Strong dynamic input-output decoupling: from linearity to nonlinearity
Huijberts, H.J.C; Nijmeijer, H.

Published in:
NOLCOS : nonlinear control systems design : 2nd IFAC symposium, Bordeaux, France, 24-26 June 1992 / Ed. Michel Fliess

Published: 01/01/1993

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Abstract

We study the strong dynamic input-output decoupling problem (SDIODP) for nonlinear systems. It is shown that, given a generically satisfied assumption, the solvability of the SDIODP around an equilibrium point is equivalent to the solvability of the same problem for the linearization of the system around this equilibrium point. We introduce the Singh compensator, a dynamic state feedback of minimal order that solves the SDIODP. It is shown that, given the assumption mentioned above, the linearization of the Singh compensator around an equilibrium point is a Singh compensator for the linearization of the original nonlinear system around this equilibrium point.

1 Introduction

Since for output regulation tasks input-output decoupled systems are relatively easy to handle, the so-called input-output decoupling problem has received a lot of attention in the literature. Today, this problem is quite well understood for nonlinear control systems, see e.g. [16], [1], [13] where a complete solution is described in terms of a dynamic compensator. An important feature of the compensator we consider in this paper, the so-called Singh compensator, is that it is a decoupling compensator of minimal dimension, see [9], and therefore it is intuitively of minimal "complexity".

Control engineers are often led to handle a specific control problem in a concrete nonlinear system by linearizing the given model around an equilibrium point, and afterwards solve, if possible, the given control problem for the linearization. We show in the present paper that at least in a local sense such an approach may be successfully used, provided the system fulfills a generically satisfied regularity assumption. In particular it follows that the decoupling Singh compensator for the nonlinear system becomes, when linearized around the equilibrium point, a decoupling compensator for the linearization. Furthermore, we have, again under the same regularity assumption, a converse of the above result: for a decoupling Singh compensator for the linearization of a nonlinear system there exists a decoupling Singh compensator for the nonlinear system having the aforementioned compensator as its linearization. The practical consequence of this is that, at least in a sufficiently small neighborhood, the linear solution of the input-output decoupling problem for the linearization of a nonlinear system will act as a first order approximate solution of the decoupling problem for the original nonlinear system. In this sense, we consider this paper as a justification of engineering practice. Of course it remains to be studied for each specific system what an acceptable size of the region of applicability of this result will be.

With the herefores mentioned relation between nonlinear and linearized input-output decouplability in mind, we conclude this paper with some remarks relating algebraic and geometric structure at infinity of the nonlinear system. These comments should be considered as a complement to the work in [11].

2 Preliminaries

Consider a square nonlinear system $P$, given by equations of the form

$$\begin{align*}
    P: \begin{cases}
    \dot{z} &= f(z) + g(z)u \\
    y &= h(z)
    \end{cases}
\end{align*}$$

with $z = \text{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$ local coordinates for the state space manifold $\mathcal{X}$, $u \in \mathbb{R}^m$ denoting the controls, and $y \in \mathbb{R}^m$ denoting the outputs. Furthermore, we assume all data to be analytic. Recall that a meromorphic function $\eta$ is a function of the form $\eta = \pi/\theta$, where $\pi$ and $\theta$ are analytic functions. Assume that the control functions $u(t)$ are $n$ times continuously differentiable. Then define $u^{(0)} := u$, $u^{(i+1)} := (d/dt)u^{(i)}$. View $x, u, \ldots, u^{(n-1)}$ as variables and let $\mathcal{K}$ denote the field consisting of the set of rational functions of $(u, \ldots, u^{(n-1)})$ with coefficients that are meromorphic in $x$. For the system (1) we define in a natural way (with $y^{(0)} := y$)

$$y^{(k+1)} = y^{(k+1)}(x, u, \ldots, u^{(k)}) = 
\frac{\partial y^{(k)}}{\partial x} [f(x) + g(x)u] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)}$$

Note that $y, \dot{y}, \ldots, y^{(n)}$ so defined have components in the field $\mathcal{K}$. 

173
For \( k = 1, \ldots, n \), introduce the Jacobian matrices (cf. [12]):
\[
J_k(x, u, \ldots, u^{(k-1)}) = \\
\left( \begin{array}{cccc}
\frac{\partial y_1}{\partial u} & 0 & 0 & \cdots & 0 \\
\frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial u} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{\partial y_k}{\partial u} & \frac{\partial y_k}{\partial u} & \frac{\partial y_k}{\partial u} & \cdots & \frac{\partial y_k}{\partial u^{(k-1)}}
\end{array} \right)
\]
Then the rank \( \rho^*(P) \) of \( P \) is defined by (see [3])
\[
\rho^*(P) = \text{rank}_K J_n - \text{rank}_K J_{n-1}
\] (2)
Note that we always have \( \rho^*(P) \leq m \). \( P \) is said to be of full rank if \( \rho^*(P) = m \).
We next introduce Singh's algorithm for the system \( P \). This algorithm has been introduced in [16] for calculation of a left inverse of a nonlinear system. The version of Singh's algorithm presented here is taken from [3].

**Algorithm 2.1 Singh's algorithm**

**Step 0** Let \( y_0^{(0)} \) be void and define \( y_0^{(0)} := y \).

**Step k+1** For \( r, s \in \mathbb{N}, \) let \( \mathcal{I}_{rs} := \{ r, r+1, \ldots, s \} \).

Suppose that in Steps 1 through \( k \), \( \bar{y}_1^{(k)}, \ldots, \bar{y}_k^{(k)} \) have been defined so that
\[
\begin{align*}
\bar{y}_1 &= \bar{a}_1(x) + \bar{b}_1(x)u \\
\vdots \\
\bar{y}_k^{(k)} &= \bar{a}_k(x, \{ \bar{y}_i^{(j)} \mid i \in \mathcal{I}_{k-1}, j \in \mathcal{I}_k \}) \\
&\quad + \bar{b}_k(x, \{ \bar{y}_i^{(j)} \mid i \in \mathcal{I}_{k-1}, j \in \mathcal{I}_{k-1} \})u \\
\bar{y}_k^{(k)} &= \bar{y}_k^{(k)}(x, \{ \bar{y}_i^{(j)} \mid 1 \leq i \in \mathcal{I}_k, j \in \mathcal{I}_k \})
\end{align*}
\]
Suppose also that the matrix \( \tilde{B}_k := [\bar{y}_1^{(k)}, \ldots, \bar{y}_k^{(k)}]^T \) has full rank equal to \( \rho_k \), where the rank is taken with respect to the field of rational functions of \( \{ y_i^{(j)} \mid 1 \leq i \leq k-1, 1 \leq j \leq k-1 \} \) with coefficients in the field of meromorphic functions of \( x \). Then calculate
\[
\bar{y}_k^{(k+1)} = \frac{\partial y_k^{(k)}}{\partial x} [f(x) + g(x)u] + \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{\partial y_k^{(j)}}{\partial y_i^{(j)}} \bar{y}_k^{(j+1)}
\]
and write this as
\[
\begin{align*}
\bar{y}_k^{(k+1)} &= a_{k+1}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \}) \\
&\quad + b_{k+1}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \})u
\end{align*}
\]
Define \( B_{k+1} := [\tilde{B}_k^{T}, \tilde{B}_{k+1}^{T}]^T \), and \( \rho_{k+1} := \text{rank} B_{k+1} \) where the rank is taken with respect to the field of rational functions of \( \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \} \) with coefficients in the field of meromorphic functions of \( x \). Permute, if necessary, the components of \( \bar{y}_k^{(k+1)} \) so that the first \( \rho_{k+1} \) rows of \( B_{k+1} \) are linearly independent. Decompose \( \bar{y}_k^{(k+1)} \) as \( \bar{y}_k^{(k+1)} = (\bar{y}_k^{(k+1)})^{(k+1)} + (\bar{y}_k^{(k+1)})^{(k+2)} \), where \( \bar{y}_k^{(k+1)} \) consists of the first \( (\rho_{k+1} - \rho_k) \) rows. Since the last rows of \( B_{k+1} \) are linearly dependent on the first \( \rho_{k+1} \) rows, we can write
\[
\begin{align*}
\bar{y}_1 &= \bar{a}_1(x) + \bar{b}_1(x)u \\
\vdots \\
\bar{y}_k^{(k+1)} &= \bar{a}_{k+1}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \}) \\
&\quad + \bar{b}_{k+1}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \})u \\
\bar{y}_k^{(k+1)} &= \bar{y}_k^{(k+1)}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{k+1}, j \in \mathcal{I}_{k+1} \})
\end{align*}
\]
Finally, set \( \tilde{B}_{k+1} := [\tilde{B}_k^{T}, \tilde{B}_{k+1}^{T}]^T \).

We associate a notion of regularity with Singh's algorithm in the following way (see [2] for a related, but somewhat different notion of regularity).

**Definition 2.2** Consider the nonlinear system \( P \) and let a point \( x_0 \in \mathcal{X} \) be given. We call \( x = (x_0, 0) \) a strongly regular point for \( P \) if for each application of Singh's algorithm to \( P \) we have
\[
\text{rank}_K \tilde{B}_k(x_0, 0) = \rho_k \quad (k = 1, \ldots, n) \quad (3)
\]
If \( x = (x_0, 0) \) is a strongly regular point for \( P \), the rank of \( P \) can be calculated by evaluating \( J_1 \) and \( J_{n-1} \) at a single point (cf. [6]):
\[
\rho^*(P) = \text{rank}_K J_n(x_0, 0) - \text{rank}_K J_{n-1}(x_0, 0) \quad (4)
\]

**3 The nonlinear SDIODP and linearization**

In this section a strong version of the dynamic input-output decoupling problem is studied. For a nonlinear system \( P \), define the relative degrees \( r_i \) \((i = 1, \ldots, m)\) as the smallest \( k \in \mathbb{N} \) for which
\[
\frac{\partial y_i^{(k)}}{\partial u} \neq 0 \quad (5)
\]
If all relative degrees are finite, define the decoupling matrix \( A(x) \) with entries \( a_{ij}(x) = (\partial y_i^{(r_j)}/\partial u_j)(x) \), \( (i, j) \in \mathcal{I}_{1,m} \). A system \( P \) is said to be input-output decoupled if each of its inputs influences only one of its outputs. The system is said to be strongly input-output decoupled (see [15]) if all relative degrees are finite, its decoupling matrix is an invertible diagonal matrix, and
\[
\frac{\partial y_i^{(k)}}{\partial u_j} = 0 \quad (i = 1, \ldots, m; j \neq i; k \geq r_i + 1) \quad (6)
\]
Definition 3.3 Consider a nonlinear system $P$ and let $x_0 \in \mathcal{X}$ be given. Then the strong dynamic input-output decoupling problem (SDIODP) is said to be solvable around $x_0$ if there exist an integer $\nu$, a dynamic state feedback $Q$ on $\mathbb{R}^\nu$ of the form

$$
Q \left\{ \begin{array}{l}
\dot{z} = \alpha(x, z) + \beta(x, z)v \\
u = \gamma(x, z) + \delta(x, z)v
\end{array} \right.
$$

with $z \in \mathbb{R}^\nu$, $v \in \mathbb{R}^m$ denoting the new controls and $\alpha, \beta, \gamma, \delta$ analytic, a neighborhood $U \subset \mathcal{X}$ of $x_0$ and an open subset $Z$ of $\mathbb{R}^\nu$ such that the system $P \circ Q$ restricted to $U \times Z$ is strongly input-output decoupled.

In [3] it was shown that if $(x_0, 0)$ is a strongly regular point for $P$, the SDIODP solvable around $x_0$ if and only if $\rho^*(P) = m$ (see also [1], [13]).

We now present a special sort of dynamic state feedback that solves SDIODP around strongly regular points for $P$. This dynamic state feedback is obtained via Singh's algorithm and we call it a Singh compensator. The Singh compensator is obtained as follows (see [7] and also [16]). Consider the nonlinear system $P$ and let $(x_0, 0)$ be a strongly regular point for $P$. Furthermore assume that $\rho^*(P) = m$. Apply Singh's algorithm to $P$. This yields at the $n$-th step:

$$
\hat{Y}_n = \hat{A}_n(x, \{y_i^{(j)} | i \in I_{1n-1}, j \in I_m\})
$$

$$
+ \hat{B}_n(x, \{y_i^{(j)} | i \in I_{1n-1}, j \in I_m\})u
$$

where $\hat{Y}_n = (\hat{y}_1^T \ldots \hat{y}_n^T)^T$ and $\hat{A}_n$ is an $m$-vector with entries $\hat{a}_k$. Moreover there exist a neighborhood $U \subset \mathcal{X}$ of $x_0$, and a neighborhood $Y_0$ of the point $(\hat{y}_0^T | i \in I_{1n-1}, j \in I_m) = 0$ such that $\hat{B}_n$ is invertible on $U \times Y_0$. Then on $U \times Y_0$ (8) yields in particular:

$$
u = \hat{B}_n^{-1}[\hat{Y}_n - \hat{A}_n]
$$

For $i = 1, \ldots, m$, let $\gamma_i$ be the lowest time-derivative and $\delta_i$ be the highest time-derivative of $y_i$ appearing in (9). Then we can rewrite (9) as

$$
u = \phi_1(x, \{y_i^{(j)} | i \in I_{1m}, j \in I_{\gamma_i-1}\})
$$

$$
+ \sum_{i=1}^m \phi_{2i}(x, \{y_i^{(j)} | i \in I_{1m}, j \in I_{\gamma_i-1}\})y_i^{(\delta_i)}
$$

for certain vector-valued functions $\phi_1, \phi_{2i}$ ($i = 1, \ldots, m$). Let $z_i$ ($i = 1, \ldots, m$) be a vector of dimension $\delta_i - \gamma_i$ and consider the system:

$$
\begin{align*}
\dot{z}_i &= A_i z_i + B_i v_i \quad (i = 1, \ldots, m) \\
u &= \phi_1(x, z_1, \ldots, z_m) + \\
&\quad \sum_{i=1}^m \phi_{2i}(x, z_1, \ldots, z_m)v_i
\end{align*}
$$

with $(A_i, B_i)$ in Brunovsky canonical form. Then (10) is called a Singh compensator and the construction of a Singh compensator (10) gives that for the compensated system we have that $y_1, \ldots, y_{m-1}$ are independent of the new controls and that $y_i^{(\delta_i)} = v_i$ ($i = 1, \ldots, m$). Thus the decoupling matrix of the compensated system is given by $A(x) = I_m$, and (6) holds. Hence any Singh compensator (10) around a strongly regular point $x_0$ solves the SDIODP around $x_0$. In [9] it was shown that in fact a Singh compensator is a dynamic state feedback of minimal order solving the SDIODP.

Let $x_0 \in \mathcal{X}$ be an equilibrium point for $P$, i.e., $f(x_0) = 0$. Assume (without loss of generality) that $h(x_0) = 0$. Let the (Jacobian) linearization $LP$ of $P$ around $x_0$ be given by

$$
LP \left\{ \begin{array}{l}
\dot{\xi} = F\xi + Gu \\
\eta = H\xi
\end{array} \right.
$$

In the sequel we make the following assumption:

Assumption 3.4 $(x, y) = (x_0, 0)$ is a strongly regular point for $P$.

We investigate the connection between the solvability of the SDIODP for $P$ and $LP$. A first result is:

Proposition 3.5 Consider a square nonlinear system $P$. Let $x_0 \in \mathcal{X}$ an equilibrium point for $P$ that satisfies Assumption 3.4. Then $\rho^*(LP) = \rho^*(P)$.

Proof By (4) we have that

$$
\rho^* = \text{rank}_{\mathbb{R}} J_n(x_0, 0) - \text{rank}_{\mathbb{R}} J_{n-1}(x_0, 0)
$$

Analogously to Section 2, define Jacobian matrices $J_k^e$ ($k = 1, \ldots, n$) for the linearized plant $LP$. It is easily checked that

$$
J_k^e =
\begin{pmatrix}
HG & 0 & 0 & \cdots & 0 \\
HFG & HG & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
HF^{k-1}G & HF^{k-2}G & \cdots & HG
\end{pmatrix}
$$

We show that $J_k(x_0, 0) = J_k^e$ ($k = 1, \ldots, n$), i.e., for $k = 1, \ldots, n$, $\ell = 0, \ldots, k-1$ we have

$$
\frac{\partial y^{(k)}}{\partial u^{(\ell)}}(x_0, 0) = HF^{k-\ell-1}G
$$

First it is shown that for $k = 0, 1, 2, \ldots$
For $k = 0, 1$ (15) is immediate. Assume that (15) holds for $k = 1, \ldots, \ell - 1$. Then
\[
\frac{\partial y^{(\ell)}}{\partial x}(x_0, 0) = \frac{\partial}{\partial x}\left(\frac{\partial y^{(\ell-1)}}{\partial x}[f(x) + g(x)u] + \sum_{r=0}^{\ell-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0)\right)
\]
\[
= \frac{\partial y^{(\ell-1)}}{\partial x}(x_0, 0)\frac{\partial f}{\partial x}(x_0) + \sum_{r=0}^{\ell-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0) = H F^{\ell-1} F = H F^{\ell}
\]
Hence (15) holds for $k = 1, 2, \ldots$. Next we show that for $k = 1, 2, \ldots, n$,
\[
\frac{\partial y^{(k)}}{\partial u}(x_0, 0) = H F^{k-1} G (16)
\]
For $k = 1$ (16) is trivially satisfied. Assume that (16) holds for $k = 0, \ldots, \ell - 1$. Then, using (15), we have:
\[
\frac{\partial y^{(k)}}{\partial u}(x_0, 0) = \frac{\partial}{\partial u}\left(\frac{\partial y^{(k-1)}}{\partial u}[f(x) + g(x)u] + \sum_{r=0}^{k-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0)\right)
\]
\[
= \frac{\partial y^{(k-1)}}{\partial u}(x_0, 0)\frac{\partial f}{\partial x}(x_0) + \sum_{r=0}^{k-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0) = H F^{k-1} G (17)
\]
Let $k \in \{1, 2, \ldots, n\}, \ell \in \{1, \ldots, k - 1\}$. Then:
\[
\frac{\partial y^{(k)}}{\partial u}(x_0, 0) = \frac{\partial}{\partial u}\left(\frac{\partial y^{(k-1)}}{\partial u}[f(x) + g(x)u] + \sum_{r=0}^{k-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0)\right)
\]
\[
= \frac{\partial y^{(k-1)}}{\partial u}(x_0, 0)\frac{\partial f}{\partial x}(x_0) + \sum_{r=0}^{k-2} \frac{\partial y^{(r+1)}}{\partial u(r)} u^{(r+1)}(x_0, 0) = H F^{k-1} G (18)
\]
(17) and (18) establish that (14) holds for $k = 1, 2, \ldots, n, \ell = 0, \ldots, k - 1$ and hence for $k = 1, \ldots, n$ we have $J_k(x_0, 0, \ldots, 0) = J_k^f$. Together with (12) this establishes that $\rho^*(LP) = \rho^*(P)$. ■

As an immediate consequence of Proposition 3.5 we have:

**Theorem 3.6** Consider a square nonlinear plant $P$. Let $x_0 \in X$ be an equilibrium point that satisfies Assumption 3.4 for $P$. Denote the linearization of $P$ around $x_0$ by LP. Then the SDIOD is solvable for $P$ around $x_0$ if and only if it is solvable for LP. ■

**Remark 3.7** The result of Theorem 3.6 can be found in [10]. It generalizes a result of [4], where a similar result was obtained for the for the strong input-output decoupling problem via static state feedback. In this case, the Singh compensator reduces to a regular static state feedback that renders $\Delta^*$, the maximal locally controlled invariant distribution in $\text{Ker} dh$, invariant. The result is then established by showing that $\Delta^*(x_0) = \Psi^*$ (or rather: $\Delta^*(x_0)$ can be identified with $\Psi^*$), where $\Psi^*$ is the maximal controlled invariant subspace in $\text{Ker} H$ for $LP$ (cf. [4]). It seems likely that also the result of Theorem 3.6 may be proved in a similar way by employing the same techniques as in Section 4 of [7]. Also the approximation result of [5] is connected with the result of Theorem 3.6.

We now investigate the connection between Singh compensators for $P$ and LP. In what follows we use the following lemma, which can easily be verified:

**Lemma 3.8** Consider the equation
\[
\dot{Y} = \tilde{A}(Z) + \tilde{B}(Z)U (19)
\]
where $\tilde{A}(Z) = 0, \tilde{A}(Z) = 0, \tilde{B}(Z)$ has full row rank on a neighborhood of $Z_0$, and each of the rows of $\tilde{B}(Z)$ is linearly dependent on the rows of $\tilde{B}(Z)$. Moreover, consider the linearization of (19) around $(Z, U, \tilde{Y}, Y) = (Z_0, 0, 0, 0)$:
\[
\dot{\tilde{Y}}^t = \tilde{A}^t Z^t + \tilde{B}^t U^t
\]
\[
\dot{Y}^t = \tilde{A}^t Z^t + \tilde{B}^t U^t (20)
\]
Let $\tilde{B}^t(Z)$ be a right inverse of $\tilde{B}(Z)$ on a neighborhood of $Z_0$ where $\tilde{B}(Z)$ has full row rank. Then (19) and (20) yield:
\[
U = \tilde{B}^t(Z)[\tilde{Y} - \tilde{A}(Z)]
\]
\[
\dot{Y} = \tilde{A}(Z) + \tilde{B}(Z)\tilde{B}^t(Z)[\tilde{Y} - \tilde{A}(Z)] (21)
\]
\[
\dot{U}^t = \tilde{B}^t Z^t[\dot{\tilde{Y}}^t - \tilde{A}^t \tilde{Z}^t]
\]
\[
\dot{\tilde{Y}}^t = \tilde{A}^t Z^t + \tilde{B}^t \tilde{B}^t[\dot{\tilde{Y}}^t - \tilde{A}^t \tilde{Z}^t] (22)
\]
where (22) can be obtained by linearizing (21) around $(Z, U, \tilde{Y}, Y) = (Z_0, 0, 0, 0)$.

**Lemma 3.9** Consider a square nonlinear plant $P$ of full rank. Let $x_0$ be a strongly regular equilibrium point satisfying Assumption 3.4. Let $Q$ be a Singh compensator for $P$ around $x_0$. Then $(x, z) = (x_0, 0)$ is an equilibrium point for $P \circ Q$.

**Proof** See [6]. ■

**Proposition 3.10** Consider a square nonlinear system $P$ of full rank. Let $x_0 \in X$ be an equilibrium point satisfying Assumption 3.4. Apply Singh’s algorithm to $P$, yielding a reordering $\tilde{y}_1, \ldots, \tilde{y}_n$ of the outputs, such that
\[
\tilde{y}_k^{(j)} = \tilde{a}_k(x, \{\tilde{y}_i^{(j)} | i \in I_{t_k-1}, j \in I_{t_k}\}) + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} | i \in I_{t_k-1}, j \in I_{t_k}\})u
\]
\[
\tilde{y}_k^{(k)} = \tilde{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} | 1 \leq i \leq k, 1 \leq j \leq k\}) (23)
\]
Then there is an application of Singh’s algorithm to LP that results in the same reordering of the outputs and where the result of Singh’s algorithm applied to LP can alternatively be obtained by linearizing the result of Singh’s algorithm applied to $P$ around $(x, u, \{\tilde{y}_i^{(j)} | 1 \leq i \leq n, 1 \leq j \leq n\}) = (x_0, 0, \ldots, 0)$. 176
Proof We first consider $k = 1$. We have:

$$\dot{y} = \frac{\partial h}{\partial x}(x)[f(x) + g(x)u] = a_1(x) + b_1(x)u \quad (24)$$

where $\text{rank}_{c} b_1(x) = p_1$. By Assumption 3.4 there exists a neighborhood of $x_0$ such that $\text{rank}_{c} b_1(x) = p_1$ on this neighborhood. After a possible permutation of the outputs (24) yields:

$$\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
\bar{a}_1(x) + \bar{b}_1(x)u \\
\bar{a}_2(x) + \bar{b}_2(x)u
\end{pmatrix} \quad (25)$$

where $\bar{b}_1$ has full row rank $p_1$ on a neighborhood of $x_0$. Let $\bar{b}_1^*(x)$ be a right inverse of $\bar{b}_1(x)$ on this neighborhood. Then (25) yields

$$\dot{y}_1 = \bar{a}_1(x) + \bar{b}_1(x)\bar{b}_1^*(x)(\dot{y}_1 \cdots \dot{a}_1(x)) \quad (26)$$

Now consider $LP$. We have:

$$\dot{z} = HF + HG \dot{u} =: a'_1 \xi + b'_1 u \quad (27)$$

where

$$a'_1 = HF = \frac{\partial}{\partial x}(\frac{\partial h}{\partial x}(x_0)) = \frac{\partial a_1}{\partial x}(x_0) \quad (28)$$

$$b'_1 = HG = \frac{\partial h}{\partial x}(x_0)g(x_0) = b_1(x_0) \quad (29)$$

Employing the same permutation of the outputs as for $P$, (27) yields

$$\begin{pmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{pmatrix} =
\begin{pmatrix}
\bar{a}_1 \xi + \bar{b}_1^* u \\
\bar{a}_2 \xi + \bar{b}_2^* u
\end{pmatrix} \quad (30)$$

where, by Assumption 3.4, $\bar{b}_1^*$ has full row rank $p_1$. Note that by (28) and (29) $\bar{a}_1 = (\partial \bar{a}_1/\partial x)(x_0)$, $\bar{a}_2 = (\partial \bar{a}_1/\partial x)(x_0)$, $\bar{b}_1^* = \dot{b}_1(x_0)$, $\bar{b}_2^* = \ddot{b}_2(x_0)$. Let $\ddot{b}_1^*$ be a right inverse of $\bar{b}_1^*$. Then (30) yields

$$\dot{\eta}_1 = \ddot{a}_1 \xi + \ddot{b}_1^* \dot{\eta}_1 \quad (31)$$

$$\dot{\eta}_2 = \ddot{a}_2 \xi + \ddot{b}_2^* \dot{\eta}_2 \quad (32)$$

Then by Lemma 3.8, (31) can be obtained by linearizing (26) around $(x, u, \dot{y}_1)$ = $(x_0, 0, 0)$ and hence our claim holds for $k = 1$. Now consider the case $k = 2$. We have:

$$y_1^{(2)} = \frac{\partial \dot{y}_1}{\partial x}(f(x) + g(x)u + \frac{\partial \dot{y}_1}{\partial y_1} y_1^{(2)}) =: a_2(x, \dot{y}_1, y_1^{(2)}) + b_2(x, \dot{y}_1)u \quad (33)$$

and

$$\eta_1^{(2)} = \frac{\partial \dot{\eta}_1}{\partial \xi}(F \xi + Gu) + \frac{\partial \dot{\eta}_1}{\partial \eta_1} \eta_1^{(2)} =: a_2^t \xi + a_2^t \eta_1^{(2)} + b_2^t u$$

We find from (26),(31),(32),(33):

$$a_2^a = \frac{\partial a_1}{\partial \xi} F = (a_1^t - a_1^t)F =$$

$$\frac{\partial a_1}{\partial \xi}(x_0) - \frac{\partial a_1}{\partial \xi}(x_0) \frac{\partial f}{\partial \xi}(x_0) =$$

$$\frac{\partial a_2}{\partial \xi} \frac{\partial a_1}{\partial \xi} f(x_0) =$$

$$\frac{\partial a_2}{\partial \xi} \frac{\partial a_1}{\partial \xi} f(x_0) \quad (34)$$

$$a_2^a = \frac{\partial a_1}{\partial \xi} \eta_1^{(2)} = \bar{b}_1 \bar{b}_1^* = \bar{b}_1(x_0) \bar{b}_1^* = \frac{\partial \dot{y}_1}{\partial \eta_1}(x_0) \quad (35)$$

$$b_2^t = \frac{\partial \dot{\eta}_1}{\partial \xi} (\eta_1^{(2)}) = \frac{\partial a_2}{\partial \xi} G = (a_1^t - a_1^t)g(x_0) =$$

$$\frac{\partial \dot{y}_1^{(2)}}{\partial \eta_1}(x_0) = b_2(x_0, 0) \quad (36)$$

From (34),(35),(36) it follows that $\ddot{\eta}_1^{(2)}$ can be obtained by linearizing $\ddot{y}_1^{(2)}$ around $(x, u, \dot{y}_1, y_1^{(2)}) = (x_0, 0, 0, 0)$. Then, using Lemma 3.8, it can be shown that our claim hold for $k = 2$. Proceeding as above, it can be shown that the claim also holds for $k = 3, 4, \ldots, n$.

Remark 3.11 For nonlinear systems, an algebraic structure at infinity ([11]) and a geometric structure at infinity ([14]) can be defined. For linear systems both structures at infinity coincide ([11]). A consequence of Proposition 3.10 is that the algebraic structure at infinity of $P$ coincides with that of $LP$. This could lead to the conjecture that for a nonlinear system $P$ satisfying Assumption 3.4 the algebraic and geometric structures at infinity coincide. However, this is not the case, as follows from the example $x_1 = u_1$, $x_2 = x_3 + u_4 u_1$, $x_3 = u_2$, $x_4 = u_3$, $x_5 = x_6$, $x_6 = v_3$, $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$. A straightforward calculation shows that this system has 3 geometric zeros at infinity of orders 2, 2, 1 respectively and it has 3 algebraic zeros at infinity of orders 3, 2, 1 respectively.

From Proposition 3.10 we may draw the following conclusion:

Theorem 3.12 Consider a square nonlinear system $P$ of full rank. Let $x_0 \in X$ be an equilibrium point satisfying Assumption 3.4 and let $LP$ denote the linearization of $P$ around $x_0$. Then:

(i) The linearization of a Singh compensator for $P$ is a Singh compensator for $LP$.

(ii) Conversely, every Singh compensator for $LP$ is a first order approximation of a Singh compensator for $P$ around $x_0$. 

177
Theorem 3.12 has the following practical implication. In engineering practice one often studies a specific control problem by addressing the problem on the linearization around a given working point. Of course it then remains questionable if the control that was designed to solve the control problem for the linearization of the system is a approximate (first order) approximation of a control that solves the control problem for the original nonlinear system. In [4] an example for the strong input-output decoupling problem via static state feedback was given where indeed this was not the case. However, from Theorem 3.12 it follows that any Singh compensator for LP (which is a dynamic state feedback that solves the SDIODP for LP) is a first order approximation of a dynamic state feedback that solves the SDIODP for P.

References


