Modeling and parametric uncertainty characterization of a dual-stage Hard Disk Drive actuator

Citation for published version (APA):

Document status and date:
Published: 01/01/2002

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 12. May. 2019
Modeling and parametric uncertainty characterization of a dual-stage Hard Disk Drive actuator

J.J.M. van Helvoort
DCT report: 2002-26

Eindhoven, May 2002

J.J.M. van Helvoort, student id. 457601

Coach: Prof. dr. ir. R.A. de Callafon
Prof. dr. ir. M. Steinbuch
Abstract

It is important, e.g. for the design of a robust controller, to have an accurate
description of the variations and uncertainties of the frequency response of the system
that is to be controlled. This description should be as little conservative as possible,
describing all possible frequency responses, nothing more and nothing less.

In this research a structured uncertainty model is posed. In this model, the
perturbations of the parameters of the transfer function around a nominal value are
described. With this model the frequency responses of 36 measurements of a dual-
stage Hard Disk Drive actuator are tried to describe, with the restraint that the total
model should be as little conservative as possible. These frequency responses are all
slightly different, e.g. because of variations in the arms of the reader head and the
supporting structure.

To achieve as little conservatism as possible, the model is reparameterized and the
correlation between the parameters of the model is regarded to reveal dependencies.
These dependencies make it possible to use a lower parametric uncertainty dimension,
but to still be able to describe all frequency responses. Therefore the conservatism is
decreased. Finally the model is validated and the effect of the individual perturbations
is regarded.
Chapter 1 Introduction

§1.1 Introduction

Recently research is done in the field of the dual-stage Hard Disk Drive (HDD), in order to expand its data capacity. The dual-stage actuation of the head makes it possible to control the radial motion at a higher frequency, leading to a decrease in position error (the tracks can be put closer together) and an increase in capability to follow the tracks quicker (the spindle speed can be increased).

Although the arms of the head and the E-block (the supporting structure for the arms) are manufactured with great precision, there will always be some differences, however small. These differences affect the frequency behavior of the actuator. At the same time, the fact that in general several disks are mounted on top of each other will ensure that there are differences between the frequency responses of all heads.

§1.2 Experimental data of dual-stage actuator

For the evaluation of the estimation and uncertainty characterization discussed in this report, frequency response measurements of the head (see figure 1.1) are available to derive the uncertainty model of the system. With the assumption that with these 36 frequency responses indeed the bounds in the uncertainty are reached, the model defining the uncertainty can be determined.

![Figure 1.1 Measured amplitude bode plot (top) and phase plot (bottom) of 36 dual-stage suspensions](image)

As can be seen in figure 1.1, all dual-stage actuators, although similarly constructed, exhibit differences in dynamical behavior. These differences are to be modeled.


§1.3 Problem definition
Goal of this research is to estimate a non-conservative parametric uncertainty description for a low order dynamical model of a dual-stage Hard Disk Drive actuator that captures the variations in the measured dynamical behavior as closely as possible, using a systematic procedure.

§1.4 Background to parametric uncertainty modeling
The differences between frequency responses can be defined as uncertainties around a nominal behavior. Because these uncertainties are bounded, it is possible to describe them using these bounds. To construct an efficient controller design, it is necessary to have an accurate model and description of the uncertainties of the system, as little conservative as possible. With an accurate model and description of the uncertainties the performance of the controller can be maximized.

With the multiplicative uncertainty method (see figure 1.2), first a nominal model is derived and afterwards an upper bound on the difference between the nominal model and the data sets is set. This upper bound represents the uncertainty of the entire system, so with the multiplicative method all uncertainty is 'lumped' in one parameter, as can be seen in figure 1.2. In this figure, the block chart is drawn for a plant using the multiplicative method. $W(\omega)$ is a frequency dependent weighting function and $G(\omega)$ is the plant without uncertainty. $\Delta$ is the lumped uncertainty parameter.

Because the uncertainty is lumped in one parameter, this method has the drawback that it is conservative, for according to this model the system is free to do whatever it wants between the upper and lower bound. Of course this is not reality.

![Figure 1.2 Schematics of multiplicative uncertainty method](image)

The structured uncertainty method (see figure 1.3) allows the description of variations in the resonance modes of the model, thus providing the possibility to describe differences between frequency responses. By doing so most variations in the frequency response can be eliminated, leaving only minor variations to the multiplicative method (there will always remain minor variations, due to measurement errors, model incorrectness, approximations or simply coincidence).
As can be seen in figure 1.3, with the structured uncertainty method the uncertainty is not lumped in one parameter, but consists of several parameters $\delta$. These parameters are multiplied by scaling factors $w_i$ and then with a feedback loop added back to the system.

§1.5 Parametric uncertainty characterization

To obtain parameter values for the parametric uncertainty description, fits are made of all individual measurements of the frequency response of the dual-stage Hard Disk Drive actuator.

To be able to make the model less conservative, the model is re-parameterized by isolating the resonance modes.

Next the nominal values and the variations of the (new) parameters are calculated. In order to decrease conservatism, the linear correlations between the parameters are calculated. With these correlations it is possible to rewrite the model in a lower parametric uncertainty dimension, as is done.

Of course the model has to be validated after these steps to see whether or not it is still capable of fitting all measurements and to see how conservative the model is.

Afterwards the model can be rewritten into LFT-format in order to be able to implement the model in a controller.
Chapter 2  Approach to parameter estimation

To find the parameters that describe the measurements as are shown in figure 1.1 best, fits have to be made of the measured frequency responses of the dual-stage Hard Disk Drive actuator. In order to make good fits, a good model has to be chosen, together with its orders of the numerator and the denominator.

§2.1 Model parameterization

In figure 1.1 all 36 frequency responses of the system, obtained from measurements, are shown. It is clear from this figure, that there are three resonance modes that are dominant, i.e. those at 25 Hz, 5.0 kHz and 9.0 kHz. At 6.2 kHz and 17 kHz are also two resonance modes, but these are smaller than the dominant three and moreover they vary widely in amplitude with the different frequency response measurements that are made. For these reasons only the three dominant resonance modes are regarded.

To be able to describe three resonance modes, a sixth order model has to be selected. Experience shows that the best fit is obtain with a sixth order denominator and a third order numerator, i.e. with 3 steps time delay (see appendix A). Furthermore, when the perturbation of the parameters is taken into account, it is preferable to write the denominator as a series of three second order pieces, each describing one resonance mode. See also appendix A.

The model, with three isolated resonance modes, generally looks like:

\[ G(z, \theta) = \frac{\theta_1 z^{-3} + \theta_2 z^{-4} + \theta_3 z^{-5} + \theta_4 z^{-6}}{(1 + \theta_5 z^{-1})(1 + \theta_6 z^{-1})(1 + \theta_7 z^{-2})} \]  

(2.1)

where \( \theta = [\theta_1 \ \theta_2 \ \ldots \ \theta_{10}] \) indicates the \( \mathbb{R}^{1x10} \) real valued parameters to be estimated and the variable \( z \) denotes the complex z-transform variable in the discrete time model.

§2.2 Curve fitting of models

Fits are made of each individual measurement using a discrete time least square fitting routine, with weighting factor \( \frac{1}{G(\omega)} \) (where \( G(\omega) \) is the frequency response measurement) to examine relative error in stead of absolute error.

The models obtained are re-parameterized to isolate the resonance modes and to find better correlations between parameters.

When all fits are made, for every parameter 36 values are available (one for every measurement), the variations of those describe the uncertainties of the dual-stage Hard Disk Drive actuator. These variations are described by assigning an upper and a lower bound to the parameters, in between which the parameter values have to be.
A selection of parameter values is shown in table 2.1. It can easily be demonstrated that the maximum value for \( \theta_2 \) is -0.1021, as is reached in fit 26, and the minimal value is -0.1568, as is reached in fit 5. These values are the upper and lower bound respectively for parameter \( \theta_2 \).

For parameter \( \theta_8 \), the maximum value is 0.9900 (fit 26) and the minimal value is 0.9689 (fit 29).

Table 2.1 Values of parameters \( \theta_2 \) and \( \theta_8 \) as obtained from fits of the frequency response measurements of the dual-stage Hard Disk Drive actuator.

<table>
<thead>
<tr>
<th>fit</th>
<th>( \theta_2 )</th>
<th>( \theta_8 )</th>
<th>fit</th>
<th>( \theta_2 )</th>
<th>( \theta_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1040</td>
<td>0.9728</td>
<td>19</td>
<td>-0.1165</td>
<td>0.9872</td>
</tr>
<tr>
<td>2</td>
<td>-0.1062</td>
<td>0.9866</td>
<td>20</td>
<td>-0.1379</td>
<td>0.9820</td>
</tr>
<tr>
<td>3</td>
<td>-0.1533</td>
<td>0.9785</td>
<td>21</td>
<td>-0.1373</td>
<td>0.9856</td>
</tr>
<tr>
<td>4</td>
<td>-0.1284</td>
<td>0.9870</td>
<td>22</td>
<td>-0.1417</td>
<td>0.9710</td>
</tr>
<tr>
<td>5</td>
<td>-0.1568</td>
<td>0.9851</td>
<td>23</td>
<td>-0.1441</td>
<td>0.9867</td>
</tr>
<tr>
<td>6</td>
<td>-0.1505</td>
<td>0.9725</td>
<td>24</td>
<td>-0.1123</td>
<td>0.9896</td>
</tr>
<tr>
<td>7</td>
<td>-0.1431</td>
<td>0.9823</td>
<td>25</td>
<td>-0.1495</td>
<td>0.9775</td>
</tr>
<tr>
<td>8</td>
<td>-0.1214</td>
<td>0.9852</td>
<td>26</td>
<td>-0.1021</td>
<td>0.9900</td>
</tr>
<tr>
<td>9</td>
<td>-0.1251</td>
<td>0.9863</td>
<td>27</td>
<td>-0.1482</td>
<td>0.9744</td>
</tr>
<tr>
<td>10</td>
<td>-0.1086</td>
<td>0.9823</td>
<td>28</td>
<td>-0.1326</td>
<td>0.9694</td>
</tr>
<tr>
<td>11</td>
<td>-0.1285</td>
<td>0.9877</td>
<td>29</td>
<td>-0.1381</td>
<td>0.9689</td>
</tr>
<tr>
<td>12</td>
<td>-0.1145</td>
<td>0.9830</td>
<td>30</td>
<td>-0.1277</td>
<td>0.9721</td>
</tr>
<tr>
<td>13</td>
<td>-0.1164</td>
<td>0.9812</td>
<td>31</td>
<td>-0.1290</td>
<td>0.9869</td>
</tr>
<tr>
<td>14</td>
<td>-0.1379</td>
<td>0.9886</td>
<td>32</td>
<td>-0.1080</td>
<td>0.9807</td>
</tr>
<tr>
<td>15</td>
<td>-0.1075</td>
<td>0.9804</td>
<td>33</td>
<td>-0.1217</td>
<td>0.9734</td>
</tr>
<tr>
<td>16</td>
<td>-0.1483</td>
<td>0.9705</td>
<td>34</td>
<td>-0.1450</td>
<td>0.9790</td>
</tr>
<tr>
<td>17</td>
<td>-0.1272</td>
<td>0.9762</td>
<td>35</td>
<td>-0.1139</td>
<td>0.9737</td>
</tr>
<tr>
<td>18</td>
<td>-0.1522</td>
<td>0.9741</td>
<td>36</td>
<td>-0.1307</td>
<td>0.9795</td>
</tr>
</tbody>
</table>

§2.3 Full size parametric uncertainty

In figure 2.1 the measurements are drawn, together with the estimated model with full size parametric uncertainty. Full size parametric uncertainty implies that in

\[ G^*(z, \theta^*) = \left\{ G^*(z, \theta^*) \right\} \]

with

\[ \theta^*_i = \tilde{\theta}_i + \delta_i \]

all \( \delta_i \) are independent of each other. It should be noted that

\[ \delta_i = [-1,1] \]

for \( i = 1, 2, \ldots, 10 \).

As can be observed from figure 2.1, still too much conservatism is included in the model, especially at low frequencies, for the model allows responses that don't really occur. The band of possible frequency responses is too wide, although all parameters only can vary between their upper and lower bounds.
Figure 2.1 Comparison between 36 measured frequency responses of a dual-stage HDD actuator (green) and estimated model with full size parametric uncertainty (red), see equation 2.1 thru 2.3, and $\delta_i = \{-1,1\}$ for $i = 1, 2, ..., 10$.

Obviously too much freedom is given to the perturbations, for every single fit is very accurate, but when all parameters of the model are perturbed independently, responses occur that don’t occur in the measured data. Obviously some correlation exists between the perturbations. By making the parameter perturbations dependent on one another, as is done in the next chapter, this correlation is taken into account.
Chapter 3  Parametric uncertainty reduction

In a mechanical system like the dual-stage HDD, it is highly likely that correlation exists between pole- and zero-locations. Whenever one resonance mode moves or the damping of it changes, the locations or damping of the zeros and other resonance modes are likely to change as well, at least for several combinations. The clue is to find these combinations and to determine the way these parameters depend on one another. A linear regression scheme is used to detect linear dependencies between the model parameters.

Once the dependencies between the parameters are known, these parameters can be made linear dependent on one parameter, a 'master parameter'. This decreases the parametric uncertainty dimension, leading to a less conservative model because fewer perturbations are allowed with the same model order.

§3.1 Normalization of the parameters

In order to eliminate scaling factors and to be able to easily compare the parameters and their correlation, the parameters need to be normalized. After normalizing, the parameter values vary from \(-1\) to \(1\), as is shown in table 3.1.

<table>
<thead>
<tr>
<th>fit</th>
<th>(\theta_{2,\text{nor}})</th>
<th>(\theta_{8,\text{nor}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9330</td>
<td>-0.6326</td>
</tr>
<tr>
<td>2</td>
<td>0.8525</td>
<td>0.6758</td>
</tr>
<tr>
<td>3</td>
<td>-0.8738</td>
<td>-0.0952</td>
</tr>
<tr>
<td>4</td>
<td>0.0369</td>
<td>0.7144</td>
</tr>
<tr>
<td>5</td>
<td>-1.0000</td>
<td>0.5327</td>
</tr>
<tr>
<td>6</td>
<td>-0.7721</td>
<td>-0.6648</td>
</tr>
<tr>
<td>7</td>
<td>-0.5019</td>
<td>0.2730</td>
</tr>
<tr>
<td>8</td>
<td>0.2937</td>
<td>0.5448</td>
</tr>
<tr>
<td>9</td>
<td>0.1596</td>
<td>0.6461</td>
</tr>
<tr>
<td>10</td>
<td>0.7639</td>
<td>0.2699</td>
</tr>
<tr>
<td>11</td>
<td>0.0354</td>
<td>0.7838</td>
</tr>
<tr>
<td>12</td>
<td>0.5478</td>
<td>0.3315</td>
</tr>
<tr>
<td>13</td>
<td>0.4778</td>
<td>0.1682</td>
</tr>
<tr>
<td>14</td>
<td>-0.3082</td>
<td>0.8656</td>
</tr>
<tr>
<td>15</td>
<td>0.8029</td>
<td>0.0892</td>
</tr>
<tr>
<td>16</td>
<td>-0.6901</td>
<td>-0.8546</td>
</tr>
<tr>
<td>17</td>
<td>0.0830</td>
<td>-0.3068</td>
</tr>
<tr>
<td>18</td>
<td>-0.8329</td>
<td>-0.5063</td>
</tr>
<tr>
<td>19</td>
<td>0.4738</td>
<td>0.7359</td>
</tr>
<tr>
<td>20</td>
<td>-0.3088</td>
<td>0.2360</td>
</tr>
<tr>
<td>21</td>
<td>-0.2877</td>
<td>0.5829</td>
</tr>
<tr>
<td>22</td>
<td>-0.4498</td>
<td>-0.8034</td>
</tr>
<tr>
<td>23</td>
<td>-0.5355</td>
<td>0.6882</td>
</tr>
<tr>
<td>24</td>
<td>0.6265</td>
<td>0.9621</td>
</tr>
<tr>
<td>25</td>
<td>-0.7357</td>
<td>-0.1912</td>
</tr>
<tr>
<td>26</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>27</td>
<td>-0.6871</td>
<td>-0.4798</td>
</tr>
<tr>
<td>28</td>
<td>-0.1152</td>
<td>-0.9560</td>
</tr>
<tr>
<td>29</td>
<td>-0.3157</td>
<td>-1.0000</td>
</tr>
<tr>
<td>30</td>
<td>0.0621</td>
<td>-0.7012</td>
</tr>
<tr>
<td>31</td>
<td>0.0154</td>
<td>0.7105</td>
</tr>
<tr>
<td>32</td>
<td>0.7835</td>
<td>0.1124</td>
</tr>
<tr>
<td>33</td>
<td>0.2829</td>
<td>-0.5739</td>
</tr>
<tr>
<td>34</td>
<td>-0.5682</td>
<td>-0.0454</td>
</tr>
<tr>
<td>35</td>
<td>0.5701</td>
<td>-0.5532</td>
</tr>
<tr>
<td>36</td>
<td>-0.0452</td>
<td>0.0020</td>
</tr>
</tbody>
</table>

Table 3.1 Values of parameters \(\theta_2\) and \(\theta_8\) after being normalized. Minimum value is \(-1\), maximum value is \(+1\).

This is achieved by implementing:

\[
\theta_{i,\text{nor}} = \left(\theta_i - \bar{\theta}_i\right) / \bar{\theta}_i, \quad i = 1, 2, \ldots, 10
\]  

(3.1)

with

\[
\bar{\theta}_i = \frac{1}{2}\left(\max_{n=1,36}(\theta_i^n) + \min_{n=1,36}(\theta_i^n)\right), \quad i = 1, 2, \ldots, 10
\]  

(3.2)

and

\[
\bar{\theta}_i = \frac{1}{2}\left(\max_{n=1,36}(\theta_i^n) - \min_{n=1,36}(\theta_i^n)\right), \quad i = 1, 2, \ldots, 10
\]  

(3.3)
§3.2 Parameter dependency and reduction

Once the normalized parameters are stacked into a matrix, the linear regression scheme can be applied. The linear regression scheme tries to minimize the second norm of the error $E$ (i.e. $\|E\|_2$) in:

$$\Theta_i - H_{m,n}T = E$$

(3.4)

In equation 3.4, $\Theta_i$ contains all 36 normalized $\theta_{i,\text{nor}}$ for $i = \{1, 2, \ldots, 10\}$.

$$\Theta_i = [\theta_{1,\text{nor}}^1 \theta_{2,\text{nor}}^2 \ldots \theta_{10,\text{nor}}^{36}]^T$$

(3.5)

$H_{m,n}$ contains a selection of the other parameters, i.e. $H_{m,n}$ contains a selection of $[\theta_{m,\text{nor}}, \theta_{n,\text{nor}}, \ldots]$ where $m \neq n \neq i$, $m = \{1, 2, \ldots, 10\}$ and $n = \{1, 2, \ldots, 10\}$. A column with 1’s complements $H$.

$$H_{m,n} = [\Theta_m \Theta_n \ldots 1]$$

(3.6)

If, for instance, $i$ is chosen to be 2 and $m$ is 8 ($n$ is omitted), the linear correlation between parameter $\theta_2$ and $\theta_8$ over the 36 measurements is calculated. $T$ is the coherence vector, the aim of the linear regression scheme. The least squares solution (minimizing $\|E\|_2$) of equation 3.4 is:

$$T = (H_{m,n}^T H_{m,n})^{-1} (H_{m,n}^T \Theta_i).$$

(3.7)

If all elements (except for the last one) in column-vector $T$ are close to 1 or -1, strong correlation exists between the investigated parameters. At least, this holds when only the correlation between two parameters is investigated, for then a value of $T$ close to 1 or -1 means that if the first parameter is perturbed, the second is perturbed almost as much (possibly in the opposite direction).

When the correlation between more then two parameters is investigated, the remaining error should be taken into account instead of the values of $T$ (but even then really large values for the elements in $T$ should be distrusted). This significant difference with the two-parameter case was only distinguished at the end of the research. Because the results with linear dependencies between only two parameters appeared to be satisfactory, no further attention is paid to correlation between more parameters.

Because only linear correlation is investigated and considered, the resulting model will still be linear in the parameter perturbations, making it possible to still represent the model in a linear fractional transformation (LFT) format. See reference [1].

§3.3 Reduced size parametric uncertainty

Equation 3.7 is solved for all parameter combinations (for up to five parameters at a time). But from now on, only the correlation between two parameters will be regarded.

The boundary for the correlation is set to 0.2, because this turned out to be a good threshold (of course a 0.2 boundary implies non-perfect correlation as well, but the offset is still small enough to ensure at least very good correlation). So whenever the values in the first row of $T$ is closer to 1 or -1 than 0.2, perfect linear correlation is assumed and otherwise any linear correlation is neglected. The results are shown in table 3.2.

<table>
<thead>
<tr>
<th>Group 1</th>
<th>$\theta_1, \theta_2, \theta_3, \theta_4$ and $-\theta_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 2</td>
<td>$\theta_9$ and $-\theta_{10}$</td>
</tr>
</tbody>
</table>

Table 3.2 Correlated parameters (See equation 2.1 for parametric definitions)
Only correlation between two parameters is regarded, so $\theta_1$ has good correlation with $\theta_2$ and with $\theta_3$ and with $\theta_4$ and so on). See also appendix B for values of the correlation analysis.

For each group of correlated parameters, a 'master parameter' $\xi$ is established. This is for each of the 36 fits the mean value of the normalized parameters at that particular fit. The parameters are made linear dependent of $\xi$, resulting in the formula:

$$\tilde{\theta}_i = \tilde{\theta}_i + \tilde{\theta}_i \cdot \xi$$

with $i = \{1,2,3,4,7,9,10\}$ and $p = \begin{cases} 1 & i = \{1,2,3,4,7\} \\ 2 & i = \{9,10\} \end{cases}$ and $\tilde{\theta}_i$ and $\tilde{\theta}_i$ from equation 3.2 and 3.3. The results can be seen in figure 3.1 and 3.2.

![Figure 3.1 First group of correlated (normalized) parameters together with the first master parameter $\xi_1$](image)
Figure 3.2 Second group of correlated (normalized) parameters together with the second master parameter \( \xi_2 \)

The parametric uncertainty dimension is decreased, because \( \hat{\theta}_i \) is the approximation of the real parameter \( \theta_i \), and just a linear function of a master parameter, as are several other parameters. Only 2 master parameters are needed to determine the values of 7 parameters, reducing the parametric uncertainty dimension from \( \mathbb{R}^{10} \) to \( \mathbb{R}^4 \), thus resulting in only 5 independent perturbations (i.e. \( \delta_i \) to \( \delta_5 \)) that are allowed in the reduced size parametric uncertainty model.
§3.4 Validation of reduced size parametric uncertainty model

To proof the accuracy of the model with lower parametric uncertainty dimension, for several fits the reduced size parametric uncertainty model (R₅-model) is compared to the original data and to the full size parametric uncertainty model (R¹₀-model). In figure 3.3 the plots of several of these comparisons are shown.

Figure 3.3 Comparison of measured frequency responses (green) in amplitude bode plot (top) and phase plot (bottom) with full size parametric uncertainty (R¹₀-model, blue) and model with reduced size parametric uncertainty (R₅-model, red) for several measurements.

It is obvious from figure 3.3 that as well the R¹₀-model as the R₅-model approximate the original data closely. Of course the R¹₀-model has the best approximation, but it has twice as much freedom as the R₅-model. It is at the resonance modes (especially in the phase plot) that the R₅-model shows some deviation, but even here the approximation is still pretty good (taken into account the reduced size). Figure 3.3 shows that indeed the parametric uncertainty can be decreased. If now simulations are made with this reduced size parametric uncertainty model, it should be less conservative then the full size parametric uncertainty model.
Chapter 4 simulations with reduced size model

In a model with structured uncertainty the nominal value of the parameters, \( \bar{\theta} \), will be changed by a perturbation varying between \( -\bar{\theta} \) and \( \bar{\theta} \), so

\[
\theta = \bar{\theta} + \bar{\theta} \cdot \delta
\]  

with \( \delta \in [-1,1] \).

Of course, to be least conservative, \( \bar{\theta} \) is wanted to be as small as possible. It can be shown that this is achieved by

\[
\bar{\theta} = \min_{\theta} \max_{m=1...36} |\theta^m - \bar{\theta}|
\]  

with \( \theta^m \) the value of the m\textsuperscript{th} measurement for the parameter \( \theta \). Consequently,

\[
\bar{\theta} = \max_{m=1...36} |\theta^m - \bar{\theta}|
\]  

These appear to be the same formulas as used for the normalization in chapter 3 (i.e. equation 3.2 and 3.3), so

\[
\bar{\theta}_i = \frac{1}{2} \left( \max_{m=1...36} (\theta^m_i) + \min_{n=1...36} (\theta^n_i) \right), \quad i = 1, 2, \ldots, 10
\]  

and

\[
\bar{\theta}_i = \frac{1}{2} \left( \max_{m=1...36} (\theta^m_i) - \min_{n=1...36} (\theta^n_i) \right), \quad i = 1, 2, \ldots, 10
\]  

§4.1 Model formulation

After curve fitting, reparameterization and correlation analysis, the final model can be formulated as follows:

\[
G(z, \theta, \delta) = \frac{B(z, \theta, \delta)}{A_1(z, \theta, \delta) \cdot A_2(z, \theta, \delta) \cdot A_3(z, \theta, \delta)}
\]  

with

\[
B(z, \theta, \delta) = (\bar{\theta}_1 + \bar{\theta}_1 \cdot \delta_1) z^{-3} + (\bar{\theta}_2 + \bar{\theta}_2 \cdot \delta_2) z^{-4} + (\bar{\theta}_3 + \bar{\theta}_3 \cdot \delta_3) z^{-5} + (\bar{\theta}_4 + \bar{\theta}_4 \cdot \delta_4) z^{-6}
\]  

\[
A_1(z, \theta, \delta) = 1 + (\bar{\theta}_7 + \bar{\theta}_7 \cdot \delta_7) z^{-1} + (\bar{\theta}_8 + \bar{\theta}_8 \cdot \delta_8) z^{-2}
\]  

\[
A_2(z, \theta, \delta) = 1 + (\bar{\theta}_9 + \bar{\theta}_9 \cdot \delta_9) z^{-1} + (\bar{\theta}_5 + \bar{\theta}_5 \cdot \delta_5) z^{-2}
\]  

and

\[
A_3(z, \theta, \delta) = 1 + (\bar{\theta}_10 + \bar{\theta}_10 \cdot \delta_5) z^{-1} + (\bar{\theta}_10 - \bar{\theta}_10 \cdot \delta_5) z^{-2}
\]  

In these equations, the variable \( z \) denotes the complex \( z \)-transform variable in the discrete time model, \( \bar{\theta}_i \) denotes the nominal value of the \( i \)\textsuperscript{th} parameter and \( \bar{\theta}_i \) denotes the maximum magnitude of the perturbation on the \( i \)\textsuperscript{th} parameter. \( \delta_7 \) thru \( \delta_9 \) represent the perturbations acting on the system. See appendix C for the block chart and LFT (Linear Fractional Transformation) representations.

Now only the numerical values for the parameters have to be calculated from the fits, obtained from the data. The values for parameters 2 and 8 are listed in table 4.1. See appendix E for all numerical values. With these values for the parameters and different values for \( \delta \), simulations can be performed in order to see whether this new representation still fits the data well.
Modeling and parametric uncertainty characterization
of a dual-stage Hard Disk Drive actuator
J.J.M. van Helvoort

\[ \begin{array}{c|c}
\bar{\theta}_2 & -0.1289 \\
\bar{\theta}_2 & 0.0245 \\
\bar{\theta}_6 & 0.9795 \\
\bar{\theta}_6 & 0.0105 \\
\end{array} \]

Table 4.1 Numerical values of \( \bar{\theta}_i \) and \( \bar{\theta}_i \) for \( i = \{2, 8\} \)

§4.2 Validation of model and uncertainties

Using these values for the parameters, simulations of the frequency response of the model are made with different values of \( \delta_1 \) thru \( \delta_5 \). Figure 4.1 shows the simulation results. For these simulations, only the values \(-1, 0, 1\) are used for \( \delta_1 \) thru \( \delta_5 \), to explore the upper and lower bounds of the frequency response. This results in \( 2^5 \) simulations, shown in the figure in red.

![Simulation results](image)

Figure 4.1 Comparison of measured frequency responses (green) with frequency responses obtained from simulations (red). In these simulations, only the values \(-1, 0, 1\) are used for \( \delta_1 \) thru \( \delta_5 \), to explore the bounds of the frequency responses.

It can be observed from figure 4.1 that indeed the simulation still fits the data quite well for all different combinations of \( \delta \) and that the entire width of the measured frequency responses is covered. Of course there are still some deviations between the measured frequency responses and those obtained from the simulations. A possible way to get rid of the remaining errors would be to use the multiplicative method in addition to the structured uncertainty method.
§4.3 Conservatism of uncertainty description

Because the model is parameterized with isolated poles and because the correlations between the parameters are taken into account while constructing the final model, the perturbations have very nice properties. When only one $\delta$ is perturbed and the others are held constant (at 0 for instance), these nice properties can be visualized. It can be observed from figure 4.2 thru figure 4.6 that every $\delta$ excites either the damping of a resonance mode or the frequency at which the resonance mode occurs, independent from the other $\delta$'s. This way, all data sets can be approximated very closely.

Figure 4.2 Simulated frequency responses with $\delta_1 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

Figure 4.3 Simulated frequency responses with $\delta_2 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

Figure 4.4 Simulated frequency responses with $\delta_3 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.
Modeling and parametric uncertainty characterization of a dual-stage Hard Disk Drive actuator

J.J.M. van Helvoort

Figure 4.5 Simulated frequency responses with $\delta_t = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

Figure 4.6 Simulated frequency responses with $\delta_s = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

§4.4 Uncertainty in Linear Fractional Transformation representation

In order to make this model easy to be implemented, it has to be rewritten into an LFT (Linear Fractional Transformation) representation. In a LFT, the $\delta$'s are taken outside the main matrix and with a feedback loop coupled back to the model, see figure 4.7.

![Schematic representation of a Linear Fractional Transformation](image)

Figure 4.7 Schematic representation of a Linear Fractional Transformation

The values of the $\delta$'s are easy accessible. See appendix D on how to do the rewriting into an LFT representation (see also references [1], [3], [4] and [5]). The LFT representation itself can be found in appendix C, together with the full notation of the transfer function of the final model.
§4.3 Conservatism of uncertainty description

Because the model is parameterized with isolated poles and because the correlations between the parameters are taken into account while constructing the final model, the perturbations have very nice properties. When only one $\delta$ is perturbed and the others are held constant (at 0 for instance), these nice properties can be visualized. It can be observed from figure 4.2 thru figure 4.6 that every $\delta$ excites either the damping of a resonance mode or the frequency at which the resonance mode occurs, independent from the other $\delta$'s. This way, all data sets can be approximated very closely.

**Figure 4.2** Simulated frequency responses with $\delta_1 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

**Figure 4.3** Simulated frequency responses with $\delta_2 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.

**Figure 4.4** Simulated frequency responses with $\delta_3 = \{-1, 0, 1\}$ and the other $\delta$'s held constant at zero. A detail is shown on the right.
Chapter 5  Precis

§5.1 Conclusion

When approximating the measurements with a low order model ($6^{th}$ order), every fit will have slightly different parameter values to compensate for a slightly different frequency response of the measurement. These differences have to be taken into account when constructing a non-conservative model. When simply the average of all modeled frequency responses is used and an upper and lower bound are set, too much freedom is given to the model (for it is free to move between the upper and lower bound), so it will be to conservative. (Multiplicative method)

Conservatism is decreased if not the variation of the frequency response is considered, but the variation of the parameters responsible for it. But even then there's conservatism included in the model. (Structured uncertainty method)

Major improvement occurs when the dependencies between the parameter variations are taken into account, what is done by looking at linear correlation. The number of independent perturbations of the model, responsible for the variations of the parameters, can consequently be reduced from 10 to 5, leading to less freedom in the model and hence to a less conservative model.

Although a quantitative measure for the remaining conservatism of the model was omitted, still some qualitative statements can be made. It is obvious that with the model, in which the parametric dependencies are taken into account, indeed the major variations can be modeled very accurate, leaving only minor variations, which, for instance, can be modeled by an additional multiplicative method description. This results in a smaller band of possible frequency responses and therefor in a less conservative description of the system.

The final model has a very systematic notation (i.e. all resonance modes are parameterized isolated, but the correlation between the parameters is taken into account nevertheless), making the perturbations to have the nice property to all excite either the damping of a resonance mode or the frequency at which it occurs, independent of one another. Hence it is obvious that all occurring frequency responses can be approximated with little conservatism.
§5.2 Recommendations
Although good results are obtained, of course there is always room for extension and improvement.
First of all, and most important, the quality of the model and the achieved conservatism should be quantified. Looking at the sensitivity plot of the combined structured uncertainty and multiplicative methods, and comparing this to the original sensitivity plot of the measurements can do this. The difference between the upper bounds of these two plots is a measure for the conservatism (and therefore also for the quality) of the obtained model.
In order to improve the structured uncertainty model, the model could be expanded. More resonance modes, even those that are less significant could be taken into account. This way the fits will approximate the measurements better, but one has to be careful not to introduce additional conservatism.
Another way to make the model less conservative is to find better ways to describe and calculate the dependencies between the parameters. This may include a reparameterization or just a different way of making one parameter dependent on another. With this model, only linear correlations are taken into account, for linear dependencies are easiest to include in an LFT description. But maybe even non-linear dependencies can be modeled using the LFT description. And also dependencies between more than two parameters can be investigated, like as is described in section 3.2.
### Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(z, \theta)$</td>
<td>frequency response</td>
</tr>
<tr>
<td>$G(\omega)$</td>
<td></td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>$i^{th}$ real valued parameter in transfer function</td>
</tr>
<tr>
<td>$z$</td>
<td>Complex $z$-transform variable in the discrete time model</td>
</tr>
<tr>
<td>$\bar{\theta}_i$</td>
<td>mean value of $i^{th}$ parameter, calculated using equation 3.2</td>
</tr>
<tr>
<td>$\tilde{\theta}_i$</td>
<td>maximum perturbation of $i^{th}$ parameter, calculated using equation 3.3</td>
</tr>
<tr>
<td>$\delta_i$</td>
<td>$i^{th}$ normalized perturbation</td>
</tr>
<tr>
<td>$\theta_{i,\text{nor}}$</td>
<td>$i^{th}$ normalized parameter (maximum value = 1, minimum value = -1), $m^{th}$ measurement</td>
</tr>
<tr>
<td>$T$</td>
<td>coherence vector</td>
</tr>
<tr>
<td>$\xi_i$</td>
<td>$i^{th}$ real valued ‘master parameter’ as defined in section 3.3</td>
</tr>
<tr>
<td>$\hat{\theta}_i$</td>
<td>approximation of value of $i^{th}$ parameter, using the master parameter $\xi$</td>
</tr>
<tr>
<td>$b_i$</td>
<td>$i^{th}$ real valued parameter in numerator of transfer function</td>
</tr>
<tr>
<td>$a_i$</td>
<td>$i^{th}$ real valued parameter in denominator of transfer function</td>
</tr>
</tbody>
</table>
References
Appendix A Determination of model order

§A.1 Sixth order denominator

In the frequency response of the system, three resonance modes are dominant (as can be seen from figure 1.1) i.e. those at 25 Hz, 5.0 kHz and 9.0 kHz. At 6.2 kHz and 17 kHz are also two resonance modes, but these are smaller than the dominant three and moreover they vary widely in amplitude with the different frequency response measurements that are made. For these reasons only the three dominant resonance modes are regarded.

To be able to describe three resonance modes, a sixth order model has to be selected, resulting in a sixth order denominator.

§A.2 Sixth order numerator

To include maximum freedom in a sixth order model, besides a sixth order denominator also a sixth order numerator has to be selected.

However, it can be shown that in a model with a sixth order numerator and a sixth order denominator, as is depicted in equation A.1, a static gain can be subtracted, leaving a model with a fifth order numerator and a sixth order denominator, see equation A.2 thru A.3.

\[
G(z,a,b) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + a_5 z^{-5} + a_6 z^{-6}} 
\]  

\[
G(z,a,b) = b_0 + \frac{B(z,a,b)^*}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + a_5 z^{-5} + a_6 z^{-6}} 
\]  

with

\[
B(z,a,b)^* = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6} 
\]  

\[
- b_0 (1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + a_5 z^{-5} + a_6 z^{-6}) 
\]  

\[
(b_1 - a_1 b_0) z^{-1} + (b_2 - a_2 b_0) z^{-2} + (b_3 - a_3 b_0) z^{-3} + (b_4 - a_4 b_0) z^{-4} 
\]  

\[
+ (b_5 - a_5 b_0) z^{-5} + (b_6 - a_6 b_0) z^{-6} 
\]  

As can be clearly seen from equation A.3, this model structure implies that all numerator parameters are correlated with the static gain \( b_0 \), what is undesirable. A solution is to simply let out the static gain, i.e. to fit a model with a fifth order numerator.

§A.3 Fifth order numerator

New fits are made with a model with a fifth order numerator. These in themselves are accurate, but when a nominal model is stated a new problem occurs. For in the nominal model, the resonance mode at 25 Hz isn’t fitted properly, as can be seen in figure A.1. Of course this is not satisfactory, so a solution has to be found.
But when the parameters are perturbed independently, the accuracy becomes worse (showing how delicate the parameter values of the low-frequent pole are). See figure A.3. When the correlation of the parameters of the low-frequent pole, as is calculated in section 3.3, is taken into account, there is a major improvement. See figure A.4.
Figure A.3 Comparison of measured frequency responses (green) with frequency responses obtained from simulations (red). In these simulations, only the values $-1$ and $1$ are used for the normalized perturbations.

Figure A.4 Comparison of measured frequency responses (green) with frequency responses obtained from simulations (red). In these simulations, only the values $-1$ and $1$ are used for the normalized perturbations and the perturbations of the parameters of the low-frequent pole are made dependent on each other.
§A.5 Third order numerator

An other issue is that the values of the first two parameter in the fifth order numerator are very small in comparison to their perturbations and even change sign with different measurements, as can be seen in figure A.5. It might be a smart just to leave them out and to take a third order numerator.

\[
G(z, a, b) = \frac{b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6}}{(1 + a_1 z^{-1} + a_2 z^{-2})(1 + a_3 z^{-1} + a_4 z^{-2})(1 + a_5 z^{-1} + a_6 z^{-2})} \quad (A.5)
\]

Figure A.5 Plot of parameter values of parameter \(b_1\) (blue) and \(b_2\) (green), as defined in equation A.4, for every fit. It can be seen that the perturbation of the parameters is large compared to their mean value.

As can be seen in figure A.6, the model with the structure as is shown in equation A.5, is even less conservative then the model with the structure as is shown in equation A.4 (See figure A.4).
Figure A.6 Comparison of measured frequency responses (green) with frequency responses obtained from simulations with third order numerator (red). In these simulations, only the values $-1$ and $1$ are used for the normalized perturbations.

The final model has the structure as is shown in equation A.6.
Appendix B  Exact correlation results

As explained in Chapter 3, correlation analysis is necessary to decrease conservatism. Linear correlation between parameters can be calculated using equation 3.4 thru 3.7 (see section 3.2):

\[ \Theta_i - H_{m,n}T = E \]  \hspace{1cm} (3.4)

\[ \Theta = \begin{bmatrix} \theta_{i,\text{nor}}^1 & \theta_{i,\text{nor}}^2 & \cdots & \theta_{i,\text{nor}}^{36} \end{bmatrix}^T \]  \hspace{1cm} (3.5)

\[ H_{m,n} = \begin{bmatrix} \Theta_m & \Theta_n & \cdots & 1 \end{bmatrix} \]  \hspace{1cm} (3.6)

\[ T = \left( H_{m,n}^T H_{m,n} \right)^{-1} \left( H_{m,n}^T \Theta_i \right) \]  \hspace{1cm} (3.7)

\( T \) is the coherence vector, if all elements (except for the last one) of this vector are close to 1 or -1, great correlation exists between the investigated parameters. In this report, only correlation between two parameters is investigated. The results are shown in table B.1. It should be noted that the correlation between \( \theta_i \) and \( \theta_j \) is similar to the correlation between \( \theta_i \) and \( \theta_k \).

<table>
<thead>
<tr>
<th>( \theta_i )</th>
<th>( \theta_j )</th>
<th>( T(1,1) )</th>
<th>( \theta_i )</th>
<th>( \theta_j )</th>
<th>( T(1,1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-0.9490</td>
<td>4</td>
<td>5</td>
<td>-0.1257</td>
</tr>
<tr>
<td>3</td>
<td>-0.7933</td>
<td></td>
<td>6</td>
<td>-0.2110</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.6603</td>
<td></td>
<td>7</td>
<td>0.8694</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.3030</td>
<td></td>
<td>8</td>
<td>-0.1677</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.1439</td>
<td></td>
<td>9</td>
<td>0.0328</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.8078</td>
<td></td>
<td>10</td>
<td>-0.0289</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.2902</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.0557</td>
<td></td>
<td>5</td>
<td>6</td>
<td>0.0446</td>
</tr>
<tr>
<td>10</td>
<td>-0.0501</td>
<td></td>
<td>7</td>
<td>-0.0508</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-0.8658</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.7707</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.2355</td>
<td></td>
<td>6</td>
<td>7</td>
<td>0.0277</td>
</tr>
<tr>
<td>6</td>
<td>0.0806</td>
<td></td>
<td>8</td>
<td>0.0986</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.8711</td>
<td></td>
<td>9</td>
<td>0.0572</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.2813</td>
<td></td>
<td>10</td>
<td>-0.0580</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.0558</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.0502</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-0.9743</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1121</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.0345</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.9888</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.2691</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.0543</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.0487</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.1 Results of correlation analysis as is defined in equation 3.4 thru 3.7

When a bound of 0.2 around 1 and -1 is set, the following groups of correlated parameters is determined:

<table>
<thead>
<tr>
<th>Group 1</th>
<th>( \theta_1, \theta_2, \theta_3, \theta_4 ) and ( -\theta_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 2</td>
<td>( \theta_9 ) and ( -\theta_{10} )</td>
</tr>
</tbody>
</table>

Table 3.2 Correlated parameters (See equation 2.1 for parametric definitions)
Appendix C TF, block chart, LFT

§C.1 Transfer function representation
The transfer function of the final model is defined as follows:

\[ G = \frac{B(z, \theta, \delta)}{A_1(z, \theta, \delta) \cdot A_2(z, \theta, \delta) \cdot A_3(z, \theta, \delta)} \]  

(4.4)

with

\[ B(z, \theta, \delta) = (\bar{\theta}_1 + \tilde{\theta}_1 \cdot \delta_1) z^{-3} + (\bar{\theta}_2 + \tilde{\theta}_2 \cdot \delta_1) z^{-4} + (\bar{\theta}_3 + \tilde{\theta}_3 \cdot \delta_1) z^{-5} + (\bar{\theta}_4 + \tilde{\theta}_4 \cdot \delta_1) z^{-6} \]  

(4.5)

\[ A_1(z, \theta, \delta) = 1 + (\bar{\theta}_5 + \tilde{\theta}_5 \cdot \delta_1) z^{-1} + (\bar{\theta}_6 + \tilde{\theta}_6 \cdot \delta_2) z^{-2} \]  

(4.6)

\[ A_2(z, \theta, \delta) = 1 + (\bar{\theta}_7 - \tilde{\theta}_7 \cdot \delta_3) z^{-1} + (\bar{\theta}_8 + \tilde{\theta}_8 \cdot \delta_4) z^{-2} \]  

(4.7)

and

\[ A_3(z, \theta, \delta) = 1 + (\bar{\theta}_9 + \tilde{\theta}_9 \cdot \delta_5) z^{-1} + (\bar{\theta}_{10} - \tilde{\theta}_{10} \cdot \delta_6) z^{-2} \]  

(4.8)

§C.2 Block chart
A representation of the block chart of the final model is given in figure C.1. It should be mentioned that \( \theta_i = \bar{\theta}_i + \tilde{\theta}_i \cdot \delta_1 \).

![Figure C.1 Block chart of final model](image)

Observing figure C.1, three subsections can be distinguished, all with a second order denominator. One of the subsections has a third order numerator, resulting from the third order numerator in the transfer function.

§C.3 Linear Fractional Transformation
A representation of the Linear Fractional Transformation (LFT), which can be used for implementation in a controller, is given in figure C.2. An extensive description of the matrices is given in equation C.1 thru C.5.
Figure C.2 Schematic representation of a Linear Fractional Transformation

The matrices of the LFT representation, as are shown in figure C.2, are, for the transfer function as mentioned in equation 4.4 thru 4.8, defined in equation C.1 thru C.5

\[ G_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (C.1)

\[ G_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (C.2)

\[ G_{21} = \begin{bmatrix} -\tilde{\theta}_5 & -\tilde{\theta}_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{\theta}_7 & 0 & 0 & 0 & -\tilde{\theta}_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (C.3)

\[ G_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Modeling and parametric uncertainty characterization
of a dual-stage Hard Disk Drive actuator
J.J.M. van Helvoort

\[
G_{22} = \begin{bmatrix}
-\bar{\theta}_5 & -\bar{\theta}_6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -\bar{\theta}_7 & -\bar{\theta}_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \bar{\theta}_1 & \bar{\theta}_2 & \bar{\theta}_3 & \bar{\theta}_4 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(C.4)

\[
\Delta = \begin{bmatrix}
\delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 \\
\end{bmatrix}
\]

(C.5)

It should be mentioned that this LFT representation is not minimal. For further implementation, it should be rewritten into minimal format (2 states less).
Appendix D Formulating an LFT representation

The formulation of a Linear Fractional Transformation (LFT) representation, as is shown in figure D.1, has proven to be hard. An LFT representation should be of the lowest possible dimension (See reference [1]). The LFT as is proposed in this report has the correct representation, but is not of the lowest possible dimension, due to the fact that a model with a third order numerator and only second order denominators has been selected. But it still should be possible to derive a representation of the lowest possible order.

Figure D.1 Schematic representation of a Linear Fractional Transformation

§D.1 Constructing the matrices

The input-output equation, belonging to the LFT representation as shown in figure D.1, is given in equation D.1. See also reference [1] and [2].

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = \begin{bmatrix} \Delta \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix} G_{21} \\
G_{22} \\
\end{bmatrix} \begin{bmatrix} (I - G_{11}\Delta)^{-1} \\
G_{12} \\
\end{bmatrix} \begin{bmatrix} x \\
u \\
\end{bmatrix}
\]  

(D.1)

For the model, as is derived in this report, The uncertainty matrix \( \Delta \) only appears linear and not inverted, so

\[ G_{11} = 0 \]  

(D.2)

Furthermore, since all \( \delta \)'s also only appear linear, \( \Delta \) is said to be a diagonal matrix with \( \delta \) on the diagonal (in this model, for \( \delta_1 \) and \( \delta_2 \) are used for more than one parameter, they also appear more then once on the diagonal of \( \Delta \). See appendix C)

\[ \Delta = diag(\delta_1, ..., \delta_n) \]  

(D.3)

\( G_{22} \) represents the system behavior without variations of the parameters. \( G_{22} \) will be filled with the nominal values of the parameters, like a State Space representation of the system. With \( G_{12} \) and \( G_{21} \) the variation of the parameters is added to the system. A straightforward way to do this is to fill \( G_{21} \) with the magnitudes of the perturbations and to define with matrix \( G_{12} \) how the perturbations act on the parameters. See also appendix C.

§D.2 Non-minimal LFT

A problem arose while constructing the matrices, for it was not clear how to represent the State Space format of the system in the lowest possible dimension. \( G_{22} \) was adjusted (see appendix C), but the representation no longer is of the lowest possible dimension. An asset of this representation nevertheless is, that all parameters appear nicely isolated in the matrices, see appendix C.
Appendix E  Numerical values of parameters

§E.1 Final model

For the completeness, the final model, as is arisen in this report, is given below (as in section 4.1)

\[ G(z, \theta, \delta) = \frac{B(z, \theta, \delta)}{A_1(z, \theta, \delta) \cdot A_2(z, \theta, \delta) \cdot A_3(z, \theta, \delta)} \]

with

\[ B(z, \theta, \delta) = c \cdot \delta + i \delta^2 \cdot \delta + i \delta^3 \cdot \delta + i \delta^4 \cdot \delta \]
\[ A_1(z, \theta, \delta) = 1 + (\tilde{\theta}_1 + \tilde{\theta}_2 \cdot \delta)^{-1} + (\tilde{\theta}_3 + \tilde{\theta}_4 \cdot \delta)^{-2} \]
\[ A_2(z, \theta, \delta) = 1 + (\tilde{\theta}_5 - \tilde{\theta}_6 \cdot \delta)^{-1} + (\tilde{\theta}_7 + \tilde{\theta}_8 \cdot \delta)^{-2} \]

and

\[ A_3(z, \theta, \delta) = 1 + (\tilde{\theta}_9 + \tilde{\theta}_10 \cdot \delta)^{-1} + (\tilde{\theta}_11 - \tilde{\theta}_12 \cdot \delta)^{-2} \].

§E.2 Numerical values

In table E.1 the numerical values, determined as discussed in this report, are given.

| \( \tilde{\theta}_1 \) | -0.0516 | \( \tilde{\theta}_2 \) | 0.0096 |
| \( \tilde{\theta}_3 \) | -0.1289 | \( \tilde{\theta}_4 \) | 0.0245 |
| \( \tilde{\theta}_5 \) | -0.1025 | \( \tilde{\theta}_6 \) | 0.0200 |
| \( \tilde{\theta}_7 \) | -0.0252 | \( \tilde{\theta}_8 \) | 0.0051 |
| \( \tilde{\theta}_9 \) | -0.2389 | \( \tilde{\theta}_10 \) | 0.0936 |
| \( \tilde{\theta}_11 \) | 0.9649 | \( \tilde{\theta}_12 \) | 0.0193 |
| \( \tilde{\theta}_13 \) | -1.3858 | \( \tilde{\theta}_14 \) | 0.1152 |
| \( \tilde{\theta}_15 \) | 0.9795 | \( \tilde{\theta}_16 \) | 0.0105 |
| \( \tilde{\theta}_17 \) | -1.9987 | \( \tilde{\theta}_18 \) | 0.0003 |
| \( \tilde{\theta}_19 \) | 0.9988 | \( \tilde{\theta}_20 \) | 0.0003 |

Table E.1 Table with values of parameters of final model.