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Influence of thermal mismatch on the energy release of bimaterial joints
Part 1: Thermal strain energy and interactions

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# Contents

1 Introduction ........................................... 3  
   1.1 General part ........................................... 3  
   1.2 Outline of the present study ......................... 5  
   1.3 Notation and symbols ................................... 6  
      1.3.1 Notation ........................................... 6  
      1.3.2 Symbols ........................................... 7  

2 Description of the system .............................. 9  
   2.1 Geometry and materials ................................. 9  
   2.2 Thermo-mechanical modeling .............................. 10  
   2.3 Basic equations ......................................... 12  
   2.4 Variational principles .................................. 12  
   2.5 Helmholtz free energy ................................... 15  
      2.5.1 General part ........................................ 15  
      2.5.2 Expansion around reference state at $T = T_r$ .... 15  
      2.5.3 Expansion around reference state at $T < T_r$ .... 16  

3 The elastic energy ....................................... 18  
   3.1 Expansion around state at $T_r$ ......................... 18  
      3.1.1 General part ........................................ 18  
      3.1.2 Elastic energy due to thermal mismatch ............ 21  
      3.1.3 Elastic energy due to the load .................... 23  
      3.1.4 Elastic interaction energy ......................... 25  
   3.2 Expansion at operating temperature .................... 27  
      3.2.1 General part ........................................ 27  
      3.2.2 Elastic energy due to thermal mismatch ............ 27  
      3.2.3 Elastic energy due to the loading ................. 30  
      3.2.4 Elastic interaction energy ......................... 31  
   3.3 Conclusions of chapter 3 ............................... 32  

4 The interaction energy .................................... 34  
   4.1 Methodological considerations ......................... 34  
   4.2 Analysis of the loading mechanism ..................... 36
CONTENTS

4.3 Derivation of the interaction ........................................... 38
  4.3.1 Case I: cooled and loaded system ............................... 39
  4.3.2 Case II: cooled but unloaded system ......................... 40
  4.3.3 Case III: cooled and loaded bodies with equal thermal ex-
      pansion coefficients ........................................... 40
  4.3.4 Interaction energy ................................................ 42

5 Summary and conclusions .................................................. 46
Chapter 1

Introduction

1.1 General part

With the increasing use of technical ceramics the need to bond these materials to metals also increases. To achieve a bond, several technologies exist or are being developed [1]. Typical of most of these technologies is that the joint is established at relatively elevated temperatures, meaning that the temperature at which a joint should function is lower than the temperature at which it is fabricated. Since the thermal expansion coefficients of the materials constituting the joint are generally different, the temperature difference mentioned above gives rise to residual stresses in and near the interface between the materials. In some cases these stresses reach values so high that the component fails during the cooling down period following the joining process. Even if the component survives the cooling process it is clear that thermal residual stresses may lead to a considerable weakening of the component, that is to say to a reduction of the mechanical load the component can withstand during function.

In an energy based approach to mechanical failure a necessary condition for the onset of crack growth is that the crack extension force or energy release rate equals the crack resistance force [2]. The crack extension force is proportional to minus the derivative of the total potential energy of the system to the crack length. Part of this energy is the elastic or strain energy present in the system. If thermal residual stresses are present and the system is also loaded externally, the elastic energy derives from two independent sources: one caused by the mismatch of thermal expansion coefficients and one caused by the external mechanical loading. In addition to that interaction may occur. Indeed, even for linear systems where the superposition principle holds only elastic states (stresses, strains, etc.) can be superposed but for energies this is usually not possible. Generally we therefore have for the total elastic energy $U_{el}$ the splitting:

$$U_{el} = U_e + U_{th} + U_{int}$$ (1.1)

with $U_e$ the elastic energy caused by the external loading, $U_{th}$ the elastic energy
associated with the thermal residual stresses and $U_{int}$ the elastic interaction energy associated with the simultaneous presence of the residual and the externally caused stress fields. Now, according to a theorem of Colonnetti (see Kröner [17]) and Eshelby [3], the interaction energy for elastic fields due to external loading and due to internal stress is zero. Since residual stress caused by thermal mismatch is a system of internal stress one would expect $U_{int}$ to be zero. The vanishing of $U_{int}$ would be important from a practical point of view because it means that the influence of the thermal mismatch on the energy release rate can be investigated independently of the particular loading.

However, again according to Eshelby [3], the system of internal stress does interact with the external loading because an interaction with the loading mechanism is to be expected. As stated above, the crack extension force or the energy release rate is derived from the total potential energy of the system which is the difference between the mechanical potential energy of the load supplying mechanism and the elastic energy present in the system. So, the whole system (cracked body plus loading mechanism) rather than only the cracked body should be considered. With reference to Fig. 1.1 it is the energy of the region enclosed by the dotted line that is of interest. Obviously, an interaction between thermal residual stresses and the loading mechanism is of practical importance since it contributes to the energy release rate. To describe such an interaction between loading mechanism and thermal residual stresses one would expect the interaction energy to be proportional to the applied loading, for example a force parameter $f$, and also proportional to the displacements $v$ caused solely by the thermal mismatch:

$$E_{int} \propto f v.$$  

Eshelby's treatment [3] uses in fact these considerations. However, because his treatment is rather on an abstract level the physical basis for such an interaction energy remains unclear. Generally the load is applied after the bond was established and after the system has been cooled down. Since this means that the displacements $v$ occur without any load acting on the system, it is not clear why a term proportional to $fv$ should be included in the energy. So it makes sense to investigate this matter in more detail.

A further reason to study the influence of the energy associated with thermal mismatch on the energy release rate is that in literature it is stated by some authors that residual stresses cannot drive crack growth. Wang and Suo [15] state: 'A remarkable feature common to all thin-layer sandwiches is that the residual stress in the layer does not drive the crack, because the strain energy stored in the layer due to residual stress is not released in the process of cracking.' Since the authors give no further information concerning the type of the residual stress they have in mind, the statement seems to be totally unqualified. From an experimental point of view this statement seems to contradict the fact that sometimes joints do not survive the cooling process at all.
1.2 Outline of the present study

In chapter 2 the system that is considered is described in detail. Important here is to keep in mind that such a description does not only involve shape and material properties but also its history. After all, at the time joints are loaded they have previously gone through a certain thermal process and it is precisely this process
which leads to thermal residual stresses. The knowledge of the history of the
system is indispensible to obtain a thermomechanical model of the system. It is
found that the Helmholtz free energy of the system, considered as a function of the
temperature $T$ and the finite strain tensor $D$, is the thermodynamic potential
most suitable for our purpose because, mechanically, a joint can be modeled
as a constraint in a variational problem involving the Helmholtz free energy.
Therefore, chapter 2 contains also a section about the free energy in the small
strain approxiamation and a section in which variational problems are treated
briefly.

Chapter 3 is about the question how elastic energy for thermal stresses has
to be defined and whether the interaction elastic energy $U_{int}$ is indeed zero. The
(methodological) question how elastic energy caused by thermal mismatch has to
be defined, comes to mind if one realizes that on the one hand the basic definition
of elastic energy is:

$$U = \int \sigma(\epsilon)d\epsilon$$

(\(\sigma\) is the stress and \(\epsilon\) the strain), whereas on the other hand processes leading
to thermal stress without any accompanying strain do exist [5]. For example,
cooling or heating of an isotropic body, thereby rigidly constraining the entire
external surface. For these processes, the definition given above predicts zero
elastic energy, which is, for physical reasons, obviously wrong.

Chapter 4 deals with the question whether an interaction between the loading
mechanism and the system of internal stress due to thermal mismatch does exists
and the physical basis, if any, of such an interaction.

Finally, in chapter 5 the most important results will be summarized and con­
cclusions will be drawn.

1.3 Notation and symbols

1.3.1 Notation

- Boldface lower case roman letters like $x$ or $u$ indicate vectors in the vector
  space associated with the three dimensional space $\mathbb{R}^3$.

- Boldface roman capitals like $D$ or $M$ always indicate second order tensors
  where the term second order tensor is used as a synonym for linear trans­
 formations from $\mathbb{R}^3$ into $\mathbb{R}^3$. Transposition is indicated by the superscript
c. In addition, the two boldface Greec symbols $\epsilon$ and $\sigma$ denote always the
  small strain tensor and the Cauchy stress tensor.

- Boldface calligrafic capitals like $L$ denote fourth order tensors, where a
  fourth order tensor is defined as a linear transformation on the space of all
second order tensors. Normal (non-boldface) capitals like \( B \) or \( S \) generally can have any meaning.

- A single fixed Cartesian coordinate system is used with (unit) base vectors \( e_1, e_2 \) and \( e_3 \).

- Components of vectors or tensors with respect to this coordinate system are indicated by normal lower case or upper case letters supplied with the appropriate number of indices. For example, the components of the vector \( u \) are \( u_i \) with \( i = 1, 2 \) or 3. Similarly the components of the second order tensor \( M \) are \( M_{ij} \) with \( i, j = 1, 2, 3 \) or the components of \( L \) are \( L_{ijkl} \) with \( i, j, k, l = 1, 2, 3 \).

- A single dot (.) denotes contraction over one and a double dot (:) contraction over two indices. Therefore if \( a \) and \( b \) are vectors, \( A \) and \( B \) second order tensors and \( L \) a fourth order tensor we have (Einstein summation convention employed):

\[
\begin{align*}
\mathbf{a} \cdot \mathbf{b} &= a_i b_i, \\
A \cdot \mathbf{b} &= A_{ij} b_j, \\
A : \mathbf{B} &= A_{ij} B_{ij}, \\
\mathcal{L} : A &= \mathcal{L}_{ijkl} A_{kl}, \\
A : \mathcal{L} : \mathbf{B} &= A_{ij} \mathcal{L}_{ijkl} B_{kl}.
\end{align*}
\]

- The symbol \( \ast \) denotes the vector product.

- The symbol \( \nabla \) denotes the gradient operator. To denote the variables this operator is suffixed with a symbol:

\[
\nabla_x \mathbf{u} = \partial u_i / \partial x_j
\]

- The symbol \( \text{div} \) denotes the divergence operator:

\[
\text{div} \sigma = \partial \sigma_{ij} / \partial x_j
\]

1.3.2 Symbols

The most important symbols are given in the list below. Occasionally, other symbols not included here are used. Their meaning will be given in the text where they first appear.
\( T \): temperature
\( T_r \): reference temperature
\( \theta = T - T_r \): temperature difference
\( x, y \): position of material points
\( r, u, v, w \): displacements
\( I \): second order unit tensor
\( D \): finite strain tensor
\( \epsilon \): small strain tensor
\( S \): first Piola-Kirchhoff stress tensor
\( \sigma \): Cauchy stress tensor
\( M \): stress temperature tensor
\( I \): fourth order unit tensor
\( \mathcal{L} \): fourth order tensor of elasticity constants
\( \delta_{ij} \): Kronecker delta function
\( \rho \): mass density
\( \gamma \): Helmholtz surface free energy
\( \psi \): Helmholtz free energy of the bulk (per unit of mass)
\( \Phi \): Helmholtz free energy of the interface
\( B_i \): volume of body \( i \) in at the reference temperature
\( \partial B_i \): surface of \( B_i \)
\( \Gamma \): surface at which bodies are joined (at reference temperature)
\( \Gamma_s \): magnitude of \( \Gamma \)
\( S_i \): part of \( \partial B_i \) at which traction is applied
\( S_B^{(i)} \): magnitude of \( \partial B_i / \Gamma \)
\( t_i \): traction system working at \( S_i \)
Chapter 2

Description of the system

2.1 Geometry and materials

We consider two bodies $B_1$ and $B_2$ with boundary surfaces $\partial B_1$ and $\partial B_2$ respectively. Each body consists of a homogeneous material, but the materials of the two bodies differ with respect to thermal and mechanical properties. The bodies are bonded along part of their boundaries. See figure 2.1. The joint, that is, the surface parts the two bodies have in common is denoted by $\Gamma$ and its magnitude by $\Gamma_s$. On the surface parts $S_i$ operate the external traction systems $t_i$, $i = 1, 2$. The traction systems are assumed to be prescribed and selfequilibrating (total external force and external moments are zero). The system as a whole is not supported in any way. Note that this case also includes a system for which the external forces and moments are not selfequilibrating provided the system is supported in a static determinate manner because in that case the reaction

![Diagram of bimaterial joint under operating conditions](image)
forces and moments in the supports are completely known and the supports can dropped if instead the reactions are introduced as additional loads.

\section*{2.2 Thermo-mechanical modeling}

The system as shown in figure 2.1 operates at the temperature $T$. The joint, however, is generally made at a different temperature $T_m$ with $T \leq T_m$.

**Assumption 1**

\textit{Prior to bonding, and at $T_m$, the bodies are stress free and the surfaces that are to be joined are plane.}

After manufacturing the joint the whole system is cooled down to its operating temperature $T$. We consider joints made by brazing or solid state bonding in which a thin interlayer has been used. Generally the interlayer material is rather ductile because one want to prevent the buildup of residual stresses through plastic flow of the interlayer material. Mechanically this means that thermal residual stresses only build up after a certain temperature $T_r$ is passed. Typically one would expect $T_r$ to be of the order of the recrystallization temperature of the interlayer material \cite{4}. For many metals the stress needed to deform the material plastically is quite small for temperatures above $T_r$. For all practical purposes one would therefore expect that during the cooling process a significant buildup of thermal residual stresses only occurs after the recrystallization temperature has been passed. To mimic this behaviour for the bimaterial system considered here, we make

**Assumption 2**

\textit{Thermal stresses exist only for temperature below $T_r$, and in the range $T \leq T_r$ no stress relief takes place solely caused by thermal stresses.}

As a corollary of this assumption we have that at $T = T_r$ the bodies are stress free. Therefore, this state can be taken as a stress free reference state.

After producing the joint (joining and cooling) several other processes may follow, for example surface grinding. Because the focus in this report is on the thermal residual stresses, we assume that these processes do not lead to other residual stress systems. Basically this means that bodies of the same shape and material, not bonded together but otherwise subjected to the same thermal and processing history, remain stress free. Therefore:

**Assumption 3**

\textit{In the absence of bonding, stress free states exist in each of the bodies for all $T \leq T_r$.}

Finally, an assumption has to be made with respect to the precise moment the traction is applied.
Assumption 4

The load is applied at some temperature $T$ and this temperature is held fixed afterwards.

As thermodynamical variables we take the finite strain tensor $D$ and the temperature $T$. Therefore, the Helmholtz free energy is the obvious choice for the thermodynamic potential to be used.

Now, let $\psi_i(D_i, T)$ ($i = 1, 2$) denote the Helmholtz free energy per unit of mass of each of the bodies, and $\Phi(T)$ the Helmholtz free energy per unit bond surface$^1$ associated with the bond and $\gamma_i(T)$ the Helmholtz surface free energies$^2$ of the bodies for the nonbonded surface regions (the regions $\partial B_i \setminus \Gamma$). Then the total free energy $F_s$ of the loaded or unloaded system is:

$$F_s = \sum_{i=1}^{2} \left[ \rho_i \psi_i(D_i, T) \, dV + \int_{\partial B_i \setminus \Gamma} \gamma_i(T) \, dS \right] + \Phi(T) \Gamma_s.$$  \hfill (2.1)

with $\rho_i$ the density of the materials at temperature $T_r$. Note that the surface free energies and the free energy associated with the bond are assumed to be independent of the strain. Strictly speaking this might not be correct [6], but as we are primarily interested in the bulk situation, this assumption was made. Note also that the influence of loading and thermal stress is through the dependence of $\psi_i$ on the actual strain field $D_i$.

The loading mechanism is assumed to be independent of the temperature. The free energy of the loading mechanism is:

$$F_l = -\sum_{i=1}^{2} \int_{S_i} t_i \cdot w_i \, dS.$$  \hfill (2.2)

Here $w_i$ is the displacement of the surface points of $S_i$ due to the loading. This is an important point because the loading is applied after the cooling process has been completed. Specifically, compared to the reference state at $T_r$, the material points in each of the bodies displace in two steps. The first displacement step $v_i$ occurs during the cooling period and the second after the load is applied. It is only during the second step the loading mechanism performs work, resulting in an additional displacement $w_i$ which is superposed on the displacement $v_i$ already present.

We consider only slow processes and restrict ourselves to situations where no body forces or heat sources are present and assume that the temperature distribution is uniform at any time (system in thermal contact with a large heat bath at temperature $T$). Therefore, inertia effects can be neglected and the thermomechanical equations decouple. Mathematically speaking this means that the temperature serves as a parameter in the mechanical equations.

$^1$Although two surfaces — each with an area $\Gamma_s$ — are bonded, the bond or interface surface is taken to be $\Gamma_s$ and not $2\Gamma_s$.

$^2$Not to be confused with the Gibbs surface energy which is also often indicated by $\gamma$. 

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2.2. THERMO-MECHANICAL MODELING

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with $\rho_i$ the density of the materials at temperature $T_r$. Note that the surface free energies and the free energy associated with the bond are assumed to be independent of the strain. Strictly speaking this might not be correct [6], but as we are primarily interested in the bulk situation, this assumption was made. Note also that the influence of loading and thermal stress is through the dependence of $\psi_i$ on the actual strain field $D_i$.

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We consider only slow processes and restrict ourselves to situations where no body forces or heat sources are present and assume that the temperature distribution is uniform at any time (system in thermal contact with a large heat bath at temperature $T$). Therefore, inertia effects can be neglected and the thermomechanical equations decouple. Mathematically speaking this means that the temperature serves as a parameter in the mechanical equations.

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$^2$Not to be confused with the Gibbs surface energy which is also often indicated by $\gamma$. 

2.3 Basic equations

Let $\mathbf{x}$ denote the positions of the material point in the reference state at $T = T_r$, and $\mathbf{u}(\mathbf{x}, T)$ the displacements of these points. The deformation gradient $\mathbf{F}$ is:

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{x} \mathbf{u}$$

with $F_{kl} = \delta_{kl} + u_{k,l}$, where $\delta_{kl}$ is the Kronecker delta symbol and $u_{k,l} = \partial u_k / \partial x_l$.

The finite strain tensor or Lagrange strain tensor $\mathbf{D}$ is a function of $\mathbf{F}$. Specifically:

$$\mathbf{D} = \frac{1}{2} (\mathbf{F}^\epsilon \mathbf{F} - \mathbf{I}).$$

The first Piola-Kirchhoff stress tensor $\mathbf{S}$ is:

$$\mathbf{S} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{D}}$$

with $\rho$ the density in the reference state. The basic mechanics equations [5] are as follows:

**Unloaded state**

- $\text{div}\mathbf{S} = 0$ (in each of the materials),
- $\mathbf{S} \cdot \mathbf{n} = 0$ $\mathbf{n}$ = outward normal;  (at the nonbonded surfaces),
- Continuity of traction across the interface,
- Continuity of the displacements across the interface.

**Loaded state**

- $\text{div}\mathbf{S} = 0$ (in each of the materials),
- $\mathbf{S} \cdot \mathbf{n} = t_i$ at $S_i$,
- $= 0$ at $\partial \mathcal{B}_1 \setminus (\Gamma \cup S_1 \cup S_2)$,
- Continuity of traction across the interface,
- Continuity of the displacements across the interface.

2.4 Variational principles

For both the unloaded and loaded system the basic equations can be derived from a variational principle [9]. In view of the aim of this report it is illuminating to show this because at a later stage this variational principle is used in a slightly different manner than is usually the case.

The Helmholtz free energy density is a function of $\mathbf{D}$. Since $\mathbf{D}$ is a function of $\mathbf{F}$ and $\mathbf{F}$ in turn a function of $\nabla \mathbf{x} \mathbf{u}$ we can also say that the Helmholtz free
2.4. VARIATIONAL PRINCIPLES

energy density is a function of $\nabla_x u$. From eq. (2.1) we see that the contribution $F_b$ of the bulk of the system to the total free energy $F_s$ is:

$$F_b[\nabla_x u] = \sum_{i=1}^2 \left[ \int_{\delta_i} \rho_i \psi_i(D_i, T) \, dV \right]$$

where the square brackets on the lefthand side of this equation express that $F_b$ is a functional of $\nabla_x u$. The displacement functions $u$ that can be used in such an expression must be admissible functions which means that they must meet certain conditions. Apart from certain smoothness properties, which are of no concern here, the most important condition is that they must be continuous across the interface because this is what a joint mechanically is about. Now, let $C$ denote the class of admissible displacements and let $u$ be any displacement function of this class. Next, let $\varepsilon f$ ($\varepsilon$ a scalar and $f$ a vector) be an admissible variation of $u$, meaning that

if $u \in C$ then also $(u + \varepsilon f) \in C$ for arbitrary value of the scalar $\varepsilon$.

Specifically this means that $f$ is also continuous across the interface. For fixed, but otherwise arbitrary, $u$ and $f$ the bulk free energy $F_b[\nabla_x u + \varepsilon \nabla_x f]$ can be considered as a function of $\varepsilon$ and expanded in a series of ascending powers of this around the value $\varepsilon = 0$. One finds

$$F_b[\nabla_x u + \varepsilon \nabla_x f] = F_b[\nabla_x u] + \varepsilon \left( \frac{dF_b}{d\varepsilon} \right)_{\varepsilon=0} + \frac{\varepsilon^2}{2} \left( \frac{d^2F_b}{d\varepsilon^2} \right)_{\varepsilon=0} + \cdots.$$ 

Using the identity:

$$\rho \frac{d\psi(\nabla_x u + \varepsilon \nabla_x f)}{d\varepsilon} = \text{div}(S \varepsilon \cdot f) - f \cdot (\text{div} S),$$

and the divergence theorem one finds (neglecting second and higher order terms):

$$F_b[\nabla_x u + \varepsilon \nabla_x f] = F_b[\nabla_x u] - \varepsilon \sum_{i=1}^2 \int_{\delta_i} f_i.(\text{div} S_i) \, dV + \varepsilon^2 \sum_{i=1}^2 \int_{\partial \delta_i} n_i \cdot S_i \cdot f_i \, dS + O(\varepsilon^3).$$

where $n_i$ is the outward normal on $\partial B_i$, and $S_i$ is the Piola-Kirchhoff stress in $B_i$. If we now require that the displacements $u_i$ in each of $B_i$ be such that:

$$\text{div} S_i = 0 \quad \text{in } B_i,$$

$$S_i \cdot n_i = 0 \quad \text{on } \partial B_i \setminus \Gamma,$$

$$S_1 \cdot n_1 + S_2 \cdot n_2 = 0 \quad \text{on } \Gamma,$$

we see from eq. (2.14) that $F_b$ is to first order invariant for the choice of the variations $\varepsilon f_i$. The eqs. (2.15) to (2.16), combined with the continuity of the
displacements $u_i$ across the interface are precisely the mechanical equations for the unloaded case as a comparison with eqs. (2.5) to (2.8) shows. That eq. (2.17) expresses the continuity of the traction becomes evident if one takes into account that on $\Gamma$ the normals have opposite direction, that is: $n_1 = -n_2$ on $\Gamma$.

Now, let $\hat{u}_i(x, T)$ denote, in each of the bodies, the displacement solution of the eqs. (2.15) to (2.17) that also meet the continuity condition at the interface. Obviously, these solutions make the bulk free energy stationary (minimum, saddle point or maximum). The same is true for the free energy of the system because $F_s = F_b +$ surface or interface terms that are independent of the displacements, so that varying the displacements, thereby keeping the temperature fixed, does not influence the surface and interface terms. So we have

Result 1

**Displacements of the unloaded, cooled system are such that the Helmholtz free energy of the system is stationary with respect to infinitesimal variations**

The nature of the stationary points depends on the properties of $\psi_i$. One would expect, however, that $F_s$ attains a minimum because this is the case for linear elastic materials in the infinitesimal strain approximation.

For the loaded system the free energy of the loading mechanism must be taken into account. Moreover, in this case we write the total displacements $u_i$ as:

$$u_i = v_i + w_i \quad (i = 1, 2)$$

were $v_i$ is the known solution of the unloaded system. Variations are again denoted by $\varepsilon f_i$. Since $u$ must be continuous across the interface and $v$ is known to have this property there, it follows that admissible functions $w$ are also continuous: $w \in C$. So, the variations $\varepsilon f$ are in fact variations of $w$. Proceeding along the same line as before, we expand the sum of $F_b$ and $F_l$ as a powerseries in $\varepsilon$:

$$F_b[\nabla x v + \nabla x w + \varepsilon \nabla x f] + F_l[w + \varepsilon f] =$$

$$F_b[\nabla x v + \nabla x w] + F_l[w] + \varepsilon \left( \frac{dF_b}{d\varepsilon} + \frac{dF_l}{d\varepsilon} \right)_{\varepsilon=0}$$

$$+ \frac{\varepsilon^2}{2} \left( \frac{d^2F_b}{d\varepsilon^2} \right)_{\varepsilon=0} + \cdots$$

Simplifying this expression, it is found:

$$F_b[\nabla x v + \nabla x w + \varepsilon \nabla x f] + F_l[w + \varepsilon f] = F_b[\nabla x v + \nabla x w] + F_l[w]$$

$$- \varepsilon \sum_{i=1}^{2} \int_{B_i} f_i.(\text{div}(S_i)) dV + \varepsilon \sum_{i=1}^{2} \int_{\partial B_i} n_i.S_i \cdot f_i dS$$

$$- \varepsilon \sum_{i=1}^{2} \int_{\Gamma} t_i.f_i dS + O(\varepsilon^2). \quad (2.18)$$
Note that the tensors $S_i$ must be evaluated using the displacement $v_i + w_i$. As before, it is found that $F_b + F_1$ is invariant for arbitrary variations, again to first order, if the functions $w_i$ are such that the sum $v_i + w_i$ is a solution of the following equations.

\[
\begin{align*}
\text{div} S_i &= 0 \quad \text{in} \quad B_i, \\
S_i &= t_i \quad \text{on} \quad \mathcal{S}_i, \\
S_i &= 0 \quad \text{on} \quad \partial B_i \setminus (\Gamma \cup \mathcal{S}_1 \cup \mathcal{S}_2), \\
S_1.n_1 + S_2.n_2 &= 0 \quad \text{on} \quad \Gamma.
\end{align*}
\]  

(2.19) \quad (2.20) \quad (2.21) \quad (2.22)

Comparing these equations with the eqs. (2.9) to (2.12) it is found again that the displacement solutions for the loaded and cooled state make the sum of bulk free energy and the free energy of the loading mechanism stationary.

Result 2

"The additional displacements $w_i$ due to the loading are such that the free energy of the system plus the loading mechanism is stationary with respect to infinitesimal variations."

2.5 Helmholtz free energy

2.5.1 General part

So far no particular choice of the free energy density $\psi$ of the materials has been made except that it depends on the finite strain tensor $D$ and the temperature $T$. Now we assume that for temperatures below $T_r$ the materials are perfectly elastic materials [10]. So, they have the property that the material behaviour is independent of the observer [5] and that the Clausius-Duhem inequality is obeyed.

It was stated before that the whole system is stress free at the reference temperature $T_r$ and that therefore this state may serve as a stress free reference state for the whole system. However, the materials as such, that is to if they are not bonded together, have a stress free state at all temperatures below $T_r$. So, one may as well use these states as reference states.

2.5.2 Expansion around reference state at $T = T_r$

The position of the material points in the the reference state is denoted by $x$ and the mass density by $\rho_i$. Expanding the free energy in a series of powers of the temperature and the strains, and retaining terms up to second order one obtains in the usual manner [7, 11] the approximation:

\[
\rho_i \psi_i \approx \rho_i \psi_i(0, T_r) - \rho_i s_i(0, T_r) \theta - \frac{\rho_i c_i \theta^2}{2T_r} + \frac{1}{2} \epsilon_i : \mathcal{L}_i : \epsilon_i - \theta M_i : \epsilon_i
\]

(2.23)
where \( s_i(0,T_r) \) is the entropy density in the reference state, \( \theta = T - T_r \) the temperature difference, \( c_i \) the specific heat capacity, \( \epsilon_i = (\nabla_x u + (\nabla_x u)^c)/2 \) the infinitesimal strain tensor, \( L_i \) the fourth order tensor of elasticity constants:

\[
L_i = L_i(T_r) = \rho_i \frac{\partial^2 \psi_i(0,T_r)}{\partial D^2}
\]

and \( M_i \) the second order stress-temperature tensor:

\[
M_i = M_i(T_r) = \rho_i \frac{\partial^2 \psi_i(0,T_r)}{\partial D \partial \theta}
\]

If the material is isotropic one has:

\[
L = \lambda I + 2\mu I
\]

\[
M = \alpha(3\lambda + 2\mu) I
\]

with \( \lambda \) and \( \mu \) Lame's constants, \( \alpha \) the coefficient of thermal expansion, \( I \) the second order unit tensor and \( I \) the fourth order unit tensor.

The disadvantage of using this expansion is that the values of the elastic constants and the coefficients of thermal expansion have to be known at the reference temperature \( T_r \) and it may very well be that in practice these values are not available at this temperature.

### 2.5.3 Expansion around reference state at \( T < T_r \)

Let, as before, \( x \) and \( \rho_i \) denote the position of the material points and the mass density respectively in the reference state at \( T_r \). If, without joining them, the two bodies are cooled down to the temperature \( T \), the material points move to new positions \( y \) and these positions may act as new material coordinates. As the bodies are homogeneous and the temperature uniform, this movement is a homogeneous function of the old positions \( x \). Assuming that at the temperature \( T = T_r \) the contact surface \( \Gamma \) passes through the origin the new material positions can be written as:

\[
y_i(x_i, \theta) = Q_i(\theta).\left(I + A_i(\theta)\right).x_i
\]

in which \( Q_i \) are rotation tensors that are determined later on and the tensors \( A_i \) determine the pure shape changes of the bodies. Geometrically, the mappings (2.24) consist of a pure shape change followed by a rigid rotation around the origin. Because: \( y_i(x_i, 0) = x_i \) it is required that \( Q_i(0) = I \) and \( A_i(0) = 0 \). The matrices \( A_i \) depend not only on the temperature but also on the material properties of the two bodies, and the sum \( I + A_i \) is — for physical reasons— regular for all \( \theta \).

Setting:

\[
G_i = \nabla_x y_i = Q_i.(I + A_i)
\]
the new mass density $\bar{\rho}_i$ becomes

$$\bar{\rho}_i = \frac{\rho_i}{\det G_i} = \frac{\rho_i}{\det(I + A_i)}$$

$$= \rho_i \det((I + A_i))^{-1}.$$  \hfill (2.25)

Because of application of the continuity constraint at the joint or because of an external loading the points $y$ undergo a displacement $\bar{u}(y)$, leading to the (infinitesimal) strains $\varepsilon = (\nabla_y \bar{u} + (\nabla_y \bar{u})^\varepsilon)/2$. An expansion of the free energy density into a power series involves now only the strains. Up to and including second order terms one obtains

$$\bar{\rho}_i \psi_i \approx \bar{\rho}_i \psi_i(0, T) + \frac{1}{2} \bar{\varepsilon}_i : \bar{\mathcal{C}}_i(T) : \bar{\varepsilon}_i.$$  \hfill (2.26)

Here again use was made of the fact that $\partial \psi / \partial \varepsilon$ vanishes in the reference state. The fourth order tensor of elasticity constants $\mathcal{C}_i$ is now a function of the operating temperature $T$.

Note that when calculating the total bulk free energy the integrations run over the transformed bodies:

$$\bar{\mathcal{B}}_i(\theta) = y_i(\mathcal{B}_i, \theta).$$

Because not only the bodies transform but also the surfaces, the following notation is used:

$$\partial \bar{\mathcal{B}}_i(\theta) = y_i(\partial \mathcal{B}_i, \theta),$$

$$\bar{\Gamma}_i(\theta) = y_i(\Gamma, \theta).$$

Generally, $\bar{\Gamma}_1(\theta) \neq \bar{\Gamma}_2(\theta)$ but $\bar{\Gamma}_1(0) = \bar{\Gamma}_2(0) = \Gamma$.

Let the points of the contact zone (at $T = T_r$) be denoted by $\xi$. Under the mappings (2.24) these points map onto

$$\Xi_i(\xi, \theta) = y_i(\xi, \theta) \quad \text{with} \quad \Xi \in \bar{\Gamma}_i.$$  \hfill (2.27)

The constraint that a joint be present is now:

$$\Xi_1 + \bar{u}_1(\Xi_1) = \Xi_2 + \bar{u}_2(\Xi_2),$$

with $\Xi_i = y_i(\xi, \theta)$ as given by eq. (2.24).
Chapter 3

The elastic energy of an externally stressed bimaterial body

3.1 Free energy expanded around state at $T_r$

3.1.1 General part

We use the results of section 2.4 (variational principles), the small strain approximation of the free energy density (expansion around the reference state at $T_r$) in the material description. During the cooling stage we have to minimize the free energy of the system or, equivalently, the free energy of the bulk:

$$F_b = \sum_{i=1}^{2} \int \left( \frac{1}{2} \varepsilon_i : \mathbf{L}_i : \varepsilon_i - \theta \mathbf{M}_i : \varepsilon_i \right) dV$$

(3.1)

where $\varepsilon_i = \{\nabla \varepsilon u_i + (\nabla \varepsilon u_i)^c\}/2$ is the small strain tensor, $\mathbf{L}_i$ the fourth order tensor of elasticity coefficients, $\theta = T - T_r$ and $\mathbf{M}_i$ the stress-temperature tensor. The class of admissible displacements should be continuous across the interface. Alternatively, this condition is posed as a constraint so that by using the method of Lagrange multipliers one can also look for stationary points of

$$\hat{F}_b = F_b + \int_{\Gamma} \lambda. (u_1 - u_2) dS$$

Now we have freed the displacements of the condition that they must be continuous by the introduction of the three Lagrange parameters (which should also be varied) that constitute the components of $\lambda$. Let $\varepsilon f_i$ and $\varepsilon \mu$ denote variations of the displacements and the Lagrange parameters respectively. Again expanding $\hat{F}_b$ in a series of powers of $\varepsilon$, the first variation $\delta \hat{F}_b$ of $\hat{F}_b$ which is the term linear
in \( \varepsilon \), is found to be:

\[
\delta F_b = -\varepsilon \sum_{i=1}^{2} \int_{\Omega_i} f_i \cdot \text{div} \sigma_i \, dV + \varepsilon \sum_{i=1}^{2} \int_{\partial \Omega_i} f_i \cdot \sigma_i \cdot n_i \, dS + \varepsilon \int_{\Gamma} \left( \mu_1 (u_1 - u_2) + \lambda (f_1 - f_2) \right) \, dS
\]

where \( n_i \) is the outward normal on \( \Omega_i \) and

\[
\sigma_i = \mathcal{L}_i : \varepsilon_i - \partial M_i
\]

(3.2)

Requiring that the first variation \( \delta F_b \) vanishes for arbitrary choice of \( F_i \) and \( \mu \), the following equations are obtained.

\[
\text{div} \sigma_i = 0 \quad \text{in } \Omega_i
\]

(3.3)

\[
\sigma_i = 0 \quad \text{on } \partial \Omega_i \setminus \Gamma
\]

(3.4)

\[
u_i = u \quad \text{on } \Gamma
\]

(3.5)

\[
\sigma_1 \cdot n_1 = -\lambda \quad \text{on } \Gamma
\]

(3.6)

\[
\sigma_2 \cdot n_2 = \lambda \quad \text{on } \Gamma
\]

(3.7)

Note that the Lagrange multiplier \( \lambda \) is the traction on the interface. On \( \Gamma \) the normals \( n_1 \) and \( n_2 \) have opposite direction. See figure 3.1. This fact can be used to eliminate the \( \lambda \) from the problem because upon setting \( m = -n_1 = n_2 \) the traction conditions 3.6 and 3.7 can be combined to:

\[
\sigma_1 \cdot m = \sigma_2 \cdot m \quad \text{on } \Gamma
\]

which means that the traction should be continuous across the interface.

The variational problem treated before can be formulated in a different, but equivalent manner. To do so we write

\[
u_i = r_i + v_i
\]
where the \( r_i' \)'s are known functions. Substitution of this expression in eq. (3.1) one finds:

\[
F_b = \sum_{i=1}^{2} \left( \frac{1}{2} \epsilon \langle \nabla_x r_i \rangle : \mathcal{L}_i : \epsilon \langle \nabla_x r_i \rangle - \theta M_i : \epsilon \langle \nabla_x r_i \rangle \right) dV
\]

\[
+ \sum_{i=1}^{2} \epsilon \langle \nabla_x v_i \rangle : \left( \mathcal{L}_i : \epsilon \langle \nabla_x r_i \rangle - \theta M_i \right) dV
\]

\[
+ \sum_{i=1}^{2} \frac{1}{2} \epsilon \langle \nabla_x v_i \rangle : \mathcal{L}_i : \epsilon \langle \nabla_x v_i \rangle dV
\]

(3.8)

Instead of \( u_i \) the functions to be determined are now the \( v_i \)'s. The requirement that the displacements be continuous across the interface becomes a constraint on the functions \( v_i \):

\[
v_1 - v_2 = r_2 - r_1
\]

(3.9)

The idea is to minimize expression (3.8) by varying \( v_i \) subject to the constraint (3.9).

Since the first term in the right hand side of (3.8) does not depend on \( v_i \) it does not play a role in the variational process. The second term can be made to vanish identically if for the functions \( r_i \) the solutions of the equations

\[
\mathcal{L}_i : \epsilon \langle \nabla_x r_i \rangle - \theta M_i = 0
\]

are used. These solutions always exist and they are of the form:

\[
r_i = \theta (\mathcal{L}_i^{-1} : M_i)(x - x_{0i}) + a_i(x)
\]

(3.10)

where \( x_{0i} \) is an arbitrary constant vector and \( a_i \) an arbitrary strainless rigid displacement; that is, a displacement field with

\[
\nabla_x a_i + (\nabla_x a_i)^c = 0
\]

Using the result (3.11) yields

\[
\epsilon \langle \nabla_x r_i \rangle : \mathcal{L}_i : \epsilon \langle \nabla_x r_i \rangle = \theta^2 M_i : \mathcal{L}_i^{-1} : M_i
\]

So, the part \( F_{br} \) of \( F_b \) that depends solely on \( r_i \) is:

\[
F_{br} = \sum_{i=1}^{2} \left( \frac{1}{2} \epsilon \langle \nabla_x r_i \rangle : \mathcal{L}_i : \epsilon \langle \nabla_x r_i \rangle - \theta M_i : \epsilon \langle \nabla_x r_i \rangle \right) dV
\]

\[
= -\frac{1}{2} \theta^2 \sum_{i=1}^{2} \left( V_i M_i : \mathcal{L}_i^{-1} : M_i \right)
\]

(3.12)
in which $V_1$ and $V_2$ are the volumes of the bodies at $T = T_r$. Since the matrix of elasticity constants $L_i$ is positive, the term $F_{br}$ is always negative. For isotropic materials one finds:

$$M_i = 3 \alpha_i K_i$$

$$L_i^{-1} : M_i = \alpha_i I$$

$$K_i = \lambda_i + \frac{2}{3} \mu_i$$

$$F_{br} = -\frac{3}{2} \theta^2 (\alpha_2^2 K_1 V_1 + \alpha_2^2 K_2 V_2)$$

(3.13)

with $\lambda_i, \mu_i$ Lame's constants, $K_i$ moduli of compression and $\alpha_i$ the coefficients of thermal expansion of the two materials.

Summarizing the results obtained until now it is found that:

$$F_b = F_{br} + \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \epsilon(\nabla_x v_i) : L_i : \epsilon(\nabla_x v_i) dV$$

(3.14)

whereby in each material $v_i$ is the difference between the actual displacement $u_i$ and the known displacement $r_i$ (defined by eq. (3.11)) and $F_{br}$ (see eq. (3.12)) independent of $v_i$. Continuity of the displacement across the interface is accounted for by requiring that:

$$v_1 - v_2 = r_2 - r_1$$

$$= \theta \left( L_2^{-1} : M_2 - L_1^{-1} : M_1 \right) \xi$$

$$+ \alpha_2(\xi) - \alpha_1(\xi) + \theta k \quad \forall \xi \in \Gamma$$

(3.15)

with

$$k = L_1^{-1} : M_1 x_{01} - L_2^{-1} : M_2 x_{02}.$$

### 3.1.2 Elastic energy due to thermal mismatch

In chapter 2 (Result 1) it was concluded that the actual displacements $u_i$ of the cooled but unloaded system are such that the Helmholtz free energy $F_s$ of the system is stationary with respect to infinitesimal variations that are continuous across the interface. Using the results of the previous section and eq. (2.1) one finds:

$$F_s = F_{bu} + \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \epsilon(\nabla_x v_i) : L_i : \epsilon(\nabla_x v_i) dV$$

$$+ \text{functions of temperature.}$$

For the unloaded system the solutions $v_i$ of the variational problem $F_s \to \text{stationary}$
subject to the constraint (3.15) are such that with the definition of

\[ \sigma_i = \varepsilon(\nabla_x v_i) \quad \text{(thermal stress)} \]  

(3.16)

one finds:

\[ \text{div} \sigma_i = 0 \quad \text{in } B_i, \quad (3.17) \]

\[ \sigma_i \cdot n_i = 0 \quad \text{on } \partial B_i \setminus \Gamma, \quad (3.18) \]

\[ \sigma_1 \cdot m = \sigma_2 \cdot m \quad \text{on } \Gamma \text{ (continuity of traction),} \quad (3.19) \]

\[ v_1 - v_2 = r_2 - r_1 \quad \text{(continuity of displacements).} \quad (3.20) \]

After setting \( u_i = r_i + v_i \), it is readily verified that the eqs. (3.16 to 3.20) are
the same as the eqs. (3.2) to (3.7). Quite as it should be.

The functional

\[ \frac{1}{2} \sum_{i=1}^{2} \int_{B_i} \varepsilon(\nabla_x v_i) : \mathcal{L}_i : \varepsilon(\nabla_x v_i) \, dV \quad (3.21) \]

is convex because the elasticity tensors \( \mathcal{L}_i \) are positive. So, the solutions \( v_i \) minimize the free energy \( F_s \). However, the minimum is a constrained one, meaning that this minimum is larger than the minimum found if no constraint is applied. Since the thermal residual stresses occur only as a result of the constraint it is logical to consider the rise of the bulk free energy due to constraint application as the elastic energy due to thermal mismatch. If \( F_b \) is minimized without any constraint one finds again eq. (3.17), and that eq. (3.18) should hold on the complete boundary \( \partial B_i \) of \( B_i \). Although the solutions \( v_i \) in this case are not unique, they all have the property the functional (3.21) attains its absolute minimum value which is zero. Therefore, for the unconstrained case one has

\[ F_b = F_{br} \]

whereas for the constrained case it is found that

\[ F_b = F_{br} + \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \varepsilon(\nabla_x \hat{v}_i) : \mathcal{L}_i : \varepsilon(\nabla_x \hat{v}_i) \, dV > F_{br}. \]

Here, \( \hat{v}_i \) is the solution of the eqs. (3.16) to (3.20).

Consequently we have:

Result 3

The elastic energy due to thermal mismatch is

\[ U_{th} = \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \varepsilon(\nabla_x \hat{v}_i) : \mathcal{L}_i : \varepsilon(\nabla_x \hat{v}_i) \, dV \quad (3.22) \]
3.1. **EXPANSION AROUND STATE AT** $T_R$

The thermal stress $\hat{\sigma}_i$ is:

$$\hat{\sigma}_i = \mathcal{L}_i : \epsilon(\nabla_x \hat{v}_i)$$  \hspace{1cm} (3.23)

So, eq. (3.22) can also be written as:

$$U_{th} = \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \hat{\sigma}_i : \mathcal{L}_i^{-1} : \hat{\sigma}_i \, dV$$  \hspace{1cm} (3.24)

To conclude this section, consider again the problem about the definition of $U_{th}$ mentioned in chapter 1. Although the state at $T = T_R$ is a valid and suitable reference state, it is not the strain associated with the displacements with respect to this state that is of interest. It is rather the strain $\epsilon(\nabla_x \hat{v}_i)$ resulting from the constraint that is of importance. So, one finds

$$\epsilon(\nabla_x \hat{v}_i) \quad \text{and conclude that the definition of } U_{th} \text{ according to (3.22) or (3.24) is in agreement with the basic definition of elastic energy.}

3.1.3 **Elastic energy due to the load**

Application of the traction on the parts $S_i$ of the boundaries $\partial B_i$ gives rise to an additional displacement $w_i$ which is to be superposed on the displacements $\hat{u}_i = \mathbf{r}_i + \hat{v}_i$ already present. According to Result 2 of chapter 2 it is now $F_s + F_1$ that has to be minimized. Again it does not matter if we take $F_b$ instead of $F_s$ since the difference between these to quantities does not depend on the displacements. As before:

$$F_b = \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \epsilon(\nabla_x u_i) : \mathcal{L}_i : \epsilon(\nabla_x u_i) - \theta M_i : \epsilon(\nabla_x u_i) \, dV$$  \hspace{1cm} (3.25)

For the contribution $F_1$ of the loading one has:

$$F_1 = \sum_{i=1}^{2} \int_{T} \mathbf{t}_i \cdot (u_i - \hat{u}_i) \, dS$$  \hspace{1cm} (3.26)

expressing the fact that the loading is applied after the cooling period has been completed, that is, after the displacements $\hat{u}_i$ occurred. Continuity of the displacements across the interface is again enforced by means of a constraint. Now:

$$u_i = \hat{u}_i + w_i = \mathbf{r}_i + \hat{v}_i + w_i$$  \hspace{1cm} (3.27)
with \( \mathbf{r}_i \) and \( \mathbf{v}_i \), the known displacements of the previous sections. Varying the displacements \( \mathbf{u}_i \) is accomplished by varying \( \mathbf{w}_i \). Substitution of the expression (3.27) in the expression for \( F_b \) (eq. (3.25)) yields ultimately:

\[
F_b = F_{br} + U_{th} + \sum_{i=1}^{2} \int \frac{1}{2} \epsilon(\nabla_x \mathbf{w}_i) : \mathbf{L}_i : \epsilon(\nabla_x \mathbf{w}_i) \, dV
\]

\[+ \frac{2}{6} \int \epsilon(\nabla_x \mathbf{w}_i) : \mathbf{L}_i : \epsilon(\nabla_x \mathbf{v}_i) \, dV. \quad (3.28)\]

In deriving this expression the equations (3.10), (3.12) and (3.22) were used.

The first two terms of \( F_b \) are independent of the unknowns \( \mathbf{w}_i \). For that reason they do not play a role in the variational process. The third term is obviously the elastic energy associated with the deformations due to the applied load:

\[
U_{el} = \sum_{i=1}^{2} \int \frac{1}{2} \epsilon(\nabla_x \mathbf{w}_i) : \mathbf{L}_i : \epsilon(\nabla_x \mathbf{w}_i) \, dV. \quad (3.29)\]

Because both \( \mathbf{w}_i \) and \( \mathbf{v}_i \) determine the fourth term, this term is an interactional elastic energy term:

\[
U_{int} = \sum_{i=1}^{2} \int \epsilon(\nabla_x \mathbf{w}_i) : \mathbf{L}_i : \epsilon(\nabla_x \mathbf{v}_i) \, dV. \quad (3.30)\]

Using the definition of \( \mathbf{t}_i \) (eq. (3.23)) this term can be written as:

\[
U_{int} = \sum_{i=1}^{2} \int \epsilon(\nabla_x \mathbf{w}_i) : \mathbf{L}_i : \epsilon(\nabla_x \mathbf{t}_i) \, dV. \quad (3.31)\]

Because \( \mathbf{w}_i = \mathbf{u}_i - \mathbf{\hat{u}}_i \), the term \( F_i \) is:

\[
F_i = \int_{\Gamma} t_i \cdot \mathbf{w}_i \, dS. \quad (3.31)\]

Because \( \mathbf{\hat{u}}_1 = \mathbf{\hat{u}}_2 \) on \( \Gamma \), the continuity of the displacements is guaranteed if the constraint:

\[
\mathbf{w}_1 - \mathbf{w}_2 = 0 \quad \text{on} \ \Gamma \quad (3.32)\]

is applied. Defining:

\[ \mathbf{\sigma}_i^* = \mathbf{L}_i : \epsilon(\nabla_x \mathbf{w}_i) , \]

the variational problem

\[ F_b + F_i \rightarrow \text{stationary} \]
3.1. EXPANSION AROUND STATE AT $T_R$

with the constraint (3.32) leads to the equations:

$$\text{div}(\sigma_i^* + \hat{\sigma}_i) = 0 \text{ on } B_i,$$  \hspace{1cm} (3.33)

$$(\sigma_i^* + \hat{\sigma}_i).n_i = t_i \text{ on } S_i,$$ \hspace{1cm} (3.34)

$$(\sigma_i^* + \hat{\sigma}_1).n_i = 0 \text{ on } \partial B_i \setminus (\Gamma \cup S_1 \cup S_2),$$ \hspace{1cm} (3.35)

$$(\sigma_i^* + \hat{\sigma}_1).m = (\sigma_2^* + \hat{\sigma}_2).m \text{ on } \Gamma,$$ \hspace{1cm} (3.36)

$$w_1 = w_2 \text{ on } \Gamma.$$ \hspace{1cm} (3.37)

Previously it was found that (eqs. (3.17) to (3.20)) it was found that:

$$\text{div} \hat{\sigma}_i = 0 \text{ on } B_i,$$

$$\hat{\sigma}_i.n_i = 0 \text{ on } \partial B_i \setminus \Gamma \text{ and therefore also on } S_i,$$

$$\hat{\sigma}_1.m = \hat{\sigma}_2.m \text{ on } \Gamma.$$

Therefore the displacement $w_i^*$ and the associated stress $\sigma_i^*$ are solutions of the following equations.

$$\text{div} \sigma_i^* = 0, \text{ on } B_i,$$ \hspace{1cm} (3.38)

$$\sigma_i^*.n_i = t_i \text{ on } S_i,$$ \hspace{1cm} (3.39)

$$\sigma_i^*.n_i = 0 \text{ on } \partial B_i \setminus (\Gamma \cup S_1 \cup S_2),$$ \hspace{1cm} (3.40)

$$\sigma_i^*.m = \sigma_2^*.m \text{ on } \Gamma,$$ \hspace{1cm} (3.41)

$$w_i^* = w_2^* \text{ on } \Gamma.$$ \hspace{1cm} (3.42)

The value of the elastic energy $U_{el}$ due to the applied loading can be found by evaluating:

$$U_{el} = \sum_{i=1}^{2} \frac{1}{2} \int_{\Gamma_i} \epsilon((\nabla_x w_i^*) : \sigma_i^*) \, dV.$$ \hspace{1cm} (3.43)

That this expression is indeed the elastic energy we are looking for in this section can be motivated as follows. Take $\theta = 0$. Then $r_i = 0$, $\dot{v}_i = 0$ and $\hat{\sigma}_i = 0$. It then follows that $U_{el}$ is the only term that contributes to $F_b$.

3.1.4 Elastic interaction energy

The interaction energy is be found by substitution of $w_i = w_i^*$ in the expression for $U_{int}$ as defined in the previous section.

$$U_{int} = \sum_{i=1}^{2} \int_{\Gamma_i} \epsilon((\nabla_x w_i^*) : \hat{\sigma}_i) \, dV.$$ \hspace{1cm} (3.44)

The integrands can be written as:

$$\epsilon((\nabla_x w^*) : \hat{\sigma}) = w^*_k \hat{\sigma}_{kl}$$
where the Einstein summation convention is implied, the symmetry of the Cauchy stress tensor has been used and — for the sake of clarity of notation — the material identifying subscript has been dropped. Because the divergence of $\mathbf{\sigma}$ vanishes in the interior of each of the bodies it is found that:

$$w_{ki}^* \mathbf{\sigma}_{kl} = (w_k^* \mathbf{\sigma}_{kl})_l - w_k^* \mathbf{\sigma}_{kl,l},$$
$$= \text{div}(\mathbf{\sigma}^*, w^*) - w^* \cdot \text{div} \mathbf{\sigma},$$
$$= \text{div}(\dot{\mathbf{\sigma}}^*, w^*)$$

Upon integrating the last equation over the volumes, summing over the materials and using the divergence theorem one finds:

$$U_{\text{int}} = \sum_{i=1}^{2} \int_{B_i} w_i^* \cdot \mathbf{\sigma}_i \cdot \mathbf{n}_i \, dS,$$

where, as before, $\mathbf{n}_i$ is the outward normal on $B_i$. Because $\mathbf{\sigma}_i \cdot \mathbf{n}_i = 0$ on $\partial B_i \setminus \Gamma$ it follows:

$$U_{\text{int}} = \sum_{i=1}^{2} \int_{B_i} w_i^* \cdot \mathbf{\sigma}_i \cdot \mathbf{n}_i \, dS. \quad (3.45)$$

Because of the continuity of the tractions $\mathbf{\sigma}_1 \cdot \mathbf{m} = \mathbf{\sigma}_2 \cdot \mathbf{m}$ and the displacements across the interface ($w^* = w_1^* = w_2^*$ on $\Gamma$), eq. (3.45) can be written as:

$$U_{\text{int}} = \int_{\Gamma} w^* \cdot (-\mathbf{\sigma}_1 \cdot \mathbf{m} + \mathbf{\sigma}_2 \cdot \mathbf{m}) \, dS = 0.$$

Consequently it follows that globally the interaction energy vanishes. However, locally the interaction energy density need not be zero.

Finally, note that instead of eq. (3.44) one can also use:

$$U_{\text{int}} = \sum_{i=1}^{2} \int_{B_i} \epsilon(\nabla \cdot \mathbf{v}_i) : \mathbf{\sigma}_i^* \, dV$$

Following a similar derivation as above, thereby using the equilibrium equations, boundary and interface conditions for $\mathbf{\sigma}_i^*$ (eq. (3.38) to (3.41)) it follows that:

$$U_{\text{int}} = \sum_{i=1}^{2} \int_{\partial S_i} t_i \, dS + \int_{\Gamma} (\mathbf{v}_2 \cdot \mathbf{\sigma}_2^* \cdot \mathbf{m} - \mathbf{v}_1 \cdot \mathbf{\sigma}_1^* \cdot \mathbf{m}) \, dS = 0.$$

Using:

$$t^* = \mathbf{\sigma}_1 \cdot \mathbf{m} = \mathbf{\sigma}_2 \cdot \mathbf{m} \quad \text{definition of interface traction}$$

$$\mathbf{v}_2 - \mathbf{v}_1 = \mathbf{r}_1 - \mathbf{r}_2 \quad \text{on } \Gamma \quad \text{(interface constraint)},$$

and the fact that $U_{\text{int}} = 0$ one then finds that:

$$\sum_{i=1}^{2} \int_{\partial S_i} t_i \, dS = \int_{\Gamma} t^* \cdot (\mathbf{r}_2 - \mathbf{r}_1) \, dS.$$

The physical interpretation of this result is not clear but it is needed in the next chapter.
3.2 Expansion of the free energy around the reference state at the operating temperature

3.2.1 General part

Now the results of section 2.4 (variational principles) are used and applied to the free energy expanded around the reference state at the operating temperature $T$. Integrating eq. (2.26) over the volumes $B_i(\theta)$ and summing over $i$ yields:

$$F_b = \tilde{F}_{bu} + \frac{2}{\beta} \int_{\partial B(\theta)} \frac{1}{2} \epsilon_i : \tilde{\mathcal{L}}_i(T) : \tilde{\epsilon}_i d\tilde{V}$$

(3.46)

where \(\tilde{\epsilon}_i = (\nabla_y u_i + (\nabla_y u_i)^c)/2\) and

$$F_{bu} = \sum_{i=1}^2 \int_{\partial B_i(\theta)} \bar{p}_i \psi_i(0,T) d\tilde{V} = \sum_{i=1}^2 m_i \psi_i(0,T) \quad (m_i: \text{mass } B_i)$$

(3.47)

$F_{bu}$ is the total free energy of the bulk of the system if the bodies were allowed to cool totally unconstrained. The presence of the joint is now enforced by means of the constraint (c.f. eqs. (2.24) and (2.24))

$$\Xi_1(\xi, \theta) + u_1(\Xi_1(\xi, \theta)) = \Xi_2(\xi, \theta) + u_2(\Xi_2(\xi, \theta)) \quad \forall \theta \quad \text{and} \quad \forall \xi \in \Gamma$$

(3.48)

3.2.2 Elastic energy due to thermal mismatch

As stated before (Result 1, chapter 2), the actual displacements $u_i$ of the cooled but unloaded system are such that the Helmholtz free energy $F_s$ is stationary with respect to infinitesimal admissable variations of the displacements. Since the difference between $F_s$ and

$$\tilde{F}_b = \sum_{i=1}^2 \int_{\partial B_i(\theta)} \frac{1}{2} \epsilon(\nabla_y u_i) : \mathcal{L}_i(\epsilon) (\nabla_y u_i) d\tilde{V}$$

(3.49)

do not depend on the the displacements, the variational problem is now that $\tilde{F}_b$ should be minimized subject to the constraint (3.48). Again using the method of the Lagrange multipliers, with $\omega$ as the vector of Lagrange parameters one has to determine stationary points of

$$\tilde{F}_b = \tilde{F}_b + \int_{\Gamma} \omega(\xi).\left(\Xi_1 + u_1(\Xi_1) - \Xi_2 - u_2(\Xi_2)\right) d\tilde{S}.$$ 

Note that each $\Xi_i$ depends on $\xi$ and $\theta$ and that $\omega$, $u_1$ and $u_2$ are to be varied independently.
CHAPTER 3. THE ELASTIC ENERGY

Let $\varepsilon f_i$ and $\varepsilon \mu$ denote variations of the displacements and the Lagrange multipliers. Following the same procedure as before, the first variation $\delta \tilde{F}_b$ is found to be

$$\delta \tilde{F}_b = -\varepsilon \sum_{i=1}^{2} \int \tilde{f}_i \cdot \text{div} \tilde{\sigma}_i \, d\tilde{V} + \varepsilon \sum_{i=1}^{2} \int \tilde{f}_i \cdot \tilde{\sigma}_i \cdot \tilde{n}_i \, d\tilde{S}$$

$$+ \varepsilon \int_{\Gamma} \mu_i (\Xi_1 - \Xi_2 + u_1(\Xi_1) - u_2(\Xi_2)) \, dS$$

$$+ \varepsilon \int_{\Gamma} \omega_i (f_1(\Xi_1) - f_2(\Xi_2)) \, dS,$$

with $\tilde{n}_i$ the outward normal on $\tilde{B}_i(\theta)$ and

$$\tilde{\sigma}_i = \tilde{L}_i : \varepsilon_i. \quad (3.50)$$

Since the first variation $\delta \tilde{F}_b$ should vanish for arbitrary variations one obtains immediately:

$$\text{div} \tilde{\sigma}_i = 0 \quad \text{in } \tilde{B}_i(\theta), \quad (3.51)$$

$$\tilde{\sigma}_i \cdot \tilde{n}_i = 0 \quad \text{on } \partial \tilde{B}_i(\theta) \setminus \Gamma_i(\theta), \quad (3.52)$$

and the constraint (3.48). In addition to that one must meet the equation

$$\sum_{i=1}^{2} \int_{\Gamma_i} \tilde{f}_i \cdot \tilde{\sigma}_i \cdot \tilde{n}_i \, d\tilde{S} + \int_{\Gamma} \omega_i (f_1(\Xi_1) - f_2(\Xi_2)) \, dS = 0. \quad (3.53)$$

Physically, this equation expresses the continuity of the traction across the interface. To see that this is indeed the case, a more extensive discussion and rewriting is needed. Note that in the surface integrals over $\Gamma_i$, the integration variables are the $\Xi_i$'s. The integration variable in the last integral is, however, $\xi$. In body 1 a unique relation between $\xi$ and $\Xi_1$ exists. In body 2 a different, but still unique, relation exists between $\xi$ and $\Xi_2$. Specifically, specializing eq. (2.24) to surface points in the contact region, these relations are:

$$\Xi_i = Q_i(\theta). (I + A_i(\theta)). \xi \quad i = 1, 2$$

Using these unique relations one finds that, for $i = 1, 2$:

$$\int_{\Gamma_i(\theta)} f_i(\Xi_i). \tilde{\sigma}_i(\Xi_i) . \tilde{n}_i(\Xi_i) \, d\tilde{S} =$$

$$\int_{\Gamma} g_1(\Xi_i(\xi, \theta)) f_i(\Xi_i(\xi, \theta)) . \sigma_i(\Xi_i(\xi, \theta)) . \tilde{n}_i(\Xi_i(\xi, \theta)) \, dS \quad (3.54)$$

The scalars $g_1$ and $g_2$ provide the links between the surface elements $dS$ and $d\tilde{S}$ in each of the bodies. Consider the surface element $\Delta S = d\xi \, d\zeta$ at the interface
3.2. EXPANSION AT OPERATING TEMPERATURE

Figure 3.2: Change of the surface elements $dS$ into different $\Delta S$ in each of the bodies.

at the temperature $T_r$ (figure 3.2). Material points of body 1 of this element are mapped onto $\Delta S_1$ and material points of body 2 onto a different element $\Delta S_2$. The mapped elements differ not only in location but also in size because each of the bodies underwent an unconstrained shrinkage during the cooling process. Using $\xi = \xi e_1 + \zeta e_2$ and the fact that the tensors $Q_1$ and $Q_2$ are rotation tensors it is found that

$$\Delta S_i = \|(I + A_i).e_1\| \cdot \|(I + A_i).e_2\| \Delta S$$

Therefore

$$g_i = g_i(\theta) = \|(I + A_i).e_1\| \cdot \|(I + A_i).e_2\|$$

Since eq. (3.53) should be met for arbitrary choice of $f_i$, it follows by means of (3.54) that:

$$g_1 \bar{\sigma}_1 \cdot \bar{n}_1 = -\omega \quad \forall \xi \in \Gamma$$

(3.55)

and

$$g_2 \bar{\sigma}_2 \cdot \bar{n}_2 = \omega \quad \forall \xi \in \Gamma$$

(3.56)

Setting, as in section 3.1, $m = -\bar{n}_1 = \bar{n}_2$ these equations represent the continuity of the traction across the interface, because the eqs. (3.55) and (3.56) can be combined to

$$(g_1(\theta)\bar{\sigma}_1(\Xi_1(\xi, \theta)) - g_2(\theta)\bar{\sigma}_2(\Xi_2(\xi, \theta))).m(\xi) = 0 \quad \forall \theta \quad \forall \xi \in \Gamma.$$  

(3.57)

Now, let $\bar{\nu}_i$ and $\bar{\sigma}_i$ denote the solutions to the eqs. (3.48), (3.50), (3.51), (3.53), (3.55) and (3.56). Then, the elastic energy due to the thermal mismatch is:

$$U_{th} = \sum_{i=1}^{2} \int_{S_i(\theta)} \frac{1}{2} \epsilon (\nabla_y \bar{\nu}_i) : \tilde{\mathcal{C}}_i(T) : \epsilon (\nabla_y \bar{\nu}_i) d\tilde{V}$$

(3.58)
because if $F_b$ is minimized without any constraint the minimum is found to be zero.

### 3.2.3 Elastic energy due to the loading

Now one has to search for stationary points of the sum of $F_b$ and the potential energy $F_l$ of the loading mechanism

$$
\sum_{i=1}^{2} \int_{B_i(\theta)} \frac{1}{2} \epsilon(\nabla_y u_i) : \mathcal{L}_i(T) : \epsilon(\nabla_y \tilde{u}_i) d\tilde{V} - \sum_{i=1}^{2} \int_{S_i(\theta)} t_i.w_i d\tilde{S},
$$

(3.59)

again subject to the constraint (3.48). As previously, $w_i$ is the displacement due to the loading. Specifically

$$w_i = u_i - \tilde{v}_i
$$

(3.60)

where $\tilde{v}_i$ is the solution for the displacements of the unloaded system, so that varying $u_i$ is achieved by varying $w_i$. After substitution of eq. (3.60), and using the definition of $U_{th}$ and $\mathbf{\tilde{\sigma}_i}$ one finds

$$
\tilde{F}_b + \tilde{F}_l = U_{th} + \sum_{i=1}^{2} \int_{B_i(\theta)} \frac{1}{2} \epsilon(\nabla_y w_i) : \mathcal{L}_i(T) : \epsilon(\nabla_y w_i) d\tilde{V}
$$

$$\quad - \sum_{i=1}^{2} \int_{S_i(\theta)} t_i.w_i d\tilde{S} + \sum_{i=1}^{2} \int_{S_i(\theta)} \epsilon(\nabla_y w_i) : \mathbf{\tilde{\sigma}_i} d\tilde{V}.
$$

(3.61)

Since $\tilde{v}_i$ meets the constraint (3.48) and the sum $w_i + \tilde{v}_i$ should meet the same requirement, it follows that at the interface

$$w_1(\Xi_1(\xi, \theta)) = w_2(\Xi_2(\xi, \theta)) \quad \forall \theta \forall \xi \in \Gamma.
$$

(3.62)

The variational problem

$$\tilde{F}_b + \tilde{F}_l \rightarrow \text{stationary}
$$

with the constraint (3.62) gives the equations

$$\text{div}(\mathbf{\tilde{\sigma}_i} + \mathbf{\tilde{\sigma}_i}) = 0 \quad \text{in} \quad B_i,$n
$$

$$(\mathbf{\tilde{\sigma}_i} + \mathbf{\tilde{\sigma}_i}).\mathbf{n}_i = t_i \quad \text{on} \quad S_i,$n
$$

$$(\mathbf{\tilde{\sigma}_i} + \mathbf{\tilde{\sigma}_i}).\mathbf{n}_i = 0 \quad \text{on} \quad \partial B_i \setminus (\Gamma \cup \tilde{S}_1 \cup \tilde{S}_2),$$

$$(g_1(\mathbf{\tilde{\sigma}_1} + \mathbf{\tilde{\sigma}_1}) - g_2(\mathbf{\tilde{\sigma}_2} + \mathbf{\tilde{\sigma}_2}).\mathbf{m} = 0 \quad \forall \theta \forall \xi.$n

Here, the stress tensors $\mathbf{\tilde{\sigma}_i}$ are:

$$\mathbf{\tilde{\sigma}_i} = \mathcal{L}_i : \epsilon(\nabla_y \tilde{u}_i^*),$$
3.2. EXPANSION AT OPERATING TEMPERATURE

with \( \bar{w}^* \) the displacement solution of the variational problem. In view of the equations (3.51), (3.52) and (3.57) the final equations for \( \bar{w}^* \) and \( \bar{\sigma}^* \) are:

\[
\begin{align*}
\text{div} \bar{\sigma}^*_i &= 0 \text{ in } \bar{B}_i, \\
\bar{\sigma}^*_i \cdot \bar{n}_i &= t_i \text{ on } \bar{S}_i, \\
\bar{\sigma}^*_i \cdot \bar{n}_i &= 0 \text{ on } \partial \bar{B}_i \setminus (\Gamma \cup \bar{S}_1 \cup \bar{S}_2), \\
(g_1 \bar{\sigma}^*_1 - g_2 \bar{\sigma}^*_2) \cdot \bar{m} &= 0 \ \forall \theta \forall \xi.
\end{align*}
\]

(3.63)  (3.64)  (3.65)  (3.66)

For the bulk free energy it is found that

\[ F_b = U_{th} + U_{el} + U_{int}. \]

\( U_{th} \) is given by eq. (3.58). The elastic energy due to the loading is:

\[ U_{el} = \sum_{i=1}^{2} \int_{\bar{B}_i(\theta)} \frac{1}{2} \epsilon(\nabla_y \bar{w}^*_i) : \bar{L}_i(T) : \epsilon(\nabla_y \bar{w}^*_i) \ dV, \]

(3.67)

and the interaction energy

\[ U_{int} = \sum_{i=1}^{2} \int_{\bar{B}_i(\theta)} \epsilon(\nabla_y \bar{w}^*_i) : \bar{\sigma}^*_i \ dV. \]

(3.68)

3.2.4 Elastic interaction energy

Following the same derivation as in section 3.4 the interaction energy can be written as

\[ U_{int} = \sum_{i=1}^{2} \int_{\partial \bar{B}_i} \bar{w}^*_i \cdot \bar{\sigma}^*_i \cdot \bar{n}_i \ dS \]

As on the parts \( \partial \bar{B}_i \setminus \Gamma \) the tractions \( \bar{\sigma}^*_i \cdot \bar{n}_i \) vanish it follows that

\[ U_{int} = \sum_{i=1}^{2} \int_{\Gamma} \bar{w}^*_i \cdot \bar{\sigma}^*_i \cdot \bar{n}_i \ dS \]

(3.69)

The surface integrals in this equation can be transformed to surface integrals over \( \Gamma \) in the same manner as in the two previous subsections

\[ \int_{\Gamma} \bar{w}^*_i \cdot \bar{\sigma}^*_i \cdot \bar{n}_i \ dS = \int_{\Gamma} g_i \bar{w}^*_i \cdot \bar{\sigma}^*_i \cdot \bar{n}_i \ dS \]

(3.70)

With the aid of (3.70), (3.66) and the constraint (3.62) it then is found again that the interaction energy vanishes. As before, this is a global result. Locally, the interaction energy density need not be zero.
3.3 Conclusions of chapter 3

With reference to the aim of this report the main results can be summarized as below.

Result 4

Provided the displacements with respect to the reference state are considered as a sum of the contributions of the various causes (free shrinkage, application of the constraint, application of the loading) the various contributions to the elastic energy present in the joint can be studied in a decoupled manner.

Result 5

The elastic energy associated with the thermal mismatch has the form of a strain energy provided one takes the strains caused by the interface displacement constraint.

Result 6

The elastic energy associated with the loading has the classic form.

Result 7

The interaction energy for the system as a whole vanishes, although the interaction energy density may not be zero.

Result 8

The previous four results apply irrespective of the reference state one uses: the state at the reference temperature $T_r$, or that at the operating temperature $T < T_r$.

Result 9

The advantage of using the reference state at $T_r$ is that the constraint condition is relatively simple. The disadvantage is that all material properties have to be known at this temperature. Data for these quantities may not always be available. When using the reference state at the operating temperature $T$ the situation is reversed. Now, the interface constraint condition becomes temperature dependent which may present a problem in practice if finite element calculation are performed. On the other hand the advantage of using this state is that one can use the material properties at the operating temperature.
3.3. CONCLUSIONS OF CHAPTER 3

Result 10

The results of this chapter can easily be generalized to the situation of three or more materials instead of two. However, if several interlayers of different materials are used it might be possible that one single reference temperature $T_r$ no longer exists because it may happen that, because of widely differing plastic yield properties of the interlayer materials, in one interlayer plus the adjacent materials stress buildup already takes place whereas elsewhere the stress is still relieved by plastic flow. If this is the case one should — out of necessity — always use the operating temperature as reference state. Note that in this case each interface constraints depends on its own temperature difference.

Note

In the remaining two chapters it is always implied that the reference state at the temperature $T_r$ is used. The reason for this is that the mathematics is much simpler.
Chapter 4

Interaction between loading and the thermal stress field

4.1 Methodological considerations

The key to obtain expressions for the energy associated with the interaction between loading mechanism is to consider both the bimaterial body and the loading mechanism as one single system. See Fig. 1.1. After all, the distinction between system and environment is quite arbitrary and can be chosen at will. The (extended) system is mechanically isolated. Exchange of thermal energy with its environment is possible. Therefore it suffices to take for the environment a large heat bath of constant temperature $T_B = T$. The ballistic free energy $F_{bat}$, defined by

$$F_{bat} = E - T_B S$$

(E the internal energy and $S$ the entropy of the system), is minimal in the sense that for all possible variations of displacements and entropy the variation of the ballistic energy vanishes or is positive if the system is in a state of stable equilibrium [12]. Ericksen [12] shows that this statement is equivalent to minimizing the Helmholtz free energy of the system if the temperature of the system is equal to that of the heat bath: $T = T_B$.

Now, the total Helmholtz free energy $F$ of the system consists of a contribution $F_s$ (c.f. eq. (2.1)) of the bimaterial body and a contribution $F_{lm}$ of the loading mechanism. Part of the latter contribution was denoted by $F_l$ in the previous chapters. While the part $F_l$ was sufficient to determine the stresses and strains in the various cases, it is important here that all entries of the Helmholtz free energy be counted. What is missing, is a reference value much akin to the first three terms in the expansion (2.23). (More details are given later on.) So the total Helmholtz free energy of the system is:

$$F = F_s + F_{lm}$$
Now, assume that $F$ depends on the simultaneous occurrence of two physical mechanisms \( pm_1 \) and \( pm_2 \) and, of course the temperature. Symbolically:

$$ F = f(pm_1, pm_2, T) $$

To study the interaction between the two mechanisms one usually attempts to write:

$$ f(pm_1, pm_2, T) = f_1(pm_1, T) + f_2(pm_2, T) + f_3 $$

where \( f_1 \) and \( f_2 \) are the energies if only one of the two mechanisms \( pm_1 \) or \( pm_2 \) is active [3] [13]. The remaining part \( f_3 \) is then considered to be the interaction energy provided the splitting has been pushed far enough. Specifically, \( f_3 \) should be zero if either \( pm_1 \) or \( pm_2 \) is not present. The function \( f_1 \) and \( f_2 \) can be found by calculating the free energies for the case that each of the two mechanisms acts alone\(^1\), and defining the \( f_3 \) as the difference between \( f \) and the sum of \( f_1 \) and \( f_2 \). This method is not applicable if the free energy contains terms that are totally independent of the occurrence of \( pm_1 \) or \( pm_2 \). For example, if:

$$ f(pm_1, pm_2, T) = h(T) + g(pm_1, pm_2, T) \quad f_1(pm_1, T) = h(T) + g_1(pm_1, T) \quad f_2(pm_2, T) = h(T) + g_2(pm_2, T) $$

where \( h(T) \) is some function not depending on either \( pm_1 \) or \( pm_2 \), then the interaction energy is defined by:

$$ f_3 = g(pm_1, pm_2, T) - (g_1(pm_1, T) + g_2(pm_2, T)) $$

and not by the difference between \( f \) and the sum of \( f_1 \) and \( f_2 \).

The splitting procedure is by no means always simple because often it is necessary to actually change the nature of a subsystems to achieve that only one of the mechanisms is active. To obtain that only thermal stresses are present one can simply isolate mechanically the loading mechanism from the body. However, to ensure that no thermal stresses are present is quite another matter. Physically, thermal stresses arise only if three conditions are fulfilled simultaneously. The first is a difference between the actual temperature and a certain reference value, the second is a difference in thermal expansion properties of the materials and the third is the presence of a constraint that prevents free shrinkage (presence of the joint). Raising the temperature to the reference temperature \( T_r \) surely has the effect that thermal stresses vanish. Unfortunately, the idea is in fact to determine the interaction at temperatures below \( T_r \). So, this is not an acceptable way. Dropping the joint constraint is equally not acceptable because this significantly changes the load transfer across the joint because shear and tensile traction can

\(^1\)In the previous chapters, this procedure has been applied to the elastic energy without explicitly mentioning it.
no longer be transferred in that case. Consequently the only manner that remains is to change the properties of one or both of the materials. This can be done by choosing new stress-temperature matrices $\tilde{M}_i$ such that:

$$\mathcal{L}_1^{-1} : \tilde{M}_1 = \mathcal{L}_2^{-1} : \tilde{M}_2$$

Now, the two materials have the same thermal expansion properties whereby the original (different) elastic properties retain their actual value.

### 4.2 Analysis of the loading mechanism

To model the loading mechanism consider the situation as shown in the figure below. For convenience only tensile traction are applied by means of a series of small masses $\delta m_1, \delta m_2, \ldots$. The masses are attached to the body by means of inexten-

---

\[\text{Figure 4.1: Example of a simple loading mechanism.}\]

---

\[\text{A similar procedure is also followed in the theory of defects of solids \cite{13}.}\]
the thread loads more or less distributed to the body. Hence, the body is loaded with a surface traction \( t_i \) of magnitude:
\[
\|t_i\| \delta s^{(i)} = g \delta m_i
\]
where \( g \) is the acceleration of gravity. Assuming that the diameters of the pulleys are very small compared to the distance between the positions \( p_1, p_2, \ldots \) of the pulley centres and the positions \( a_1, a_2, \ldots \) of the corresponding attachment points, the direction of the traction is given by:
\[
t_i = \frac{p_i - a_i}{\|p_i - a_i\|}
\]
Defining \( l_i = \|p_i - a_i\| \) and the mass density \( \omega_i = \delta m_i / \delta s^{(i)} \), it is found that:
\[
t_i = \left( \frac{g \omega_i}{l_i} \right) (p_i - a_i)
\]
Denoting the positions of the masses by \( h_i \):
\[
h_i = p_i - (l_i - l_i) e_3
\]
the potential energy of the masses is:
\[
F_{\text{mass}} = - \sum_i \delta m_i g h_i \cdot e_3
\]
\( F_{\text{mass}} \) is the Helmholtz free energy of the loading mechanism:
\[
F_{\text{load}} = F_{\text{mass}} \quad (4.3)
\]
The discrete model described above can be generalized, so as to describe continuous traction systems including compressive stresses. One obtains, for each traction system (material identifying sub- or superscript not indicated):
\[
F_{\text{load}} = \int_{s} \omega(x) \left\{ p(x) \cdot e_3 - (L(x) - \|p(x) - \kappa(x)\|) \right\} dS \quad (4.4)
\]
where \( x \) labels the material points of the surface region \( s \), \( \kappa(x) \) the position of these points, \( \omega, L \) and \( p \) are given functions. Note that \( F_{\text{im}} \) is a functional of the position \( \kappa \): \( F_{\text{im}} = F_{\text{load}}[\kappa] \). As before the traction is:
\[
t(x; \kappa(x)) = \omega(x) \left( \frac{p(x) - \kappa(x)}{\|p(x) - \kappa(x)\|} \right)
\]
because if a variation \( \delta \kappa \) is performed the variation of \( F_{\text{load}} \) is to first order equal to:
\[
\delta F_{\text{load}} = - \int_{s} t \cdot \delta \kappa dS
\]
CHAPTER 4. THE INTERACTION ENERGY

Now, let \( \kappa_0 \) be the positions of the surface points in some reference state and \( \kappa = \kappa_0 + u \) the current positions. Expansion of \( F_{\text{load}} \) in a Taylor series obtains:

\[
F_{\text{load}}[\kappa_0 + u] = F_{\text{load}}[\kappa] - \int_\Sigma t(x; \kappa_0(x)).u(x)dS + \cdots
\]

The terms of second or higher order in \( u \) can be neglected if \( p(x) \) is chosen such that

\[
\|p - \kappa_0\| \gg \|u\|.
\]

From now on it is assumed that this condition is always met. Therefore, to any desired degree of accuracy one has:

\[
F_{\text{load}}[\kappa_0 + u] = F_{\text{load}}[\kappa] - \int_\Sigma t(x; \kappa_0).u(x)dS
\]

(4.6)

where, for convenience \( t(x; \kappa_0(x)) \) is denoted by \( t(x; \kappa_0) \). The second term on the righthand side of eq. (4.6) was denoted earlier by \( F_t \). So the first term is the reference value we are looking for.

The system considered in this report has two loading mechanism. For the remainder of this report they are assumed to be specified by means of suitable choices for \( L^{(i)}, \omega^{(i)} \) and \( p^{(i)} \). Moreover, given the functions \( \kappa_0^{(i)} \) and \( u_i \), the total Helmholtz free energy \( F_{\text{im}} \) is:

\[
F_{\text{im}}[\kappa_0^{(1)} + u_1, \kappa_0^{(2)} + u_2] = \sum_{i=1}^2 F_{\text{load}}^{(i)}[\kappa_0^{(i)} + u_i].
\]

The precise form for \( \kappa_0^{(i)} \) and \( u_i \) depends on the case that is considered. Using the expansion given above it also holds that:

\[
F_{\text{im}}[\kappa_0^{(1)} + u_1, \kappa_0^{(2)} + u_2] = F_{\text{im}}[\kappa_0^{(1)}, \kappa_0^{(2)}] - \sum_{i=1}^2 \int_{\Sigma_i} t_i(x; \kappa_0^{(i)}).u_i dS.
\]

4.3 Derivation of the interaction

The Helmholtz free energy of the (extended) system is the sum of the contributions of each of the subsystem \( F = F_s + F_{\text{im}} \). Collecting results obtained previously, the free energy for the joined, cooled and loaded bodies is:

\[
F_s = \Phi(T).\Gamma_s + \sum_{i=1}^2 \gamma_i(T).S_B^{(i)}
\]

\[
+ \sum_{i=1}^2 m_i \left\{ \psi_i(0, T_r) - s_i(0, T_r)\theta - \frac{c_i(0, T_r)\theta^2}{2T_r} \right\}
\]

\[
+ F_{br} + U_{th} + U_{el},
\]

(4.7)
with:
\[ S_B^{(i)} = \int_{\partial B_i \setminus S_i} dS \]
and where the expansion around the reference state at \( T_r \) is used. The expressions for \( F_{br}, U_{th} \) and \( U_{el} \) are according to eqs. (3.12), (3.22) and (3.29). To this the total free energy of the loading mechanism as derived in the previous section has to be added. To establish a base line value for \( F_{lm} \) we assume (quite arbitrarily but without loss of generality)
\[ \kappa_0^{(i)} = \kappa_r^{(i)} = x \quad \text{for} \quad x \in S_i \]

### 4.3.1 Case 1: cooled and loaded system

Any material point of \( B_i \) occupying the position \( x \) at the reference temperature \( T_r \) occupies after cooling, but before load application, the position \( x + r_i(x) + \hat{v}_i(x) \quad (x \in B_i) \). To apply the load, contact between loading mechanism and the body must be established. For the discrete model of the previous section this means that the masses should have been positioned and supported in such a way that stress-free attachment of the threads is possible. The load is then applied by removing the supports. In our discrete or continuous model this means that the reference functions \( F_{br} \) should be chosen as:
\[ F_{br} = \kappa_0^{(i)} = \kappa_r^{(i)} + r_i + \hat{v}_i \quad x \in S_i \]
where \( r_1 \) and \( r_2 \) represent the displacements due to unconstrained shrinkage (section 3.1) and \( \hat{v}_i \) those due to the constraint. The displacement functions are
\[ \mathbf{u}_i = w_i^*(x) \quad x \in S_i. \]

Define:
\[ F_0^{(i)} = \Phi(T_r, T_s + \sum_{i=1}^2 \gamma_i(T_r) S_B^{(i)} \]
\[ + \sum_{i=1}^2 m_i \left\{ \psi_i(0, T_r) - s_i(0, T_r) \theta - \frac{c_i(0, T_r) \theta^2}{2 T_r} \right\} \]
and (c.f. eq.(3.12))
\[ F_{br}^{(i)} = \frac{1}{2} \theta^2 \sum_{i=1}^2 \left( V_i M_i : L_i^{-1} : M_i \right) \]
and also
\[ F_{lm}^{(i)} = \sum_{i=1}^2 F_{load}^{(i)} [\kappa_r^{(i)} + w_i^*] \]
\[ = \sum_{i=1}^2 F_{load}^{(i)} [\kappa_r^{(i)}] - \sum_{i=1}^2 \int_{S_i} t_i (r_i + \hat{v}_i + w_i^*) dS, \]
4. CHAPTER 4. THE INTERACTION ENERGY

with

\[ t_i = t_i(x; \kappa_r^{(i)}) \quad \text{for} \quad x \in S_i. \]

The free energy of the system is:

\[ F^{(I)} = F_0^{(I)} + F_{br}^{(I)} + F_{lm}^{(I)} + U_{th} + U_{el}. \]  

(4.8)

4.3.2 Case II: cooled but unloaded system

The loading is not applied. It even need not be activated. So:

\[ \kappa_0^{(i)} = \kappa_{II}^{(i)} = \kappa_r^{(i)} \quad \text{and} \quad u_i(x) = 0 \quad \text{for} \quad x \in S_i \]

and:

\[ F_{lm}^{II} = \sum_{i=1}^2 F_{load}^{(i)}[\kappa_{r}^{(i)}] = \sum_{i=1}^2 F_{load}^{(ii)}[\kappa_r^{(i)}] \]

The contribution of the loading to the elastic energy of the body obviously vanishes: \( U_{el} = 0 \) but the contributions of \( F_0 \), \( F_{br} \) and \( U_{th} \) to the free energy of the body are the same as in the previous case. As a result one finds for the total free energy:

\[ F^{(II)} = F_0^{(I)} + F_{br}^{(I)} + F_{lm}^{(II)} + U_{th} \]  

(4.9)

4.3.3 Case III: cooled and loaded bodies with equal thermal expansion coefficients

Let \( \bar{M} \) denote the stress-temperature tensors of the materials. As stated before, equal thermal expansion properties means that

\[ \mathcal{L}_1^{-1} : \bar{M}_1 = \mathcal{L}_2^{-1} : \bar{M}_2 \]  

(4.10)

The result is that the displacements due to unconstrained cooling (see section 3.1.1) are now \( \bar{r} \). Application of the constraint does generally lead to additional displacements \( \bar{v} \). This is so because the solutions for \( \bar{r} \) contain arbitrary rigid body displacements. These displacements have to be subtracted because otherwise the joint constraint

\[ \bar{r}_1 + \bar{v}_1 = \bar{r}_2 + \bar{v}_2 \quad \forall x \in \Gamma \]

cannot be met. In fact the displacements \( \bar{v}_i \) are only rigid body displacements. Indeed, because of the equality of the thermal expansion properties, the constraint can be met solely by choosing suitable rigid body displacements for the \( \bar{v}_i \), the result being: \( \epsilon(\nabla x \bar{v}_i) = 0 \). So, the thermal residual stresses are absent and \( U_{th} = 0 \), which is as it should be. As the materials actually differ from those constituting the actual bodies, one should perhaps also allow for different values of surface energy, interface energy, bulk free energy and entropy at the temperature
4.3. **DERIVATION OF THE INTERACTION**

$T_r$. However, to remain as closely as possible to the original system it will be assumed that these quantities are the same. Specifically this means that

$$F_0^{(III)} = F_0^{(I)}.
$$

The contribution $F_{br}$, however, depends on the stress-temperature matrices $\hat{M}_i$ and this contribution is different from that of case I or case II. Therefore (c.f. eq. (3.12))

$$F_{br}^{(III)} = -\frac{1}{2} \theta^2 \sum_{i=1}^{2} (V_i \hat{M}_i : \mathcal{L}_i^{-1} : \hat{M}_i) \neq F_{br}^{(I)}.
$$

Upon setting

$$\hat{B} = \mathcal{L}_1^{-1} \hat{M}_1 = \mathcal{L}_2^{-1} \hat{M}_2,
$$

$$V_i = V_1 + V_2,
$$

$$\mathcal{L}_v = \frac{1}{V_i} \sum_{i=1}^{2} V_i \mathcal{L}_i,$$

one obtains

$$F_{br}^{(III)} = -\frac{1}{2} \theta^2 V_i \hat{B} : \mathcal{L}_v : \hat{B}.$$

To establish contact between the loading mechanism and the bodies one must take:

$$\kappa_0^{(i)} = \kappa_{III}^{(i)} = \kappa_r^{(i)} + \tilde{r}_i + \tilde{v}_i \quad x \in S_i.
$$

The elastic properties and the traction are the same as in case I. Therefore the displacements due to the loading and the associated elastic energy are also the same. Consequently the displacements $u_i$ needed to obtain a full description of the energy of the loading mechanism are

$$u_i = w_i^*.
$$

With

$$F_{lm}^{(III)} = \sum_{i=1}^{2} F_{load}^{(i)} [\kappa_0^{(i)} + w_i^*]
$$

$$= \sum_{i=1}^{2} F_{load}^{(i)} [\kappa_r^{(i)}] - \sum_{i=1}^{2} \int_{S_i} t_i (\tilde{r}_i + \tilde{v}_i + w_i^*) \, dS,
$$

$$= F_{lm}^{(II)} - \sum_{i=1}^{2} \int_{S_i} t_i (\tilde{r}_i + \tilde{v}_i + w_i^*) \, dS,
$$

the total energy it is then found to be:

$$F^{(III)} = F_0^{(I)} + F_{br}^{(III)} + F_{lm}^{(II)} + U_{el} - \sum_{i=1}^{2} \int_{S_i} t_i (\tilde{r}_i + \tilde{v}_i + w_i^*) \, dS. \quad (4.11)$$
4.3.4 Interaction energy

Comparison of $F_{im}^{(I)}$ with $F_{im}^{(II)}$ shows that

$$F_{im}^{(I)} = F_{im}^{(II)} - \frac{2}{\delta} \int_{S_i} t_i (r_i + \hat{v}_i + w_i^*) dS. \quad (4.12)$$

Upon setting $B_i = L_i^{-1} : M_i$ the term $F_{br}^{(I)}$ can be written as:

$$F_{br}^{(I)} = -\frac{1}{2} \vartheta^2 (\sum_{i=1}^{2} V_i B_i : L_i : B_i).$$

Note that:

$$F_{br}^{(I)} = F_{br}^{(III)} + F_{mm},$$

with

$$F_{mm} = \frac{1}{2} \vartheta^2 (\sum_{i=1}^{2} V_i (\hat{B} - B_i) : L_i : (\hat{B} + B_i))$$

and that this splitting is valid\(^3\) for any $\hat{B}, B_1$ and $B_2$. Whenever $\hat{B} = B_1 = B_2$ the mismatch term $F_{mm}$ vanishes. As $F_{br}^{(I)} = F_{br}^{(I)}$ the same splitting can also be used for case II.

Using the splitting given above and comparing all results obtained for the three cases (eqs. (4.8), (4.11) and (4.12)) it is concluded that

$$F_{com} = F_{com}^{(I)} + F_{br}^{(III)} + F_{im}^{(II)}$$

is common to all the energy expressions for the three considered cases. In particular:

$$F^{(I)} = F_{com} + F_{mm} + U_{th} + U_{el} - \frac{2}{\delta} \int_{S_i} t_i (r_i + \hat{v}_i + w_i^*) dS,$$

$$F^{(II)} = F_{com} + F_{mm} + U_{th},$$

$$F^{(III)} = F_{com} + U_{el} - \frac{2}{\delta} \int_{S_i} t_i (\hat{r}_i + \hat{v}_i + \hat{w}_i^*) dS.$$

Following the procedure outlined in section 4.1, the interaction energy $F_{int}$ is defined by:

$$F_{int} = (F^{(I)} - F_{com}) - [(F^{(II)} - F_{com}) + (F^{(III)} - F_{com})]$$

and the result is

$$F_{int} = -\frac{2}{\delta} \int_{S_i} t_i [(r_i + \hat{v}_i) - (\hat{r}_i + \hat{v}_i)] dS. \quad (4.13)$$

---

\(^3\)The only mathematical requirements are that the stress-temperature tensors and the elasticity tensors are symmetric which is always the case.
The total energy of the actual system can now be written as

\[ F^{(I)} = F_{\text{com}} + (F_{\text{mm}} + U_{\text{th}} + F_{\text{int}}) - \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot (\mathbf{r}_i + \mathbf{v}_i) \, dS + (U_{\text{el}} - \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot \mathbf{w}_i \, dS) \]

The expression for \( F_{\text{int}} \) derived above is not very convenient because it depends on the (fictitious) materials having \( \mathbf{B} \) as thermal expansion coefficients. Note however the \( F^{(I)} \) itself does not depend on \( \mathbf{B} \). Therefore we may choose \( \mathbf{B} = 0 \), and this choice yields a suitable interaction energy because

\[ \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot (\mathbf{r}_i + \mathbf{v}_i) \, dS = 0 \quad \text{if: } \hat{\mathbf{B}} = 0 \]

The reason for this is that in this case the displacements \( \mathbf{r}_i \) and \( \mathbf{v}_i \) are rigid body displacements. Because of the constraint that \( \mathbf{r}_i + \mathbf{v}_i \) be continuous across the interface it is then found that:

\[ \mathbf{r}_i + \mathbf{v}_i = \mathbf{h} + \mathbf{c} \times \mathbf{x} \]

Here the vectors \( \mathbf{h} \) and \( \mathbf{c} \) are arbitrary. The result is that

\[ \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot (\mathbf{r}_i + \mathbf{v}_i) \, dS = \mathbf{h} \cdot [\sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \, dS] + \mathbf{c} \cdot [\sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \times \mathbf{x} \, dS] = 0 \]

because the traction system is selfequilibrating. The resulting expression

\[ F_{\text{int}} = \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot (\mathbf{r}_i + \mathbf{v}_i) \, dS \quad (4.14) \]

can be rewritten using the results of section 3.2.2. There it was found that

\[ \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot \mathbf{v}_i \, dS = \int_{\Gamma} \mathbf{t}^{*} \cdot (\mathbf{r}_2 - \mathbf{r}_1) \, dS \quad (4.15) \]

with \( \mathbf{t}^{*} \) the interface traction due to the loading. Adding

\[ \sum_{i=1}^{2} \int_{S_i} \mathbf{t}_i \cdot \mathbf{r}_i \, dS \]

to both sides of eq. (4.15), using \( \mathbf{t}^{*} = -\mathbf{t}_1 = \mathbf{t}_2 \) on \( \Gamma \) and the fact that no traction is applied to \( \partial \mathcal{B}_i \backslash \{(S_i \cup \Gamma) \} \) it is found that:

\[ F_{\text{int}} = -\sum_{i=1}^{2} \int_{\partial \mathcal{B}_i} \mathbf{t}_i \cdot \mathbf{r}_i \, dS. \]
Finally, using the divergence theorem, the fact that in the interior of the bodies \( \text{div}\sigma_i^* = 0 \) and applying eq. (3.10) obtains:

\[
F_{\text{int}} = -\theta \sum_{i=1}^{2} \int_{\Omega_i} \sigma_i^* : B_i \, dV.
\]

Rewriting this expression yields:

\[
F_{\text{int}} = -\frac{\theta}{2} (B_1 + B_2) \left( \int_{\Omega_1} \sigma_1^* \, dV - \int_{\Omega_2} \sigma_1^* \, dV \right) - \frac{\theta}{2} (B_1 - B_2) \left( \int_{\Omega_1} \sigma_1^* \, dV - \int_{\Omega_2} \sigma_1^* \, dV \right).
\]

From this it follows:

\text{Result 11}

Whenever the thermal expansion coefficients are non-zero, the interaction energy is also non-zero.

 Nonetheless, only the term proportional to the mismatch \( B_1 - B_2 \) contributes to the energy release rate because the term proportional to \( B_1 + B_2 \) is independent of the crack length. To see that this must be the case we introduce first the vector function \( \mathbf{q} \) defined by:

\[
\mathbf{q}(x) = \sum_{n=1}^{3} q_n \mathbf{e}_i \quad \forall x
\]

with

\[
q_n = e_n.(B_1 + B_2).x.
\]

Both \( \mathbf{q} \) and the tractions are prescribed and therefore independent of crack length. Therefore:

\[
\sum_{i=1}^{2} \int_{\Sigma_i} \mathbf{q}.\mathbf{t}_i \, dS = \text{independent of crack length}.
\]

Then, using the interface continuity of \( \mathbf{q} \) and that of the interface traction caused by \( \sigma_1^* \), the divergence theorem and the equilibrium equations \( \text{div}\sigma_i^* = 0 \) yields:

\[
\sum_{i=1}^{2} \int_{\Sigma_i} \mathbf{q}.\mathbf{t}_i \, dS = \sum_{i=1}^{2} \int_{\Sigma_i} \mathbf{q}.\sigma_i^* \cdot \mathbf{n}_i \, dS = \sum_{i=1}^{2} \int_{\partial\Omega_i} \mathbf{q}.\sigma_i^* \cdot \mathbf{n}_i \, dS = \sum_{i=1}^{2} \left( B_1 + B_2 \right) : \sigma_i^* \, dV.
\]

The lefthand side of this equation is independent of the crack length. So, the same applies to the righthand side. Consequently the term in the interaction
energy that is proportional to $B_1 + B_2$ is also independent of crack length. It therefore follows:

*Result 12*

*Only the mismatch term in the interaction energy contributes to the energy release rate*

To conclude; the total energy is then:

$$F^{(l)} = F_{com} + F_{mm} + U_{th} + F_{int} + U_{el} - \sum_{i=1}^{2} \int_{S_i} t_i \cdot w^* dS$$  \hspace{1cm} (4.17)

with the interaction energy given by (4.16).
Chapter 5
Summary and conclusions

This report, the first in a series of two, deals with the question which energy expression has to be used when one wants to calculate the energy release rate of loaded bimaterial joints consisting of materials having different thermal expansion coefficients. The second report then treats the thermodynamic aspects of this problem and numerical aspect for computing the energy release rate of this system.

Experience tells us that the elastic energy associated with the residual stresses caused by thermal mismatch is sufficient to drive crack growth because sometimes joints fail mechanically during the cooling process after the manufacturing the joint. However, in recent literature [15] the opinion is expressed that thermal residual stresses cannot drive crack growth because the energy associated with these stresses is not released. This statement is totally at variance with experience.

According to Eshelby [3] an interaction energy should exist between a system of internal stresses, such as thermal residual stresses, and the loading mechanism. The problem is that the physical basis of such an interactional energy is not clear at all. Joints are usually fabricated without application of the operational loading. So, the deformations associated with the thermal mismatch occur in a situation without any (significant) load acting on the joint. For that reason it is not clear how the loading mechanism interacts, on an energetic level, with the system of residual thermal stresses. This report therefore aims at resolving the inconsistency mentioned above, and at investigating the physical basis of the interaction energy.

The system, with each of the materials having a stress free state at some reference temperature $T_r$, is modelled by treating the joint as a mechanical constraint on the displacements of material points adjacent to the interface surface. The displacements of any material point is determined by minimizing the Helmholtz free energy. The total displacement $u_i$, $(i = 1, 2)$ with respect to the reference state is

$$u_i = r_i + \dot{v}_i + w^*_i.$$
(i: a material identifying subscript). Each of the terms on the righthand side reflects a basic property of the system. The contribution \( r_i \) constitutes the free shrinkage of each of the material bodies eventually constituting the joint, that is to say the shrinkage of each of the components in case no joint nor a loading is present. If next the joint is established, still without a loading system being present, the material points undergo additional displacements denoted by \( \tilde{v}_i \). If, finally, the loading is applied the material points again displace over a certain distance. This is denoted by \( w_i^* \).

Since free shrinkage of a homogeneous material does not lead to stresses, the deformations associated with \( r_i \) do not lead to stresses. The other two displacements \( \tilde{v}_i \) and \( w_i^* \), however, do. Their respective stress tensors are denoted by \( \tilde{\sigma}_i \) and \( \sigma_i^* \). Physically these tensors are the residual thermal stresses and the stresses caused by the loading respectively.

The Helmholtz free energy of the bulk of the system was found to be:

\[
F_b = F_{br} + U_{th} + U_{el},
\]

(5.1)

with \( U_{th} \) and \( U_{el} \) the elastic strain energies caused by thermal mismatch and loading respectively. In the first term \( F_{br} \) contributions of miscellaneous origin are collected.

The term \( F_{br} \) depends on the free energy and entropy of both materials in the reference state, on the temperature, the displacements \( r_i \) and on material properties. This term does not contribute to the energy release rate because it does not depend on either the loading or the joint surface.

The contribution to the total elastic strain energy, caused by thermal mismatch is:

\[
U_{th} = \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \varepsilon(\nabla_x \tilde{v}_i) : \mathcal{L}_i : \varepsilon(\nabla_x \tilde{v}_i) dV,
\]

(5.2)

where \( \varepsilon \) is the strain tensor and \( \mathcal{L}_i \) the tensor of elastic constants. The solution \( \tilde{v}_i \) is found by minimizing the left-hand side of (5.2) subject to the constraint that \( r_i + \tilde{v}_i \) be continuous across the bonded interface. Since crack growth decreases the bonded area, the effect is that during crack growth the constraint is relaxed meaning that \( U_{th} \) is a decreasing function of crack length or at least non-increasing. Because \( U_{th} \) is a positive quadratic form and becomes zero after the bond has been broken completely it follows that in general the derivative of \( U_{th} \) to the crack length \( l \) is negative. Obviously, energy from this source may be released during the cracking process. Therefore it is concluded that the idea that \( U_{th} \) does not contribute to the energy release rate is not correct.

\[ ^1 \text{This can also be derived mathematically, but the derivation is given in the second report because there are some non-trivial mathematical technicalities involved that need to be discussed but are outside the scope of the present report.} \]
The term

\[ U_{el} = \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \epsilon(\nabla_{w_i}^* : L_i : \epsilon(\nabla_{w_i}^*) \, dV, \tag{5.3} \]

is the familiar strain energy caused by the external loading. The solutions \( w_i^* \) minimize:

\[ \sum_{i=1}^{2} \int_{B_i} \frac{1}{2} \epsilon(\nabla_{w_i}^* : L_i : \epsilon(\nabla_{w_i}^*) \, dV - \sum_{i=1}^{2} \int_{S_i} t_i \cdot w_i \, dS \tag{5.4} \]

\((t_i: \text{applied tractions})\) subject to the constraint that \( w_i \) be continuous across the interface.

Locally there is an interaction between the internal (residual) stress field \( \sigma_i \) and the field \( \sigma_i^* \) caused by the loading. Because the strain energy density is quadratic in the actual stresses, the interaction strain energy density contains cross terms. These terms can be written as:

\[ \sigma_i^* : \epsilon(\nabla_{w_i}^* : \nabla_{w_i}^*) = \sigma_i : \epsilon(\nabla_{w_i}^*). \]

The interaction strain energy density is not necessarily zero everywhere, but its integrated form —the total interaction strain energy— vanishes. Consequently, eq. (5.1), the Helmholtz free energy of the bulk does not contain an interaction term.

To answer the question about the existence of an interaction between the loading mechanism and the residual thermal stresses, the joint plus the loading mechanism should be considered as one single system as is indicated in Fig. 5.1. The presence of such an interaction can be found by meticulously comparing the Helmholtz free energy of three different systems thereby taking care that the initial state of the loading mechanism is the same in all three cases. The three different systems are chosen as below.

- System I consists of loading mechanism plus a loaded and cooled joint of materials with different thermal expansion properties.
- System II consists of loading mechanism plus a cooled but non-loaded joint of materials with different thermal expansion properties.
- System III consists of loading mechanism plus a loaded and cooled joint of materials with equal thermal expansion properties. The expansion coefficients are chosen to be zero.

System I is the one that is actually considered. Systems II and III are comparison systems. It is found that an interaction energy exists and is given by:

\[ F_{int} = -\sum_{i=1}^{2} \int_{S_i} t_i \cdot (r_i + \hat{v}_i) \, dS. \tag{5.5} \]
Writing $B_i = \mathcal{L}_i^{-1} : M_i$, this expression can also shown to be equal to:

\[
F_{\text{int}} = -\theta \sum_{i=1}^{2} B_i : \int_{\bar{B}_i} \sigma_i^* dV. \tag{5.6}
\]

To explain the physical basis of this interaction suppose, by way of example, that the loading is applied by attaching masses through strings to the joint as is shown in figure 4.1. The energy of the loading mechanism is the potential energy of the masses. Since this energy is only determined up to an arbitrary constant, the constant is fixed by assuming that the masses initially occupy a certain position which is the same in all three cases that are considered. Evidently, in case II were no load needs to be applied, the masses remain in their original position. For cases I and III, however, the masses generally have to displaced so as to attach the strings to the joint. However, as in case III the expansion coefficients are zero, the surface points of the system III do not displace after cooling down. So, any attachment point $a$ remains in its original position during the cooling period. In case of system I, these point displace from their original position $a$ to the new
position \( a + r(a) + \bar{v}(a) \). So, the masses have to be displaced differently for case I, as compared to case III, in order to attach the masses to the surface of the joint and this difference gives rise to a difference in potential energy which is the interaction energy given in eq. (5.6). It should be stressed that in the interaction term two aspects are present. This can be seen, if the expression for \( F_{\text{int}} \) is rewritten as

\[
F_{\text{int}} = -\frac{\partial}{2}(B_1 + B_2) : (\sum_{i=1}^{2} \int_{\mathcal{B}_i} \sigma_i^* dV) - \frac{\partial}{2}(B_1 - B_2) : (\int_{\mathcal{B}_1} \sigma_1^* dV - \int_{\mathcal{B}_2} \sigma_2^* dV). \tag{5.7}
\]

If \( B_1 = B_2 \) (equal but nonvanishing expansion coefficients), the interaction term is still nonzero. Consequently, there is always an interaction term whether the expansion properties are equal or not. This is the first aspect. If next thermal mismatch is introduced an additional term appears. This is the second aspect. As far as the energy release rate is concerned only the second term in (5.7) is important because the first is independent of crack length. Indeed, if \( B_1 = B_2 \) (equal but nonvanishing expansion coefficients) the displacements \( \hat{v}_i \) are either zero or rigid body displacements because the constraint is met automatically simply by lining up the (originally non-connected) bodies and connecting (stress free in this case) the necessary points adjacent to the interface, the result being that \( \hat{v}_i \) is dependent on crack length. As the traction systems \( t_i \), the displacements \( r_i \) and the surface regions \( S_i \) are independent of crack length anyway, it follows that in this case eq. (5.5) becomes independent of crack length.

According to Eshelby the interaction energy between a field of external stress, like that caused by the tractions \( t_i \), and internal inhomogeneities leading to displacements like \( \hat{v}_i \) is given by:

\[
F_{\text{int}}^{(\text{esh})} = -\sum_{i=1}^{2} \int_{S_i} t_i \cdot \hat{v}_i dS.
\]

Although one would expect this to be true because the interaction energy should be proportional to the loading and proportional to the displacements the inhomogeneities give rise to, the results obtained above show that this expectation is not correct. The reason is that for a system with different thermal expansion properties the displacements \( \hat{v}_i \) are not unique and contain arbitrary rigid body displacements which are different for each of the bodies. The sum \( r_i + \hat{v}_i \) is much more determined because the enforcing of the constraint at the joint ensures that on \( \Gamma: r_1 + \hat{v}_1 = r_2 + \hat{v}_2 \). That Eshelby's expression cannot be correct can also be shown as follows. Using eq. (4.15) one finds that

\[
F_{\text{int}}^{(\text{esh})} = -\int_{\Gamma} t^* \cdot (r_2 - r_1) dS.
\]

Both \( r_1 \) and \( r_2 \) contain arbitrary rigid body motion. These can be chosen such that on \( \Gamma: r_2 - r_1 = \theta (B_2 - B_1) \xi, \)
the result being that $F_{int}^{(e_{th})}$ turns out not to be independent of the choice of the coordinate system. For that reason $F_{int}^{(e_{th})}$ is not a proper interaction energy. The expression (5.5) derived here is indeed a proper energy describing the interaction between loading mechanism and the inhomogeneities present in the thermal expansion properties. It is proportional to the loading, proportional to temperature and its contribution to the energy release rate vanishes for all $\theta$ if no thermal mismatch is present.

Finally, the expression for the Helmholtz free energy of the extended system (bimaterial joint plus loading mechanism) on which the energy release rate has to be based is:

$$F_{tot} = F_{ref} + F_{surf} + U_{th} + U_{el} + F_{int} - \sum_{i=1}^{2} \int_{S_i} t_i \cdot w_i^* \, dS,$$

(5.8)
in which only the term $F_{ref}$ does not depend on the crack length $l$. In the term $F_{surf}$ all contributions from the surface or the interface are collected:

$$F_{surf} = \sum_{i=1}^{2} \gamma_i \cdot S_{iB} + \Phi \Gamma_s.$$

Here $\gamma_i$ is the Helmholtz surface free energy of material $i$ and $\Phi$ the Helmholtz free energy of the interface between the two materials. The terms $U_{th}$, $U_{el}$ and $F_{int}$ are as given previously in this section. Note that $F_{surf}$ contributes to the energy release rate because the interface surface $\Gamma_s$ decreases and the non-interface parts $S_{iB}^{(i)}$ of the surfaces $\partial B_i$ increase in magnitude if a crack starts to run.
Bibliography


