ) = 179.3</td>
<td>( N(t) - N_k(t) ) = -14.4</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -8.0</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1970</td>
<td>( N(t) ) = 175.8</td>
<td>( N_k(t) ) = 207.0</td>
<td>( N(t) - N_k(t) ) = -31.2</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -15.1</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1980</td>
<td>( N(t) ) = 184.8</td>
<td>( N_k(t) ) = 192.2</td>
<td>( N(t) - N_k(t) ) = -7.4</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -3.8</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 1990</td>
<td>( N(t) ) = 192.2</td>
<td>( N_k(t) ) = 200.0</td>
<td>( N(t) - N_k(t) ) = -7.8</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -3.9</td>
</tr>
<tr>
<td>Year</td>
<td>( t ) = 2000</td>
<td>( N(t) ) = 198.2</td>
<td>( N_k(t) ) = 205.6</td>
<td>( N(t) - N_k(t) ) = -7.4</td>
<td>( \frac{N(t) - N_k(t)}{N_k(t)} ) = -3.6</td>
</tr>
</tbody>
</table>

Table 2: Estimated and known population size and their absolute and relative differences.
These estimated and known values of \( N(t) \) are plotted in Figure 3.

We note that the estimated values of \( N(t) \) for \( t = 1860, 1870, \ldots, 1930 \) are very good: the absolute relative difference with the known values is always less than 1 %. The estimated values of \( N(t) \) for \( t = 1940, 1950, \ldots, 1970 \) are significantly worse: the average absolute relative difference is approximately 7 %. The estimated value of \( N(2000) \) is less than the known value of \( N(1970) \), so this estimation is probably very bad. This is not surprisingly: a close look at the factor of growth from 1930 tells us that it does not decrease as in the period 1860-1930, but roughly remains constant at the level holding at 1910-1930.
Values of $N_\infty$, $\lambda$ and $\hat{c}$ are listed for various sets $T$ in Table 3.

<table>
<thead>
<tr>
<th>$t_f$</th>
<th>$t_l$</th>
<th>$N_\infty$</th>
<th>$\lambda$</th>
<th>$\hat{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1820</td>
<td>1870</td>
<td>153</td>
<td>0.3354</td>
<td>1.9762</td>
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<tr>
<td>1820</td>
<td>1880</td>
<td>138</td>
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<td>1890</td>
<td>161</td>
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<td>1820</td>
<td>1900</td>
<td>163</td>
<td>0.3308</td>
<td>1.9801</td>
</tr>
<tr>
<td>1820</td>
<td>1910</td>
<td>185</td>
<td>0.3207</td>
<td>1.9935</td>
</tr>
<tr>
<td>1820</td>
<td>1920</td>
<td>187</td>
<td>0.3197</td>
<td>1.9957</td>
</tr>
<tr>
<td>1820</td>
<td>1930</td>
<td>198</td>
<td>0.3141</td>
<td>2.0068</td>
</tr>
<tr>
<td>1820</td>
<td>1940</td>
<td>186</td>
<td>0.3209</td>
<td>1.9923</td>
</tr>
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<td>1950</td>
<td>200</td>
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<td>2.0164</td>
</tr>
<tr>
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<td>1960</td>
<td>242</td>
<td>0.2884</td>
<td>2.0802</td>
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<tr>
<td>1860</td>
<td>1930</td>
<td>218</td>
<td>0.2920</td>
<td>3.3062</td>
</tr>
<tr>
<td>1860</td>
<td>1940</td>
<td>187</td>
<td>0.3172</td>
<td>3.2913</td>
</tr>
<tr>
<td>1860</td>
<td>1950</td>
<td>208</td>
<td>0.2965</td>
<td>3.3093</td>
</tr>
<tr>
<td>1860</td>
<td>1960</td>
<td>285</td>
<td>0.2502</td>
<td>3.3563</td>
</tr>
<tr>
<td>1860</td>
<td>1970</td>
<td>398</td>
<td>0.2204</td>
<td>3.3855</td>
</tr>
</tbody>
</table>

Table 3: Values of $N_\infty$, $\lambda$ and $\hat{c}$ for various sets $T$.

We note that the values of $N_\infty$, $\hat{\lambda}$ and $\hat{c}$ strongly depend on the choice of $T$. Furthermore we note that for fixed value of $t_f$, the following is observable: a relatively large value of $N_\infty$ corresponds to a relatively small value of $\lambda$ and a relatively large value of $\hat{c}$. 

9
4 Conclusions

We have modelled the size of the population of the U.S. with the following expression

\[ N(t) = \frac{e^\gamma e^{\lambda t}}{1 + \frac{e^\gamma}{N(\infty)} e^{\lambda t}} \]

based on the assumption that the size of the population converges to a certain value \( N(\infty) \) as \( t \to \infty \), caused by a factor of growth which decreases to zero in time.

We have computed values of \( N(\infty), \lambda \) and \( c \) with a least squares method using the known values of \( N(t) \) for \( t = 1860, 1870, \ldots, 1930 \). The prediction of \( N(t) \) for \( t = 2000 \) is 198.2 million, which is probably very bad because the size of the population was already 207 million in 1970. This extrapolation gives such a bad result because the factor of growth from 1930 remains roughly constant at the level holding at 1910-1930, which contradicts with our assumptions. Extrapolation gives in general bad results for a least squares method. Furthermore we have noticed that the computed values of \( N(\infty), \lambda \) and \( c \) strongly depend on the set of known values of \( N(t) \) used to compute these parameters.