Fractional calculus, Gegenbauer transformations and integral equations

by

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Summary

Starting from the Riemann–Liouville and the Weyl calculus compositions of fractional integral and fractional differential operators are studied in this paper. These composite operators and their inverses admit descriptions as integral transformations with Gegenbauer functions in their kernel. Rodrigues type formulas for Gegenbauer functions and new relations for fractional differential and integral operators are derived. Thus classical results on integral equations of Mellin convolution type are extended and unified.
1 Introduction

Mellin convolution equations, where the convolutor is the product of an algebraic function and a Gegenbauer polynomial or, more generally, a modified Legendre function, has been the subject of various papers, cf. Buschman [4], Erdelyi [9], Sneddon [13], Higgins [10] and Deans [5-6]. These convolution equations correspond to integral transformations which can be composed of operators of fractional differentiation and fractional integration. Thus existence and uniqueness of a solution can be proved. In the aforementioned papers expressions of the solutions were derived merely on the basis of properties of the Mellin transformation and tedious calculations.

In this paper we develop a more concise approach starting from new relations for fractional operators. Thus we cover the classical results and obtain considerable extensions. We remark that for a greater part this paper is based on results stated in the PhD-thesis of one of the authors, cf. [1].

Through the Riemann-Liouville and Weyl calculus fractional operators such as $(\frac{d}{dx})^\lambda$ and $(\frac{1}{x} \frac{d}{dx})^\mu$, $\lambda, \mu \in \mathbb{R}$, become meaningful. A central result in this paper is the following relation, cf. Theorem 1.3 and Theorem 2.2

\[
\left( \frac{d}{dx} \right)^{2\lambda+1} = \left( \frac{1}{x} \frac{d}{dx} \right)^\lambda x^{2\lambda+1} \left( \frac{1}{x} \frac{d}{dx} \right)^{\lambda+1}.
\]

The plan of the paper is the following.

The first two sections contain some relevant material on the Weyl fractional calculus and the Riemann-Liouville fractional calculus. In Section 3 a Rodrigues type formula for Gegenbauer functions is discussed and in Section 4 the so called Gegenbauer transformations. In Section 5 Mellin convolution equations are studied and comparisons with results from literature are presented.

Our discussion starts with the space $\mathcal{E}^+$ of all infinitely differentiable functions on $\mathbb{R}^+ = (0,\infty)$. In $\mathcal{E}^+$ we introduce the operators $\mathcal{M}_a$, $a \in \mathbb{R}$, $\mathcal{T}_p$, $p \in \mathbb{R} \setminus \{0\}$ and $\Sigma_t$, $t \in \mathbb{R}^+$ by

\[
\begin{align*}
(\mathcal{M}_a f)(x) &= x^a f(x) \\
(\mathcal{T}_pf)(x) &= f(x^p) \\
(\Sigma_t f)(x) &= f(x+t), \quad x > 0, \ f \in \mathcal{E}^+.
\end{align*}
\]

We observe that $\mathcal{M}_a \mathcal{M}_b = \mathcal{M}_{a+b}$, $\mathcal{T}_p \mathcal{T}_q = \mathcal{T}_{pq}$ and $\Sigma_t \Sigma_r = \Sigma_{t+r}$, for all admissible values of $a$, $b$, $p$, $q$, $t$ and $r$. Besides,

\begin{equation}
(1) \quad \mathcal{T}_p \mathcal{M}_a = \mathcal{M}_{pa} \mathcal{T}_p.
\end{equation}

For $\mathcal{D}$ the differentiation operator we have

\begin{equation}
(2) \quad \mathcal{D} \mathcal{T}_p = p \mathcal{M}_{p-1} \mathcal{T}_p \mathcal{D}.
\end{equation}
and so $D^n T_p$ is a linear combination of the operators $M_{jp-n} T_p D^j$, $j = 1, \ldots, n$.

## 2 The Weyl calculus

Let $S^+$ denote the space of all $f \in \mathcal{E}^+$ for which for all $k, \ell \in \mathbb{N}_0$

$$\sup_{x > 0} \left| x^k f^{(\ell)}(x) \right| < \infty .$$

Then $S^+$ consists of all restrictions to $\mathbb{R}^+$ of functions in the Schwarz space $S$ of functions on $\mathbb{R}$ of rapid decrease. The Fréchet topology in $S^+$ is brought about by the seminorms

$$p_n(f) = \sup_{x > 0} \left| x^n f(x) \right| + \sup_{x > 0} \left| f^{(n)}(x) \right| , \quad n = 0, 1, 2, \ldots .$$

Moreover, if $f \in \mathcal{E}^+$ with $p_n(f) < \infty$ for all $n \in \mathbb{N} \cup \{0\}$, then $f \in S^+$. For proofs of these results we refer to [1].

An appropriate space for establishing the Weyl calculus is the space $S^+_\tau$.

**Definition 1.1.** The space $S^+_\tau$ consists of all $f \in \mathcal{E}^+$ with the property that there is $t > 0$ such that $\Sigma_t f \in S^+$.

It follows that $f \in S^+_\tau$ if and only if $f \in \mathcal{E}^+$ with

$$\sup_{x > t} \left| x^k f^{(\ell)}(x) \right| < \infty$$

for $t > 0$ and hence $\Sigma_t f \in S^+$ for all $t > 0$.

Although we do not intend to go into topological considerations in this paper, for completeness we mention that $S^+_\tau$ is a Fréchet space with topology brought about by the semi-norms

$$q_n(f) = \sup_{x \geq \frac{1}{n}} \left| x^n f(x) \right| + \sup_{x \geq \frac{1}{n}} \left| f^{(n)}(x) \right| .$$

Moreover, if $f \in \mathcal{E}^+$ with $q_n(f) < \infty$, $n \in \mathbb{N} \cup \{0\}$, then $f \in S^+_\tau$.

The operator $M_a$, $a \in \mathbb{R}$, $T_p$, $p > 0$, and $D$ map $S^+_\tau$ onto $S^+_\tau$. For $\mu > 0$ the Weyl operator $W_\mu$ on $S^+_\tau$ is defined by

$$(W_\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t - x)^{\mu-1} f(t) dt , \quad x > 0 .$$

To obtain a fractional calculus the definition of $W_\mu$ is extended to non-positive values of $\mu$. An $n$-times repeated integration by parts in the integral expression (3) for $W_\mu f$ yields
Since the right hand side of the above equality makes sense for \( \mu > -n \), we extend the definition of the operator \( \mathcal{W}_\mu \) to non-positive values of \( \mu \) as follows

\[
(4) \quad \mathcal{W}_0 = I, \quad \mathcal{W}_{-n+\alpha} = \mathcal{W}_\alpha (-D)^n, \quad n \in \mathbb{N}, \quad 0 \leq \alpha < 1.
\]

The operators \( \mathcal{W}_\mu, \mu \in \mathbb{R} \), thus defined, map \( S^+_{\mathbb{R}} \) onto \( S^+_{\mathbb{R}} \) and satisfy the group property

\[
(5) \quad \mathcal{W}_\mu \mathcal{W}_\nu = \mathcal{W}_{\mu+\nu}, \quad \mu, \nu \in \mathbb{R}.
\]

And further \( \mathcal{W}_{-n} = (-D)^n \). For these reasons, the Weyl operators \( \mathcal{W}_\mu, \mu \in \mathbb{R} \), yield a fractional calculus for the operator \(-D\) by defining \( (-D)^\mu = \mathcal{W}_{-\mu}, \mu \in \mathbb{R} \).

Next we introduce the Erdelyi-Kober operators \( \mathcal{W}_{\mu,p}, \mu \in \mathbb{R}, \ p > 0 \) as

\[
(6) \quad \mathcal{W}_{\mu,p} = p^{-\mu} T_p \mathcal{W}_\mu T_{1/p}.
\]

For fixed \( p > 0 \), the operators \( \mathcal{W}_{\mu,p} \) establish a one-parameter group on \( S^+_{\mathbb{R}} \). By (1) and (2) we have

\[
\mathcal{W}_{-n,p} = (-M_{1-p} D)^n, \quad n = 0, 1, 2, \ldots.
\]

So the Erdelyi-Kober operators \( \mathcal{W}_{\mu,p}, \mu \in \mathbb{R} \) yield a fractional calculus for the operator \(-M_{1-p} D\).

In this paper we use the Erdelyi-Kober operators \( \mathcal{W}_{\mu,2} \) and we set \( \mathcal{W}_\mu = \mathcal{W}_{\mu,2}, \mu \in \mathbb{R} \). For \( \mu > 0 \) we have

\[
(7) \quad (\mathcal{W}_\mu f)(x) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_{-\infty}^{\infty} (t^2 - x^2)^{\mu-1} f(t) t \, dt, \quad x > 0.
\]

The operators \( \mathcal{W}_\mu \) yield a fractional calculus for the differential operator \(-M_{-1} D = -\frac{1}{2} \frac{d}{dx}\).

Next we prove some relations for the operators \( M_a, \mathcal{W}_\mu \) and \( \mathcal{W}_\lambda, a, \mu, \lambda \in \mathbb{R} \). The first one is known as the first index law.

**Theorem 1.2.** For all \( \mu, \nu \in \mathbb{R} \),

\[
\mathcal{W}_\mu M_{-\mu+\nu} \mathcal{W}_\nu = M_{-\nu} \mathcal{W}_{\mu+\nu} M_{-\mu}.
\]

**Proof.** For \( \mu = 0 \) or \( \nu = 0 \) the assertion is a trivial identity. Let \( \mu > 0 \) and \( \nu > 0 \). Then for \( f \in S^+_{\mathbb{R}} \)

\[
\mathcal{W}_\mu M_{-\mu+\nu} \mathcal{W}_\nu f = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_{-\infty}^{\infty} (t^2 - x^2)^{\mu-1} f(t) t \, dt.
\]
\[
(W_\mu M_{-(\mu+\nu)} W_\nu f)(x) = \\
= \frac{1}{\Gamma(\mu)} \frac{1}{\Gamma(\nu)} \int_0^\infty f(\tau) \left( \int_\tau^\infty (t-x)^{\mu-1}(\tau-t)^{\nu-1} t^{-(\mu\nu)} \, dt \right) \, d\tau .
\]

Using the Euler type integral (cf. [7, formula 3.199])
\[
\int_a^b (t-a)^{\mu-1} (b-t)^{\nu-1} (t-c)^{-(\mu+\nu)} \, dt =
\]
where \(c < a < b\), the assertion follows for positive \(\mu\) and \(\nu\).
So for \(\mu, \nu > 0\) we obtain by rewriting the equality
\[
M_{-(\mu+\nu)} W_\nu M_\mu = W_\mu M_{-\nu} W_{\mu+\nu}
\]
\[
M_\nu W_\mu M_{-(\mu+\nu)} = W_{\mu+\nu} M_\mu W_{-\nu}
\]
and by inverting
\[
M_\mu W_{-(\mu+\nu)} M_\nu = W_{-\nu} M_{\mu+\nu} W_{-\mu}
\]
\[
M_{-\mu} W_{-\nu} M_{\mu+\nu} = W_{-(\mu+\nu)} M_\nu W_\mu
\]
\[
M_{\mu+\nu} W_{-\mu} M_{-\nu} = W_\nu M_\mu W_{-(\mu+\nu)}.
\]
Thus all possible cases are covered. \(\square\)

The relation stated in the next theorem is fundamental for this paper.

**Theorem 1.3.** For all \(\mu \in \mathbb{R}\),
\[
W_{2\mu+1} = W_{\mu+1} M_{-2\mu-1} W_\mu .
\]

**Proof.** Having proved the relation for \(\mu > -1\) the relation for \(\mu \leq -1\) follows by replacing \(\mu\) by \(-\mu - 1\) and inverting the operators on both sides. So let \(\mu > -1\). Then for \(f \in S^+_\alpha\) and \(x > 0\)
\[
(W_{\mu+1} M_{-2\mu-1} W_\mu f)(x) = (W_{\mu+1} M_{-2\mu-1} W_{\mu+1})(W_{-1} f)(x) =
\]
\[
= \left( \frac{2^{1-\mu}}{\Gamma(\mu + 1)} \right)^2 \int_0^\infty (W_{-1} f)(t) \left( \int_t^\infty (t-x)^\mu (t^2-x^2)^\mu t^{-2\mu} \, dt \right) \, dt .
\]

Substituting \(\xi = (r^2-x^2)/(t^2-x^2)\) and using formulas [7, 2.1.3(10), 2.8(6)] the inner integral equals
\[
\int_0^t (r^2 - x^2)^\mu (t^2 - r^2)^\mu r^{-2\mu} \, dr = 2^{2\mu} B(\mu + 1, \mu + 1)(t - x)^{2\mu+1}.
\]

Then we obtain
\[
(W_{\mu+1} M_{-2\mu-1} W_\mu f)(x) = (W_{2\mu+2} W_{-1} f)(x) = (W_{2\mu+1} f)(x)
\]
and the stated identity for \( \mu > -1 \) has been settled.

**Corollary 1.3.** For \( \lambda, \mu \in \mathbb{R} \),
\[
\begin{align*}
(i) & \quad W_\mu W_\lambda = W_{\mu+2\lambda+1} M_{2\lambda-1}, \\
(ii) & \quad W_\lambda W_\mu = M_{2\lambda+1} W_{-\lambda-1} W_{\mu+2\lambda+1}.
\end{align*}
\]

### 3 The Riemann–Liouville calculus

The subspace \( S^\pm_+ \) of \( \mathcal{E}^+ \) is defined by \( S^\pm_+ = T_{-1}(S^\pm_+) \). The proof of the following characterization of \( S^\pm_+ \) is left to the reader.

**Proposition 2.1.** A function \( f \in \mathcal{E}^+ \) belongs to \( S^\pm_+ \) if and only if
\[
\lim_{x \to 0} f^{(k)}(x) = 0 \quad \text{for all} \quad k \in \mathbb{N}_0.
\]

Since \( T_{-1} \) is a bijection from \( S^\pm_+ \) onto \( S^\pm_+ \) with \( (T_{-1})^2 = I \) the operator \( M_a, a \in \mathbb{R}, T_p, p > 0 \) and \( D \) map \( S^\pm_+ \) onto \( S^\pm_+ \). Further for \( \mu \in \mathbb{R} \) we define the operator \( I_\mu \) on \( S^\pm_+ \) by
\[
(I_\mu f)(x) = \frac{1}{\Gamma(\mu + 1)} \int_0^x (\tau - x)^{\mu-1} f(\tau) \, d\tau.
\]

Applying the first index law (Theorem 1.1) we get
\[
(I_\mu I_\nu f)(x) = (M_{\mu-1} T_1 W_\mu T_{-1} M_{\mu+1})(M_{\nu-1} T_1 W_\nu T_{-1} M_{\nu+1})
\]
\[
= M_{\mu-1} T_1 (W_\mu M_{-\mu+\nu}) T_{-1} M_{\nu+1}
\]
\[
= M_{\mu-1} T_1 M_{-\nu} W_{\mu+\nu} M_{-\mu} T_{-1} M_{\nu+1}
\]
\[
= I_{\mu+\nu}, \quad \mu, \nu \in \mathbb{R}.
\]

Further, \( T_{-1} = D \) and so the operators \( I_\mu \) yield a fractional calculus for the operator \( D \) on \( S^\pm_+ \). A straightforward computation yields for all \( \mu > 0, \ f \in S^\pm_+ \) and \( x > 0 \),
\[(I_{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_{0}^{x} (x-t)^{\mu-1} f(t) dt.\]

The corresponding Erdelyi-Kober operators \(I_{\mu,p}\) on \(\mathcal{S}_{+}^{+}\) are defined by

\[I_{\mu,p} = p^{-\mu} T_{p} I_{\mu} T_{1/p}.\]

For fixed \(p > 0\), the \(I_{\mu,p}\) establish a one parameter group which yields a fractional calculus for the operator \(M_{1-p} D\) on \(\mathcal{S}_{+}^{+}\).

In this paper the operator \(I_{\mu,2}\) are of importance; for convenience we write \(I_{\mu} = I_{\mu,2}\). Then for \(\mu > 0\), \(f \in \mathcal{S}_{+}^{+}\) and \(x > 0\),

\[(I_{\mu} f)(x) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_{0}^{x} f(t)(x^2 - t^2)^{\mu-1} t dt.\]

By (1) and (2) we obtain

\[I_{\mu} = M_{2\mu-2} T_{-1} W_{\mu} T_{-1} M_{2\mu+2}.\]

Here are the analogues of Theorem 1.3 and Corollary 1.4.

**Theorem 2.2.** For all \(\mu \in \mathbb{R}\),

\[I_{2\mu+1} = I_{\mu+1} M_{-2\mu-1} I_{\mu}.\]

**Proof.** Using relations (8) and (12) and Theorem 1.3 we obtain

\[I_{2\mu+1} = M_{2\mu} T_{-1} W_{\mu+1} M_{-(2\mu+1)} W_{\mu} T_{-1} M_{2(\mu+1)}.\]

Since \(T_{-1} M_{-\gamma} = M_{-\gamma} T_{-1}, \gamma \in \mathbb{R}\), this expression equals

\[(M_{2\mu} T_{-1} W_{\mu+1} T_{-1} M_{2\mu+2}) M_{-(2\mu+1)} (M_{2\mu-2} T_{-1} W_{\mu} M_{2\mu+2})\]

which is just \(I_{\mu+1} M_{-2\mu-1} I_{\mu}.\)

**Corollary 2.3.** For all \(\lambda, \mu \in \mathbb{R}\)

(i) \(I_{\mu} I_{\lambda} = I_{\mu+2\lambda-1} I_{-\lambda+1} M_{2\lambda-1}\)

(ii) \(I_{\mu} I_{\mu} = M_{2\lambda+1} I_{-\lambda-1} I_{\mu+2\lambda+1}.\)
4 Rodriques type formulas for Gegenbauer functions

In Section 4 we study the operators $\mathcal{W}_\mu \mathcal{W}_\lambda, \mathcal{W}_\lambda \mathcal{W}_\mu, \mathcal{I}_\mu \mathcal{I}_\lambda$ and $\mathcal{I}_\lambda \mathcal{I}_\mu$. For $\lambda + \mu > 0$, they are integral operators with so called Gegenbauer functions in their kernels. To prove this, we establish formulas for Gegenbauer functions in this section, which are extensions of the Rodriques’ formula for the Gegenbauer polynomials.

For $\mu > -\frac{1}{2}$ with $\mu \neq 0$ and $n \in \mathbb{N}_0$, the Gegenbauer polynomial $C_n^\mu$ is defined by the Rodriques’ formula, cf. [7, 3.15.1(10)]

$$C_n^\mu(x) = \frac{(-2)^n \Gamma(n + \mu) \Gamma(2\mu + n)}{n! \Gamma(\mu) \Gamma(2\mu + 2n)} (1 - x^2)^{\frac{1}{2} - \mu} \left(\frac{d}{dx}\right)^n [(1 - x^2)^{n+\mu-\frac{1}{2}}].$$

The Chebyshev polynomial $T_n$ and the Legendre polynomial $P_n$ are special Gegenbauer polynomials

$$C_n^0(x) = \lim_{\mu \to 0} \frac{1}{\mu} C_n^\mu(x) = \frac{2}{n} T_n(x)$$

$$C_n^{\frac{1}{2}}(x) = P_n(x) = (2^n n!)^{-1} \left(\frac{d}{dx}\right)^n (x^2 - 1)^n.$$

The Legendre functions $P_\nu^\mu$ and $P_\nu^{-\mu}$ and the Gegenbauer function $C_\nu^\mu$, where $\nu, \mu \in \mathbb{R}$, are defined in terms of hypergeometric functions, cf. [7, 3.2(16), 3.4(1), 3.15.1(4)]

$$P_\nu^\mu(x) = \frac{2^{-\nu}}{\Gamma(1 - \mu)} \frac{(x + 1)^{\nu + \mu/2}}{(x - 1)^{\mu/2}} \, _2F_1(-\nu, -\nu - \mu; 1 - \mu; \frac{x - 1}{x + 1}), \quad x > 1,$$

$$P_\nu^{-\mu}(x) = \frac{2^{-\nu}}{\Gamma(1 - \mu)} \frac{(1 + x)^{\nu + \mu/2}}{(1 - x)^{\mu/2}} \, _2F_1(-\nu, -\nu - \mu; 1 - \mu; \frac{x - 1}{x + 1}), \quad 0 < x < 1.$$

$$C_\nu^\mu(x) = \frac{2^{-\nu-\mu+\frac{1}{2}}}{\Gamma(\mu + \frac{1}{2})} (x + 1)^\nu \, _2F_1(-\nu - \mu + \frac{1}{2}, -\nu; \mu + \frac{1}{2}; \frac{x - 1}{x + 1}), \quad x > 0.$$

For $\nu = n \in \mathbb{N}_0$ and $\mu > -\frac{1}{2}$, the function $C_n^\mu$ equals $C_n^\mu$ upto a multiplicative constant,

$$C_n^\mu = \frac{2^{1 - \mu} \Gamma(2\mu)n!}{\Gamma(n + 2\mu) \Gamma(\mu + \frac{1}{2})} C_n^\mu, \quad \mu \neq 0$$

$$C_n^0 = \sqrt{\frac{2}{\pi}} T_n.$$  

Furthermore, for $\mu > -\frac{1}{2}$

$$C_0^\mu(x) = 2^{\frac{1}{2} - \mu}/\Gamma(\mu + \frac{1}{2}).$$
To be of convenience later, we introduce the notations $C_\nu^\mu$ and $G_\nu^\mu$ denoting the restrictions of $C_\nu^\mu$ to $(1, \infty)$ and $(0, 1)$, respectively. Hence

\begin{align*}
(22) \quad & C_\nu^\mu(x) = C_\nu^\mu(x) = (x^2 - 1)^{\frac{1}{2} - \mu - \frac{1}{2}} P_{\nu + \mu - \frac{1}{2}}^{\frac{1}{2} - \mu}(x), \quad x > 1, \\
(23) \quad & G_\nu^\mu(x) = C_\nu^\mu(x) = (1 - x^2)^{\frac{1}{2} - \mu - \frac{1}{2}} P_{\nu + \mu - \frac{1}{2}}^{\frac{1}{2} - \mu}(x), \quad 0 < x < 1.
\end{align*}

Since $P_\nu^\mu = P_{\nu - 1}^{\mu}$ and $P_\nu^\mu = P_{\nu - 1}^{\mu}$, see [6, 3.3.1(1), 3.4(7)] we have

\begin{align*}
(24) \quad & C_\nu^\mu = C_{\nu - 2\mu}^\mu \quad \text{and} \quad G_\nu^\mu = G_{\nu - 2\mu}^\mu.
\end{align*}

The following integral relations generalize the Rodrigues' formula for Gegenbauer polynomials.

**Theorem 3.1.**

(i) For $\nu \in \mathbb{R}, \mu > 0, \lambda > -\frac{1}{2}$ and $x > 1$

\[
\frac{1}{\Gamma(\mu)} \int_1^x (x - t)^{\mu - 1} (t^2 - 1)^{\lambda - \frac{1}{2}} C_\nu^\lambda(t) dt = (x^2 - 1)^{\lambda + \mu - \frac{1}{2}} C_{\nu - \mu}^{\lambda + \mu}(x).
\]

(ii) For $\nu \in \mathbb{R}, \mu > 0, \lambda > -\frac{1}{2}$ and $0 < x < 1$

\[
\frac{1}{\Gamma(\mu)} \int_x^1 (t - x)^{\mu - 1} (1 - t^2)^{\lambda - \frac{1}{2}} G_\nu^\lambda(t) dt = (1 - x^2)^{\lambda + \mu - \frac{1}{2}} G_{\nu - \mu}^{\lambda + \mu}(x).
\]

**Proof.**

(i) By substitution of $t = x - (x - 1)s$ we derive consecutively

\[
\left(\frac{x - 1}{x + 1}\right)^{\lambda + \nu - \frac{1}{2}} \int_1^x (t^2 - 1)^{\lambda - \frac{1}{2}} C_\nu^\lambda(t)(x - t)^{\mu - 1} dt =
\]

\[
= \frac{2^{\nu - \frac{1}{2}}}{\Gamma(\lambda + \nu + \frac{1}{2})} \int_0^1 s^{\mu - 1} (1 - s)^{\lambda - \frac{1}{2}} (1 - \frac{x - 1}{x + 1} s)^{\lambda + \nu - \frac{1}{2}} ds
\]

\[
= \frac{2^{\nu - \frac{1}{2}}}{\Gamma(\lambda + \nu + \frac{1}{2})} \int_0^1 s^{\mu - 1} \sum_{k=0}^{\infty} \left(\frac{-\lambda - \nu + \frac{1}{2}}{\lambda + \frac{1}{2}}\right)_k \frac{(-\nu)_k}{k!} \left(\frac{x - 1}{x + 1}\right)_k (1 - s)^{\lambda + k - \frac{1}{2}} ds.
\]
\[
\cdot (1 - \frac{x - 1}{x + 1})^\lambda + \nu - k - \frac{1}{2}]ds
\]
\[
= 2^{-\lambda - \nu + \frac{1}{2}} \Gamma(\lambda + \frac{1}{2}) \int_0^1 \left[ \sum_{k=0}^{\infty} \frac{(-\lambda - \nu + \frac{1}{2})_k (-\nu)_k}{(\lambda + \frac{1}{2})_k k!} \left( \frac{x - 1}{x + 1} \right)^k (1 - s)^{\lambda + k - \frac{1}{2}} \right] ds
\]
\[
= \sum_{\ell=0}^{\infty} \frac{(-\lambda - \nu + k + \frac{1}{2})_\ell}{\ell!} \left( \frac{x - 1}{x + 1} \right)^\ell s^\ell + \mu - 1]ds
\]
\[
= 2^{-\lambda - \nu + \frac{1}{2}} \Gamma(\mu) \Gamma(\lambda + \mu + \frac{1}{2}) \sum_{k=0}^{\infty} \frac{(-\lambda - \nu + \frac{1}{2})_n}{(\lambda + \mu + \frac{1}{2})_n} \left( \sum_{k=0}^{n} \frac{(-\nu)_k (\mu)_k}{k! (n - k)!} \right) \left( \frac{x - 1}{x + 1} \right)^n
\]
\[
= 2^{-\lambda - \nu + \frac{1}{2}} \Gamma(\mu) \Gamma(\lambda + \mu + \frac{1}{2}) 2F_1(-\lambda - \nu + \frac{1}{2}, \mu - \nu; \lambda + \mu + \frac{1}{2}; \frac{x - 1}{x + 1}).
\]

We note that
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-\nu)_k (\mu)_k}{k! (n - k)!} \right) x^n = \left( \sum_{k=0}^{\infty} \frac{(-\nu)_k}{k!} z^k \right) \left( \sum_{\ell=0}^{\infty} \frac{\mu_{\ell}}{\ell!} z^\ell \right) = (1 - z)^{\nu - \mu}.
\]
so that
\[
\sum_{k=0}^{n} \frac{(-\nu)_k}{k!} \frac{(\mu)_k}{(n - k)!} = \frac{(\mu - \nu)_n}{n!}.
\]

(ii) The proof of the second integral formula runs along the same lines. 

To see that the integral relations in Theorem 3.1 generalize the Rodriques’ formula for the Gegenbauer polynomials, we take \(\mu = \nu = n\) in Theorem 3.1 and differentiate \(n\) times. Then we obtain
\[
C_n^\lambda(x) = \frac{2^{-\lambda - n + \frac{1}{2}}}{\Gamma(\lambda + n + \frac{1}{2})} (x^2 - 1)^{\frac{1}{2} - \lambda} \left( \frac{d}{dx} \right)^n [(x^2 - 1)^{\frac{1}{2} - \frac{1}{2}} - \frac{1}{2} - \frac{1}{2}] , \quad x > 1,
\]
\[
G_n^\lambda(x) = \frac{2^{-\lambda - n + \frac{1}{2}}}{\Gamma(\lambda + n + \frac{1}{2})} (1 - x^2)^{\frac{1}{2} - \lambda} \left( - \frac{d}{dx} \right)^n [(1 - x^2)^{\frac{1}{2} - \frac{1}{2}} - \frac{1}{2} - \frac{1}{2}] , \quad 0 < x < 1,
\]
which are in correspondence with (13).

Other Rodriques type formulas for Gegenbauer functions are presented in the following theorem.

**Theorem 3.2.**

(i) For \(\nu \in \mathbb{R}, n \in \mathbb{N}_0, \lambda > n - \frac{1}{2}\) and \(x > 1\)
\begin{align*}
C_{\nu+n}^{\lambda,n}(x) &= (x^2 - 1)^{n-\lambda+\frac{1}{2}} \left( \frac{d}{dx} \right)^n[(x^2 - 1)^{-\frac{1}{2}} C_{\nu}^{\lambda}(x)].
\end{align*}

(ii) For \( \nu \in \mathbb{R}, n \in \mathbb{N}_0, \lambda > n - \frac{1}{2} \) and \( 0 < x < 1 \),
\begin{align*}
G_{\nu+n}^{\lambda,n}(x) &= (1 - x^2)^{n-\lambda+\frac{1}{2}} \left( -\frac{d}{dx} \right)^n[(1 - x^2)^{-\frac{1}{2}} G_{\nu}^{\lambda}(x)].
\end{align*}

**Proof.** For \( n = 0 \) the formulas are obvious. For \( n \in \mathbb{N} \) they follow by setting \( \mu := n, \lambda := \lambda - n \) and \( \nu := \nu + n \) in Theorem 3.1 and by differentiating the integral relation thus obtained, \( n \)-times. \( \square \)

## 5 Gegenbauer transformations

In preparation for the introduction of the so called Gegenbauer transformations we show that for \( \lambda + \mu > 0 \) the operator products \( W_{\mu} W_{\lambda}, W_{\lambda} W_{\mu}, I_{\mu} I_{\lambda} \) and \( I_{\lambda} I_{\mu} \) are integral operators with kernels described by Gegenbauer functions.

**Theorem 4.1.** Let \( \lambda, \mu \in \mathbb{R} \) with \( \lambda + \mu > 0 \).

Then for \( f \in S_{++} \),
\begin{enumerate}
  \item [(i)] \( (W_{\mu} W_{\lambda} f)(x) = \int_0^\infty t^{1-\mu} (t^2 - x^2)^{\lambda+\mu-1} G_{-\mu}^{\lambda+\mu-\frac{1}{2}} (x/t)f(t)dt, \quad x > 0, \)
  \item [(ii)] \( (W_{\lambda} W_{\mu} f)(x) = \int_0^\infty x^{1-\mu} (t^2 - x^2)^{\lambda+\mu-1} C_{-\mu}^{\lambda+\mu-\frac{1}{2}} (t/x)f(t)dt, \quad x > 0 \).
\end{enumerate}

And for \( f \in S_{+}^+ \)
\begin{enumerate}
  \item [(iii)] \( (I_{\mu} I_{\lambda} f)(x) = \int_0^\infty t^{1-\mu} (x^2 - t^2)^{\lambda+\mu-1} C_{-\mu}^{\lambda+\mu-\frac{1}{2}} (x/t)f(t)dt, \quad z > 0, \)
  \item [(iv)] \( (I_{\lambda} I_{\mu} f)(x) = \int_0^\infty x^{1-\mu} (x^2 - t^2)^{\lambda+\mu-1} G_{-\mu}^{\lambda+\mu-\frac{1}{2}} (x/t)f(t)dt, \quad x > 0. \)
\end{enumerate}

**Proof.** First suppose \( \lambda, \mu > 0 \).

Starting from the definitions of \( W_{\mu} \) and \( W_{\lambda} \) and by (21) we obtain
\begin{align*}
(W_{\mu} W_{\lambda} f)(x) &= \frac{2^{1-\lambda}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty \left( \int_0^\infty f(t)(t^2 - r^2)^{\lambda-1}t \, dt \right)(r - x)^{\mu-1}dr \\
&= \frac{1}{\Gamma(\mu)} \int_0^\infty \left( \int_0^t (t^2 - r^2)^{\lambda-1} G_{0}^{\lambda-\frac{1}{2}}(r/t)(r - x)^{\mu-1}dr \right)f(t)dt.
\end{align*}
By Theorem 3.1(ii) we obtain

\[
\frac{1}{\Gamma(\mu)} \int_{-\infty}^{t} (t^2 - r^2)^{\lambda-1} G_{-\mu}^{\lambda+\mu-\frac{1}{2}}(r/t)(r-x)^{\mu-1}dr = \\
(t^2 - x^2)^{\lambda+\mu-1} G_{-\mu}^{\lambda+\mu-\frac{1}{2}}(x/t)t^{-\mu}.
\]

Hence for \( \lambda, \mu > 0 \) the stated integral relation is established.

Next suppose \( \lambda > 0, \mu \leq 0 \) with \( \lambda + \mu > 0 \). Choose \( n \in \mathbb{N} \) so large that \( \mu + n > 0 \). Using the previous result and Theorem 3.2(ii),

\[
(W_{\mu}W_{\lambda}f)(x) = W_{-n}(W_{\mu+n}W_{\lambda})f)(x) \\
= (-\frac{d}{dx})^n \int_{-\infty}^{\infty} (t^2 - x^2)^{\lambda+\mu+n-1} G_{-\mu-n}^{\lambda+\mu+n-\frac{1}{2}}(x/t)t^{1-\mu-n}dt \\
= \int_{-\infty}^{\infty} (t^2 - x^2)^{\lambda+\mu-1} G_{-\mu}^{\lambda+\mu-\frac{1}{2}}(x/t)t^{1-\mu}dt.
\]

Thus we have proved the stated integral relation for \( \lambda > 0, \mu \leq 0 \) with \( \lambda + \mu > 0 \).

Finally suppose \( \lambda \leq 0, \mu > 0 \) with \( \lambda + \mu > 0 \). From the identities in Corollary 1.4(i) and (24)

\[
W_{\mu}W_{\lambda} = W_{2\lambda+\mu-1}W_{-\lambda+1}M_{2\lambda-1}, \quad G_{-\mu}^{\lambda+\mu-\frac{1}{2}} = G_{-2\lambda-\mu+1}^{\lambda+\mu-\frac{1}{2}}
\]

the stated integral relation can be derived using the previous result. Relations (ii), (iii) and (iv) are proved with the same techniques. \( \square \)

The Gegenbauer transformations \( W_{\nu,\lambda} \), \( WC_{\nu,\lambda} \), \( IC_{\nu,\lambda} \) and \( IG_{\nu,\lambda} \) are defined as follows.

**Definition 4.2.** For \( \nu, \lambda \in \mathbb{R} \)

(i) \( W_{\nu,\lambda} = W_{-\nu}W_{\nu+\lambda+\frac{1}{2}}M_{-\nu} \)

(ii) \( WC_{\nu,\lambda} = M_{1-\nu}W_{\nu+\lambda-\frac{1}{2}}W_{1-\nu} \)

(iii) \( IG_{\nu,\lambda} = M_{1-\nu}I_{\nu+\lambda-\frac{1}{2}}I_{1-\nu} \)

(iv) \( IC_{\nu,\lambda} = I_{-\nu}I_{\nu+\lambda+\frac{1}{2}}M_{-\nu} \).
By Theorem 4.1 for $\lambda > -\frac{1}{2}$ each of the Gegenbauer transformations is an integral operator.

**Theorem 4.3.** Let $\nu, \lambda \in \mathbb{R}$ with $\lambda > -\frac{1}{2}$.
For $f \in S^+_\nu$,

(i) \( (WG_{\nu,\lambda}f)(x) = \int_{-\infty}^{\infty} (t^2 - x^2)^{\lambda - \frac{1}{2}} G^\lambda_{\nu}(x/t)f(t)t \, dt \), $x > 0$,

(ii) \( (WC_{\nu,\lambda}f)(x) = x \int_{-\infty}^{\infty} (t^2 - x^2)^{\lambda - \frac{1}{2}} C^\lambda_{\nu}(x/t)f(t)t \, dt \), $x > 0$.

And for $f \in S^-_{-\nu}$,

(iii) \( (IG_{\nu,\lambda}f)(x) = x \int_{0}^{\infty} (x^2 - t^2)^{\lambda - \frac{1}{2}} G^\lambda_{\nu}(t/x)f(t) \, dt \), $x > 0$,

(iv) \( (IC_{\nu,\lambda}f)(x) = \int_{0}^{\infty} (x^2 - t^2)^{\lambda - \frac{1}{2}} C^\lambda_{\nu}(x/t)f(t)t \, dt \), $x > 0$.

For $\lambda \leq -\frac{1}{2}$, a Gegenbauer transformation is the product of a (fractional) differential operator and a Gegenbauer integral transformation.

**Theorem 4.4.** For $\nu, \lambda, \rho \in \mathbb{R}$,

(i) \( WG_{\nu,\lambda} = WG_{\nu,\lambda+\rho}(M_{\nu}W_{-\rho}M_{-\nu}) \),

(ii) \( WC_{\nu,\lambda} = (M_{1-\nu}W_{-\rho}M_{\nu-1})WC_{\nu,\lambda+\rho} \),

(iii) \( IG_{\nu,\lambda} = (M_{1-\nu}I_{-\rho}M_{\nu-1})IG_{\nu,\lambda+\rho} \),

(iv) \( IC_{\nu,\lambda} = IC_{\nu,\lambda+\rho}(M_{\nu}I_{-\rho}M_{-\nu}) \).

Corresponding to (23), viz. $C^\lambda_{\nu} = C^\lambda_{\nu-2\lambda}$, $G^\lambda_{\nu} = G^\lambda_{-\nu-2\lambda}$ we have the following counterpart for the Gegenbauer transformations.

**Theorem 4.5.** For $\nu, \lambda \in \mathbb{R}$

(i) \( WG_{\nu,\lambda} = WG_{-\nu-2\lambda,\lambda} \),

(ii) \( WC_{\nu,\lambda} = WC_{-\nu-2\lambda,\lambda} \),

(iii) \( IG_{\nu,\lambda} = IG_{-\nu-2\lambda,\lambda} \),

(iv) \( IC_{\nu,\lambda} = IC_{-\nu-2\lambda,\lambda} \).
Proof. 

(i) According to Definition 4.2 and Corollary 1.4(i) 
\[ \mathcal{W}G_{\nu,\lambda} = \mathcal{W}_{-\nu}W_{\nu+\lambda+\frac{1}{2}}M_{-\nu} = \mathcal{W}_{\nu+2\lambda}W_{-\nu-\lambda+\frac{1}{2}}M_{\nu+2\lambda} = \mathcal{W}G_{-\nu-2\lambda,\lambda}. \]

In the same way (ii), (iii) and (iv) are proved. \( \square \)

According to the definition of the Gegenbauer transformations their inverses are Gegenbauer transformations, also.

**Theorem 4.6.** For \( \nu, \lambda \in \mathbb{R} \)

(i) \[ \mathcal{W}G_{\nu,\lambda}^{-1} = \mathcal{W}C_{1-\nu,1-\lambda} \]

(ii) \[ \mathcal{W}C_{\nu,\lambda}^{-1} = \mathcal{W}G_{1-\nu,1-\lambda} \]

(iii) \[ IG_{\nu,\lambda}^{-1} = IC_{1-\nu,1-\lambda} \]

(iv) \[ IC_{\nu,\lambda}^{-1} = IG_{1-\nu,1-\lambda} \]

**Remark.** For \( \lambda \neq -\frac{1}{2} \), either \( \mathcal{W}G_{\nu,\lambda}, \mathcal{W}C_{\nu,\lambda}, IC_{\nu,\lambda} \) and \( IC_{\nu,\lambda}^{-1} \), \( IC_{\nu,\lambda}^{-1} \), \( IC_{\nu,\lambda}^{-1} \) and \( IC_{\nu,\lambda}^{-1} \) are Gegenbauer integral transformations.

**Theorem 4.7.** For \( \nu, \lambda \in \mathbb{R} \)

(i) \[ \mathcal{W}G_{\nu,\lambda}^{-1} = M_{-2\lambda-1}W_{\nu,\lambda}W_{-2\lambda-1} = W_{-2\lambda-1}W_{\nu,\lambda}M_{-2\lambda-1} \]

(ii) \[ \mathcal{W}C_{\nu,\lambda}^{-1} = M_{-2\lambda-1}W_{\nu,\lambda}W_{-2\lambda-1} = W_{-2\lambda-1}W_{\nu,\lambda}M_{-2\lambda-1} \]

(iii) \[ IG_{\nu,\lambda}^{-1} = M_{-2\lambda-1}IC_{\nu,\lambda}IC_{-2\lambda-1} = I_{-2\lambda-1}IC_{\nu,\lambda}M_{-2\lambda-1} \]

(iv) \[ IC_{\nu,\lambda}^{-1} = M_{-2\lambda-1}IC_{\nu,\lambda}IC_{-2\lambda-1} = I_{-2\lambda-1}IC_{\nu,\lambda}M_{-2\lambda-1} \]

**Proof.** We prove (i) and (ii) simultaneously. Relation (iii) and (iv) are shown similarly. Corollary 1.4(i) yields

\[ \mathcal{W}G_{\nu,\lambda}^{-1} = M_{\nu}(W_{-\nu-\lambda-\frac{1}{2}}W_{\nu}) = \]

\[ = M_{-2\lambda-1}(M_{1-\nu}W_{\nu+\lambda-\frac{1}{2}}W_{1-\nu})W_{-2\lambda-1} \]

\[ = M_{-2\lambda-1}W_{\nu,\lambda}W_{-2\lambda-1}. \]

So it suffices to show that for all \( \lambda \in \mathbb{R} \)
We prove the relations (a) and (b) only for $\lambda > -\frac{1}{2}$. Then they follow for $\lambda < -\frac{1}{2}$ by replacing $\nu$ by $1 - \nu$ and $\lambda$ by $-1 - \lambda$ and inverting the operators on both sides.

For $\lambda > -\frac{1}{2}$ and $f \in S_+^b$

\[
(W_{2\lambda+1}M_{-2\lambda-1}W_{\nu,\lambda}f)(x) =
\]

\[
= \frac{1}{\Gamma(2\lambda + 1)} \int_{0}^{\infty} t^{-2\lambda-1}(t-x)^{2\lambda} \left( \int_{t}^{\infty} (s^2 - t^2)^{-\frac{\lambda}{2}} g^\lambda_{\nu}(t/s) f(s) ds \right) dt.
\]

Take $\tau = \frac{2x}{t}$ and $\sigma = s$, then the latter expression equals

\[
= \frac{1}{\Gamma(2\lambda + 1)} \int_{0}^{\infty} (\tau^2 - x^2)^{-\frac{\lambda}{2}} G^\lambda_{\nu}(x/\tau) \tau^{-2\lambda} \left( \int_{\tau}^{\infty} f(\sigma)(\sigma - \tau)^{2\lambda} d\sigma \right) d\tau
\]

\[
= (W_{\nu,\lambda}M_{-2\lambda-1}W_{2\lambda+1}f)(x).
\]

The proof of (b) runs the same.

6 Integral equations involving Gegenbauer functions

On the basis of the theory developed in Section 4 we can solve the integral equations

\[
(25) \quad f(x) = \int_{x}^{b} (t^2 - x^2)^{\lambda} \frac{1}{2} G^\lambda_{\nu}(x/t) t^{\nu+1} g(t) dt, \quad 0 < x < b,
\]

\[
(26) \quad f(x) = x^{\nu+1} \int_{x}^{b} (t^2 - x^2)^{\lambda} \frac{1}{2} C^\lambda_{\nu}(t/x) g(t) dt, \quad 0 < x < b,
\]

for $\nu, \mu \in \mathbb{R}$, $\lambda > -\frac{1}{2}$, $0 < b \leq \infty$ and

\[
f \in S_+^b = \{ h \in S_+^b \mid h(x) = 0 \text{ for } x \geq b \}.
\]

Also we can solve the integral equations

\[
(27) \quad f(x) = \int_{0}^{x} (x^2 - t^2)^{\lambda} \frac{1}{2} C^\lambda_{\nu}(t/x) t^{\mu+1} g(t) dt, \quad x > a,
\]

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In this section we discuss the equation (26) in relation with results from literature. In our notation, (26) can be written as

\[ M_\mu W_{\nu,\lambda} g = f. \]

Hence, the solution \( g \) is given by

\[ g = W_{\nu,\lambda}^{-1} M_\mu f. \]

Now the various expressions for \( W_{\nu,\lambda}^{-1} \), established in Section 4 provide integral/differential formulas for the solution \( g \). From Theorems 4.6(ii), 4.4(i), 4.5(i) we find for \( \rho > \lambda + \frac{1}{2} \),

\[
g(t) = (W_{\nu+2\lambda-2\rho+1,-1-\lambda+\rho} M_{-\rho} W_{-\rho} M_{-\mu-1} f)(t)
\]

\[
= \int_t^b (y^2 - t^2)^{\rho-\lambda-3/2} G_{\nu+2\lambda-2\rho+1}(t/y) y^{2-\nu} W_{-\rho} \{ y^{\nu-1} f(y) \} dy , \quad 0 < t < b.
\]

From Theorem 4.7(ii) and 4.3(i) we obtain the following alternative expressions for \( g \):

\[
g(t) = (M_{-2\lambda-1} \ W_{\nu,\lambda} \ W_{-2\lambda-1} M_{-\mu} f)(t)
\]

\[
= t^{-2\lambda-1} \int_t^b (y^2 - t^2)^{\lambda-1/2} G_{\nu}^\lambda(t/y) y W_{-2\lambda-1} \{ y^{-\mu} f(y) \} dy , \quad 0 < t < b,
\]

and

\[
g(t) = (W_{-2\lambda-1} W_{\nu,\lambda} M_{-2\lambda-\mu-1} f)(t)
\]

\[
= W_{-2\lambda-1} \left\{ \int_t^b (y^2 - t^2)^{\lambda-1/2} G_{\nu}^\lambda(t/y) y^{-2\lambda-\mu} f(y) dy \right\} , \quad 0 < t < b.
\]

For specific values of the parameters \( \lambda, \nu \), the Gegenbauer function kernel in the integral equation (26) reduces to a Gegenbauer polynomial or to one of its special cases, a Legendre polynomial or a Chebyshev polynomial. Convolution integral equations with such polynomial
kernels have been treated by Li [11], Buschman [2], [3], [4], and Higgins [10]. In these papers the solution of the integral equation seems to have been found by clever guessing, supported in [3] by a formal application of the Mellin transformation. The proposed solution is then verified by substitution into the original integral equation. The verification requires the (lengthy) computation of a special convolution integral.

It is the aim of this section to show that the solutions of Li, Buschman, and Higgins, can be found systematically as special cases of our solution (29) with specific values assigned to the parameters \( \lambda, \mu, \nu, \rho, b \). In addition, we present alternative expressions for the solution of the integral equations considered, obtained by specialisation of (30) and (31).

In the course of his work in aerodynamics, Li [11] was led to an integral equation with a Chebyshev polynomial of the first kind as its kernel. He showed that the integral equation

\[
(32) \quad \int_{x}^{1} (t^2 - x^2)^{-1/2} T_n(t/x) g(t) dt = f(x), \quad 0 < x < 1,
\]

has the solution

\[
(33) \quad g(t) = -\frac{2}{\pi} \int_{t}^{1} (y^2 - t^2)^{-1/2} T_n(t/y) y^{\nu-n} \frac{d}{dy} (y^n f(y)) dy, \quad 0 < t < 1,
\]

provided \( f \) satisfies certain conditions. Here, in the case \( n = 0 \), \( T_{-1} \) is taken to be \( T_1 \), and the integral equation (32) reduces to the classical Abel integral equation.

Now assume \( f, g \in S_{-1}(\mathbb{R}^+) \) and observe that \( T_n = \sqrt{\pi/2} C_n^0 \), see (20). Then the Chebyshev transform pair (32), (33) is precisely the special case of (28), (20), with \( \lambda = 0, \mu = -1, \nu = n, \rho = 1, b = 1 \). By assigning the same values to \( \lambda, \mu, \nu, b \) in (30), (31), we obtain the following alternative expressions for \( g \):

\[
\begin{align*}
g(t) &= -\frac{2}{\pi} t^{-1} \int_{t}^{1} (y^2 - t^2)^{-1/2} T_n(t/y) y \frac{d}{dy} (y f(y)) dy, \quad 0 < t < 1, \\
g(t) &= -\frac{2}{\pi} \left\{ \int_{t}^{1} (y^2 - t^2)^{-1/2} T_n(t/y) y f(y) dy \right\}, \quad 0 < t < 1.
\end{align*}
\]

Inspired by Li’s work, Buschman [2] considered an integral equation with a Legendre polynomial as its kernel. He showed that the integral equation

\[
(34) \quad \int_{x}^{1} P_n(t/x) g(t) dt = f(x), \quad 0 < x < 1,
\]

has the solution

\[
(35) \quad g(t) = \int_{t}^{1} P_{n-2}(t/y) y^{2-n} \left( \frac{1}{y} \frac{d}{dy} \right)^2 (y^n f(y)) dy^4, \quad 0 < t < 1,
\]
provided $f$ satisfies certain conditions.

Now assume $f, g \in S_{-1}(\mathbb{R}^+)$ and observe that $\mathcal{P}_n = C_n^{1/2}$, see (15). Then the Legendre transform pair (34), (35) is recognized as the special case of (26), (29) with $\lambda = \frac{1}{2}, \mu = -1, \nu = n, \rho = 2, b = 1$. By assigning the same values to $\lambda, \mu, \nu, b$ in (30), (31), we obtain the following alternative expressions for $g$:

$$g(t) = t^{-2} \int_0^1 \mathcal{P}_n(t/y)y \left(\frac{d}{dy}\right)^2(y f(y))dy, \quad 0 < t < 1,$$

$$g(t) = \left(\frac{d}{dt}\right)^2 \left\{ \int_0^1 \mathcal{P}_n(t/y)f(y)dy \right\}, \quad 0 < t < 1.$$

Next, Buschman [3] considered an integral equation involving the Gegenbauer polynomial $C_{k/2}^n$ with $k, n \in \mathbb{N}_0, 0 \leq k < n$. For convenience we rewrite $C_{k/2}^n$ in terms of our notation $C_{n}^{k/2}$, cf. (17). Then Buschman’s integral equation takes the form

$$(36) \quad \int_0^1 (t^2 - x^2)^{(k-1)/2} C_{n}^{k/2}(t/x)g(t)dt = f(x), \quad 0 < x < 1,$$

with the solution

$$(37) \quad g(t) = \int_0^1 (y^2 - t^2)^{(k-1)/2} C_{n-k-1}^{k/2}(t/y)y^{2-n} \left(-\frac{1}{y} \frac{d}{dy}\right)^{k+1} (y^n f(y))dy, \quad 0 < t < 1,$$

provided $f$ satisfies certain conditions.

Now assume $f, g \in S_{-1}(\mathbb{R}^+)$. Then the Gegenbauer transform pair (36), (37) is precisely the special case of (26), (29) with $\lambda = k/2, \mu = -1, \nu = n, \rho = k + 1, b = 1$. The corresponding specialisation of (40), (41) yields the following alternative expressions for $g$:

$$g(t) = t^{-k-1} \int_0^1 (y^2 - t^2)^{(k-1)/2} C_{n}^{k/2}(t/y)y \left(-\frac{d}{dy}\right)^{k+1} (y f(y))dy, \quad 0 < t < 1,$$

$$g(t) = \left(-\frac{d}{dt}\right)^{k+1} \left\{ \int_0^1 (y^2 - t^2)^{(k-1)/2} C_{n}^{k/2}(t/y)y^{1-k} f(y)dy \right\}, \quad 0 < t < 1.$$

Buschman’s transform pair (36), (37) covers Li’s transform pair (32), (33) (take $k = 0$) and Buschman’s transform pair (34), (35) (take $k = 1$).

Higgins [10] studied an integral equation involving the Gegenbauer polynomial $C_{m}^{\lambda}$ with $m \in \mathbb{N}_0, \lambda > -\frac{1}{2}$. Again for convenience we rewrite $C_{m}^{\lambda}$ in terms of our notation $C_{m}^{\lambda}$, cf. (19). Then Higgins’ integral equation takes the form

$$(38) \quad \int_0^1 (t^2 - x^2)^{\lambda-1/2} C_{m}^{\lambda}(t/x)g(t)dt = f(x), \quad 0 < x < 1,$$
with the solution

\[ g(t) = \int_{t}^{1} (y^2 - t^2)^{\nu - 1/2} p_{\nu - 1/2}^{1 - \mu}(t/y) y^{\nu - 2} \mathcal{W}_{-\mu - \lambda - 1} \{ y^m f(y) \} dy, \quad 0 < t < 1, \]

where \( n, m \in \mathbb{N}, \, n < m, \, \mu = (m - n - 1)/2, \, \lambda > -\frac{1}{2}, \) provided \( f \) satisfies certain conditions.

Now assume \( f, g \in \mathcal{S}_{-1}(\mathbb{R}^+) \). Then the Gegenbauer transform pair (38), (39) is recognized as the special case of (26), (29) with \( \lambda = \lambda, \, \mu = -1, \, \nu = m, \, \rho = \lambda + (m - n + 1)/2, \, b = 1. \) The corresponding specialisation of (30), (31) yields the following alternative expressions for \( g \):

\[
\begin{align*}
g(t) &= t^{-2\lambda - 1} \int_{t}^{1} (y^2 - t^2)^{\lambda - 1/2} C_{m}^{\lambda}(t/y) y^{\lambda - 2\lambda - 1} \{ y f(y) \} dy, \quad 0 < t < 1, \\
g(t) &= \mathcal{W}_{-2\lambda - 1} \left\{ \int_{t}^{1} (y^2 - t^2)^{\lambda - 1/2} C_{m}^{\lambda}(t/y) y^{\lambda - 2\lambda - 1} f(y) dy \right\}, \quad 0 < t < 1.
\end{align*}
\]

Higgins' transform pair (38), (39) covers Buschman's transform pair (36), (37) (take \( \lambda = k/2, \, n = m - k - 1, \) so that \( \mu = k/2). \) Finally, Buschman [3] considered an integral equation with the Legendre function \( \mathcal{P}_{\nu}^{1 - \mu} \) as its kernel. He showed that the integral equation

\[ \int_{x}^{1} (t^2 - x^2)^{(\mu - 1)/2} \mathcal{P}_{\nu}^{1 - \mu}(t/x) g(t) dt = f(x), \quad 0 < x < 1, \]

has the solution

\[ g(t) = \int_{t}^{1} (y^2 - t^2)^{(\mu - n - 1)/2} \mathcal{P}_{\nu}^{1 - \mu}(t/y) y^{n - 2\nu} \left( -\frac{1}{y} \frac{d}{dy} \right)^n (y^\nu f(y)) dy, \quad 0 < t < 1, \]

where \( n \in \mathbb{N}, \, n > \mu > 0, \) provided \( f \) satisfies certain conditions.

Now assume \( f, g \in \mathcal{S}_{-1}(\mathbb{R}^+) \) and recall the relations (22), (23). The Legendre transform pair (40), (41) is precisely the special case of (26), (29) with \( \lambda := \mu - \frac{1}{2}, \, \mu := -\mu, \, \nu := \nu - \mu + 1, \, \rho = n, \, b = 1. \) By assigning the same values to \( \lambda, \mu, \nu, b \) in (30), (31), we obtain the following alternative expressions for \( g \):

\[
\begin{align*}
g(t) &= t^{-2\mu} \int_{t}^{1} (y^2 - t^2)^{(\mu - 1)/2} \mathcal{P}_{\nu}^{1 - \mu}(t/y) y^{\mu} \mathcal{W}_{-2\mu} \{ y^\nu f(y) \} dy, \quad 0 < t < 1, \\
g(t) &= \mathcal{W}_{-2\mu} \left\{ \int_{t}^{1} (y^2 - t^2)^{(\mu - 1)/2} \mathcal{P}_{\nu}^{1 - \mu}(t/y) f(y) dy \right\}, \quad 0 < t < 1.
\end{align*}
\]
From (29) it follows that the expression (41) for \( g \) remains valid for non-integral \( n > \mu \), provided that \((-y^{-1}d/dy)^n\) is replaced by \( W_n \). Thus, Buschman's transform pair (40), (41) covers Higgin's transform pair (38), (39) (take \( \mu := \lambda + \frac{1}{2}, \nu := m + \lambda - \frac{1}{2}, n := \lambda + (m - n + 1)/2, f(x) := x^{-\lambda+1/2}f(x) \)).

Sneddon [13] treated the general Mellin convolution equation which he solved by Mellin transform techniques. He recovered the previous solutions of Li, Buschman, Higgins, and Erdélyi, as special cases. However, his procedure requires a thorough knowledge of Mellin transforms.

For a detailed review of the literature on convolution integral equations with special function kernels, we refer to Srivastava and Buschman [14], [15].

More recently, Deans [5], [6] utilized the Radon transformation to determine a Gegenbauer transform pair with the Gegenbauer polynomial \( C_m^{q/2-1} \), \( m, q \in \mathbb{N}, q \geq 2 \), as its kernel. For convenience we rewrite \( C_m^{q/2-1} \) in terms of our notation \( C_m^{q/2-1} \), cf. (19). Then Deans' Gegenbauer transform pair takes the form

\[
\begin{align*}
(42) \quad f(x) &= \int_0^\infty \left( t^2 - x^2 \right)^{(q-3)/2} C_m^{q/2-1}(x/t)t \ g(t)dt, \quad x > 0, \\
(43) \quad g(t) &= t^{2-q} \int_0^\infty \left( y^2 - t^2 \right)^{(q-3)/2} C_m^{q/2-1}(y/t) \left( - \frac{d}{dy} \right)^{q-1} f(y)dy, \quad t > 0,
\end{align*}
\]

where \( m, q \in \mathbb{N}, q \geq 2 \), provided \( f \) satisfies certain conditions.

Now assume \( f, g \in \mathcal{S}(\mathbb{R}^+) \). Then the Gegenbauer transform pair (42), (43), follows from Theorems 4.3(i), 4.7(i) with \( \lambda = q/2 - 1, \nu = m, b = \infty \). Moreover Theorem 4.7(i) with \( \lambda = q/2 - 1, \nu = m \), yields the following alternative expression for \( g \):

\[
\begin{align*}
g(t) &= \left( - \frac{d}{dt} \right)^{q-1} \left\{ t \int_0^\infty \left( y^2 - t^2 \right)^{(q-3)/2} C_m^{q/2-1}(y/t)y^{1-q} f(y)dy \right\}, \quad t > 0.
\end{align*}
\]

Gegenbauer transformations arise in the study of the Radon transformations \( \mathcal{R} \) on \( \mathbb{R}^d \) in the following way. Let \( \psi \) be a function on \( \mathbb{R}^d \) of the form

\[
\psi(r\omega) = f(\tau)\mathcal{Y}_m(\omega)
\]

where \( r \geq 0 \) and \( \omega \in S^{q-1} \) (the unit sphere in \( \mathbb{R}^q \)) and where \( \mathcal{Y}_m \) is a homogeneous harmonic polynomial of degree \( m \). Then for \( p \geq 0 \) and \( \omega \in S^{q-1} \),

\[
(\mathcal{R}\psi)(p, \omega) = (2\pi)^{(q-1)/2} \mathcal{W}_{G_m, \mathcal{Y}_{q-1}} f(p)\mathcal{Y}_m(\omega)
\]

\[
= (2\pi)^{(q-1)/2} \int_0^\infty \left( p^2 - r^2 \right)^{(q-3)/2} C_m^{q/2-1}(p/r)r f(r)dr \mathcal{Y}_m(\omega).
\]

See e.g. Ludwig [12, Lemma 5.2].
References


