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A HYBRID FORMULATION FOR DETERMINING TORSION AND WARPING CONSTANTS

C. M. Menken
Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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Abstract—Theoretically, the torsion constant of a bar subjected to a torque can be calculated irrespective of its cross-sectional shape, either with a potential energy (displacement) or with a complementary energy (stress function) formulation. The potential energy formulation, however, also provides the so-called warping constant, which may be important in the case of thin-walled sections, but the complementary energy formulation does not. On the other hand, the complementary energy formulation leads to simple analytical expressions for the torsion constant of thin-walled sections, while a finite element calculation based on potential energy requires a relatively large number of elements to obtain comparable results.

Since hybrid formulations combine some potential energy features with complementary ones, a hybrid formulation was derived for the torsion problem, with the purpose of combining the advantages of both—i.e. an ultimate displacement formulation and possibly greater accuracy. Both the torsion and warping constants can be calculated consistently with this formulation.

The hybrid variational formulation that was derived, together with the general discretization procedure, can be used to formulate finite elements for calculating the torsional constants for all kinds of cross-section, whether solid or thin-walled, open or closed, symmetrical or unsymmetrical. The initial application, to rectangular solid cross-sections, resulted in an improved warping constant.

NOTATION

- \( a \): half width of rectangle
- \( A \): cross-sectional area of the bar
- \( A^e \): cross-sectional area of the \( e \)th element
- \( b \): half height of rectangle
- \( C \): constant, polar moment of inertia
- \( C_{ijkl} \): components of the elastic compliance tensor
- \( \varepsilon_{xy}, \varepsilon_{xz} \): non-zero components of the strain tensor
- \( G \): shear modulus
- \( H \): flexibility matrix; see equation (3.6)
- \( I_t \): torsion constant
- \( k \): constant in \( I_t \)
- \( k \): stiffness matrix
- \( l \): length of bar
- \( L \): shape functions for \( \psi \) on the element boundary
- \( M \): prescribed torque
- \( n_x, n_z \): components of outer normal
- \( N_\beta \): column with shape functions for \( \beta \)
- \( N_\psi \): column with shape functions for \( \psi \)
- \( N_i \): shape function for the \( i \)th coordinate
- \( p_i \): \( i \)th component of boundary traction
- \( \mathbf{p}_i \): prescribed boundary traction
- \( P \): matrix; see equations (3.2) and (3.4)
- \( \mathbf{q} \): Yamada displacement vector [10]
- \( \mathbf{q} \): column with nodal values of \( \psi \)
- \( Q \): column with generalized loads
- \( s \): coordinate along element boundary
- \( S' \): boundary of \( A' \)
- \( S'_r \): portion of the boundary of \( V' \) where the tractions are prescribed
- \( T \): matrix; see equation (3.12)
- \( \mathbf{t}_{xy}, \mathbf{t}_{xz} \): non-zero components of the stress tensor
- \( \lambda_{xy}, \lambda_{xz} \): Lagrange multipliers
- \( u, v, w \): displacements
1. INTRODUCTION

The general linear theory of the uniform torsion of prismatic bars was well formulated by the pioneers of mechanical science, de Saint-Venant [1] and Prandtl [2], according to the then main approaches to solid mechanics—an ultimate formulation in either displacements or stresses. A common feature was that a truly three-dimensional problem was split into a two-dimensional problem and a one-dimensional problem with the aid of assumptions such as rigidity of the cross-section in its plane. The two-dimensional problem includes the determination of geometric cross-sectional properties such as the de Saint-Venant torsion constant and the so-called warping constant, while the one-dimensional problem is the response problem of the remaining one-dimensional model of the bar. Analytical solutions were restricted to bars with simple cross-sections. The introduction of the computer, together with the advent of the finite element method, enabled the geometric constants to be calculated for cross-sections with complicated shapes, particularly those of extruded rods [3]. Although in analytical calculations both the displacement approach and the stress approach were used (with preference for the stress approach in the case of thin-walled sections), in computer calculations the displacement approach prevails.

The torsion problem was frequently treated in pre-computer textbooks [4–6]; rarely do publications mention finite element calculations. This is perhaps due to the fact that such calculations require the solution of the plane Laplace equation, a beloved subject of introductory books about finite element methods.

From one of the earliest publications [7], however, it appears that an accurate determination of the torsion constant requires a relatively large number of elements. The same applies to the warping constant. More striking is the observation that for some simple thin-walled cross-sections (e.g. the narrow rectangle and the slotted tube) where the analytical stress approach provides simple approximations for the de Saint-Venant torsion constant, the displacement-based finite element approach, even with 8-node isoparametric elements, requires a relatively large number of elements [3].

This prompted us to develop a hybrid torsion formulation, since hybrid formulations often perform more accurately [8]. For a consistent formulation of the torsion and warping constants, its development cannot be based on a simple transformation of the plane Laplace problem, as suggested by Martin [9], but recourse must initially be made to the full three-dimensional problem.
As far as we know, only a few hybrid torsion formulations have been published, including the following.

Yamada et al. [10] divided the bar under torsion into longitudinal prisms with a triangular cross-section; the finite elements having so-called nodes on their vertices. Their element displacement vector is

\[ \mathbf{\dot{q}} = [\theta \ w_1 \ w_2 \ w_3 \ldots]; \]

it contains, in addition to the axial displacements \( w_i \) of the nodes, the angle \( \theta \) through which the whole cross-section rotates. Since the angle of rotation \( \theta \) and the collection of axial displacements for all the elements are calculated simultaneously, the procedure does not comply with beam and bar theory, where geometric constants have to be determined separately and are subsequently used to calculate the angle of rotation \( \theta \). Moreover, an older hybrid formulation was used that required \textit{a priori} fulfillment of the dynamic boundary conditions.

In a more recent paper by Tralli [11] that is restricted to thin-walled sections, the Hellinger-Reissner principle [12] was used. The secondary warping of the individual components was taken into account by the known torsional rigidity of the plate-like components. The predominant primary torsional rigidity due to either closed parts (cells) or restrained warping is hidden in the assembled stiffness matrix. As in the aforementioned paper, the angle of rotation of the cross-section as a whole appears in the displacement vector of each individual (bar) element. Again, axial displacements and the angle of rotation are calculated simultaneously and the torsion and warping constants are not obtained explicitly. (A merit of Tralli’s work, however, is that it can deal with variations of the cross-section along the length of the beam.)

In section 2, a hybrid variational principle will be developed by combining the classical de Saint-Venant theory with an established procedure for generating variational principles. An explicit expression for the warping constant will be obtained at this stage.

Section 3 presents a general finite element discretization procedure for this variational principle, yielding the torsion constant explicitly.

A simple example is presented in order to explore the accuracy of this approach.

2. DERIVATION OF THE HYBRID FUNCTIONAL

There are several options for deriving the hybrid functional of uniform torsion.

One option is to introduce the classical assumptions of torsion theory into the general hybrid functional that was originally derived by Pian [13, 15]:

\[ \Pi_{mc} = \sum_e \left[ \int_{V_e} \frac{1}{2} C_{ijkl} I_k I_l \, dV - \int_{\partial V_e} p_i u_i \, dS + \int_{S_i} \tilde{p}_i u_i \, dS \right], \quad (2.1) \]

where \( \partial V_e \) is the entire boundary of the \( e \)th element having volume \( V_e \). In this functional, the displacements \( u_i \) only appear in the boundary integral and can be viewed as Lagrange multipliers that have been introduced exclusively on the element boundary in order to relax the continuity requirements for the inter-element tractions \( p_i \). In a number of finite element models, such as plate elements, this formulation has proved to be quite advantageous, because for the boundary displacements simple functions of the boundary coordinates can be chosen independently of the implicit internal displacements.

One objective here, however, was to calculate the warping constant \( \Gamma \) using the classical formula

\[ \Gamma = \int_A \psi^2 \, dA, \quad (2.2) \]

where \( A \) is the cross-sectional area of the bar under torsion, while the so-called warping \( \psi(y, z) \) represents the axial displacements per unit twist. The coordinates \( y \) and \( z \) are located in the cross-section. Thus, the displacements have to be defined over the whole cross-section and not only along the element boundaries.
That was why another way of constructing the hybrid formulation had to be chosen. It started with the potential energy formulation, adapted to the problem of uniform torsion, then proceeded to the hybrid formulation along the line of the so-called Friedrich transformation. Apart from making some features of the ultimate formulation clearer, it shows that the (warping) displacements occurring exclusively in the boundary integrals represent the displacements originally defined within the entire element.

Consider a homogeneous and isotropic prismatic bar of length \( l \) (Fig. 1) and introduce Cartesian coordinates in such a way that the \( x \)-axis passes through the centroids of the cross-sections. Since the determination of both the de Saint-Venant torsion constant and the warping constant is based on the situation of uniform torsion, orientation of the \( y \)- and \( z \)-axes with respect to the principal axes is irrelevant. Moreover, any longitudinal fiber can be chosen as the axis of rotation; in this case, the \( x \)-axis was chosen. The displacements in the \( x \), \( y \) and \( z \) directions are denoted by \( u \), \( v \) and \( w \) respectively, and the twist per unit length is \( \psi(x) \). When solving the torsion problem, de Saint-Venant [1] assumed that the axial displacement \( u \) was proportional to the constant twist \( \psi \) and depended on the coordinates \( y \) and \( z \) (see e.g. [4], p. 223):

\[
u = \psi(y, z),
\]

(2.3)
Hybrid determination of torsion and warping constants

where $\psi$ is the unknown warping function. The potential energy formulation requires $\psi$ to be continuous. Owing to the assumption that a cross-section does not deform in its plane, the other displacements become

$$
\begin{align*}
v &= -axz \\
w &= axy.
\end{align*}
$$

(2.4)

Implicit in these relations is the fact that rotation at $x = 0$ has been suppressed. As a consequence of equations (2.3) and (2.4), the non-vanishing strains are:

$$
\begin{align*}
e_{xy} &= \frac{1}{2}a(\psi_{xy} - z) \\
e_{xz} &= \frac{1}{2}a(\psi_{xz} + y).
\end{align*}
$$

(2.5)

A subscript preceded by a comma denotes partial differentiation with respect to the independent variable indicated by the subscript. The pertinent shear stresses are

$$
\begin{align*}
t_{xy} &= 2Ge_{xy} \\
t_{xz} &= 2Ge_{xz},
\end{align*}
$$

(2.6)

where $G$ is the shear modulus of the material. The stresses should satisfy the local equilibrium equation

$$
t_{xy,x} + t_{xz,z} = 0.
$$

(2.7)

For a bar which is free to warp when loaded with a twisting moment $M$ at $x = l$, the potential energy is

$$
\Pi_p = \frac{1}{2} \int_V \left\{ 4G(e_{xy}^2 + e_{xz}^2) \right\} dV - M\alpha l.
$$

(2.8)

If the bar is subdivided into longitudinal prisms over its whole length, the elastic energy becomes the sum of the elastic energy of the individual prisms:

$$
\Pi_p = \sum \int_{V^e} \frac{1}{2} \left\{ 4G(e_{xy}^2 + e_{xz}^2) \right\} dV - M\alpha l,
$$

(2.9)

where $V^e$ is the volume of the longitudinal prism whose cross-section contains the element $e$. The summation includes all the elements. In uniform torsion, the twist $\alpha$ is constant; therefore, the continuity requirement for the axial displacement $u(y, z)$, equation (2.3), implies that the warping function $\psi(y, z)$ must be continuous over inter-element boundaries. Compatibility in the plane of the cross-section is guaranteed by equation (2.4). The stress-hybrid formulation required can be derived by using the general transformation scheme described by Washizu [14] and Pian [15].

In the first step, subsidiary conditions (2.5) are introduced in the functional (2.9) by means of Lagrange multipliers $t_{xy}^*$ and $t_{xz}^*$:

$$
\Pi_{m1} = \sum \int_{V^e} \frac{1}{2} \left\{ 4G(e_{xy}^2 + e_{xz}^2) \right\} dV - 2 \int_{V^e} \left( t_{xy}^* \left\{ e_{xy} - \frac{1}{2}a(\psi_{xy} - z) \right\} \\
+ t_{xz}^* \left\{ e_{xz} - \frac{1}{2}a(\psi_{xz} + y) \right\} \right) dV - M\alpha l.
$$

(2.10)

In the second step, $\Pi_{m1}$ has to be stationary with respect to the strains; this leads to the relations

$$
\begin{align*}
t_{xy}^* &= 2Ge_{xy} \\
t_{xz}^* &= 2Ge_{xz}.
\end{align*}
$$

(2.11)

Comparing equation (2.11) with (2.6) shows that the multipliers can be identified with the shearing stresses. Replacing the multipliers by these stresses and adopting Hooke's law (2.6)
in order to eliminate the strains lead to the specialized Reissner functional:

$$\Pi_{mR} = \sum_{\nu} \int_{y_{\nu}} \left\{ -\frac{1}{2G} \left( t_{xx}^2 + t_{zz}^2 \right) + t_{yy} \frac{\partial}{\partial y} \left( \psi_{yy} - z \right) + t_{xz} \frac{\partial}{\partial z} \left( \psi_{xz} + y \right) \right\} \, dV \right\} - \bar{M} z l. \quad (2.12)$$

The third step requires partial integration in order to generate the local equilibrium equation:

$$\Pi_{mR} = \sum_{\nu} \int_{y_{\nu}} \left\{ -\frac{1}{2G} \left( t_{xx}^2 + t_{zz}^2 \right) - \frac{\partial}{\partial y} \left( t_{yy} + t_{yy} \right) \psi \right\} \, dV$$

$$+ \sum_{x=0}^{N} \int_{S_{x}} \left( t_{yy} n_{y} + t_{xx} n_{x} \right) \psi \, ds \, dx$$

$$+ \sum_{x=0}^{N} \int_{A_{x}} \left( -t_{zy} + t_{zz} \right) \, dA \, dx \right\} - \bar{M} z l. \quad (2.13)$$

where \( A_{x} \) is the area of the \( x \)-th element, while \( S_{x} \) is the boundary of \( A_{x} \). Now, if local equilibrium is required within each element, and the components of the stress tensor on \( S_{x} \) are expressed in the only relevant component of the stress vector

$$p_{x} = t_{yy} n_{y} + t_{xx} n_{x} \quad \text{on} \quad S_{x}. \quad (2.14)$$

then the hybrid formulation becomes

$$\Pi_{mc} = \sum_{\nu} \int_{A_{\nu}} \left\{ -\frac{1}{2G} \left( t_{xx}^2 + t_{zz}^2 \right) \right\} \, dA + \bar{M} z l \int_{S_{x}} p_{x} \psi \, ds$$

$$+ \bar{M} z l \int_{A_{x}} \left( -t_{zy} + t_{zz} \right) \, dA \right\} - \bar{M} z l. \quad (2.15)$$

Use has been made of the fact that all quantities are independent of \( x \) due to the uniform torsion.

The subsidiary conditions are:

1. continuity of the warping function \( \psi(y, z) \) on inter-element boundaries;
2. local equilibrium [equation (2.7)] within each prism.

The latter requirement can be met by introducing the scalar stress function \( \Phi(y, z) \) together with the prescription

$$t_{xy} = \Phi_{xz} \quad \text{and} \quad t_{xz} = -\Phi_{yx}. \quad (2.16)$$

The relations which make \( \pi_{mc} \) stationary with respect to all the permissible variations are:

1. compatibility within each prism;
2. a continuous stress component \( p_{x} \) between the prisms;
3. \( p_{x} = 0 \) on the stress-free longitudinal face of the bar as a whole;
4. the natural boundary condition at \( x = 1 \)

$$\bar{M} = \sum_{x} \int_{A_{x}} \left( -t_{xy} z + t_{xz} y \right) \, dA. \quad (2.17)$$

Apart from its sign, functional (2.15) represents a modified complementary energy functional with continuity of \( p_{x} \) relaxed by the axial displacements \( z \psi(s) \) on the longitudinal faces of the prisms and, on the transverse endface, the natural boundary condition (2.17) relaxed by the rotation \( z l \).

It is worth noting that, in some hybrid formulations, the stress continuity is relaxed by introducing boundary displacements that are only defined on the element boundaries but are independent of the internal displacements [15]. However, the warping function \( \Phi(s) \), although it only appears in the contour integral of equation (2.13), represents a continuous warping function \( \psi(y, z) \) that is defined over the whole area \( A \), as shown by the derivation of \( \pi_{mc} \). Thus it can be used to calculate the warping constant \( \Gamma \) with the classical formula (see
e.g. [14], p. 303)

\[ \Gamma = \sum_e \int_{A_e} \psi^2 \, dA. \]  

(2.18)

Although this formula complies with the potential energy formulation, it may be used with the hybrid formulation too. If a term accounting for the strain energy due to non-uniform torsion is added to the potential energy (2.9), this term will remain unaffected by the transformation to the hybrid formulation. As this term contains equation (2.18), the classical formula will also remain unaffected.

In the following section, the hybrid formulation will be used to construct a stress–hybrid finite element model. After assembling the elements, it will become clear that the determination of the torsion and warping constants can be performed separately and will precede the calculation of the twist \( \alpha \) as a function of torque \( M \), as for the classical displacement formulation.

3. DISCRETIZATION OF THE FUNCTIONAL

The finite element discretization proceeds along the general lines given by Pian [9]. Since the aim was to generate a formulation where the determination of torsional constants could be distinguished from the one-dimensional torsion problem, the end rotation \( \alpha_d \) and the shear modulus \( G \) will be left out of the brackets when formulating matrices.

We start by assuming a stress function distribution

\[ \Phi(y, z) = N^T(y, z)\beta, \]  

(3.1)

where column \( \beta \) contains discrete values of the element stress function. Continuity of \( \Phi \) between elements is not a requirement. Due to the boundary integral appearing in equation (2.13), nodal values of \( \Phi \) may be advantageous, in which case the so-called shape functions could be taken as components of \( N_y \). The pertinent stresses can now be expressed as

\[ \sigma = P\beta, \]  

(3.2)

where \( \sigma \) contains the relevant stresses

\[ \sigma^T = (t_{xy} \ t_{xz}), \]  

(3.3)

while matrix \( P \) is defined as

\[ P = \begin{bmatrix} \partial N^T_y / \partial z \\ -\partial N^T_x / \partial y \end{bmatrix}. \]  

(3.4)

The fact that the stresses are related to the derivatives of the stress function will affect equation (3.1):

- if a polynomial shape is chosen for equation (3.1), the constant value of \( \Phi \) can be omitted
- if a description is chosen in nodal value of \( \Phi \), column \( \beta \) should contain the appropriate differences between these nodal values, while \( N(y, z) \) should be adapted accordingly.

Substituting equation (3.2) into the first term of (2.13) gives

\[ -\frac{1}{2G} \int_{A^e} (t_{xy}^2 + t_{xz}^2) \, dA = -\frac{1}{2G} P^T H \beta, \]  

(3.5)

where

\[ H = \int_{A^e} P^T P \, dA \]  

(3.6)

with

\[ H = H^T. \]  

(3.7)

The second term of equation (2.13) represents the contribution of the prismatic faces to \( \int p_i u_i \, dS \) of (2.1). It contains the axial shear component \( p_x \) on each individual face. With the aid of equation (2.14), \( p_x \) can be expressed along each smooth side \( i \) as

\[ p_x^{(i)} = R^{(i)T} \beta. \]  

(3.8)

In accordance with the aim, the warping \( \psi(y, z) \) will be defined over the whole elemental
area as
\[ \psi(y, z) = N_b^T(y, z)q. \]  
(3.9)
This warping distribution appears on the ith element boundary as
\[ \psi^{(i)}(s) = L^{(i)T}(s)q. \]  
(3.10)
This gives
\[ \int_{S_z} p_s \psi \, ds = \int_{S_z} \beta^T Tq, \]  
(3.11)
where
\[ T = \sum_i \int_{S_z} R^{(i)T} \, ds. \]  
(3.12)
The third term of equation (2.13) represents the contribution to the term \( \int p_n u \, dS \) of (2.1) of the cross-section at \( x = l \) which has a surface \( A_z \). Due to the assumption of the rigidity of the cross-section in its plane, the in-plane displacements \( v \) and \( w \) are restricted by equation (2.4). Together with the stress distribution [equation (3.2)], this gives:
\[ \int_{A_z} \left( t_{xy}, t_{xz} \right) \left( \begin{array}{c} -z \\ y \end{array} \right) dA = \int_{A_z} \beta^T a, \]  
(3.13)
where
\[ a = \int_{A_z} P^T \left( \begin{array}{c} -z \\ y \end{array} \right) dA. \]  
(3.14)
The individual terms of functional (2.15) have now been discretized so that the total matrix representation becomes
\[ \Pi_{mc} = \int \sum \left[ -\frac{1}{2G} \beta^T H\beta + \alpha \beta^T (Tq + a) \right] - M x l. \]  
(3.15)
For the traction \( p_s \), neither is there a continuity requirement at the inter-elemental boundaries nor are there requirements at the traction-free longitudinal faces of the bar. Thus, all the stress parameters of \( \beta \) can be varied at element level. If the function (3.15) is required to be stationary with respect to the variations of \( \beta \), this yields for each element
\[ \delta \beta^T \left[ -\frac{1}{2G} H \beta + \alpha (Tq + a) \right] = 0 \]  
(3.16)
and, since \( H \) is not singular,
\[ \beta = \alpha G H^{-1} (Tq + a). \]  
(3.17)
This expression is used to eliminate the stress parameters of \( \beta \) from equation (3.15):
\[ \Pi = \alpha^2 G \sum \left[ \frac{1}{2} q^T kq - q^T Q + C \right] - M x l, \]  
(3.18)
where
\[ k = T^T H^{-1} T \]  
(3.19)
\[ Q = T^T a \]  
(3.20)
\[ C = \frac{1}{2} a^T H^{-1} a. \]  
(3.21)
Equation (3.18) has the same appearance as the common displacement formulation. In this special case, the displacement vector \( q \) only contains unknown nodal warping values. It does not contain the twist \( \alpha \). Contributions to the strain energy of displacements in the cross-sectional plane are contained in \( C \), whereas \( q^T Q \) represents the coupling between the warping and in-plane displacements. Again, a plane problem has to be solved in order to find the unknown warping values. Comparing equation (3.18) with the potential energy expression for a uniformly twisted bar,
\[ \Pi = \frac{1}{2} Gl, \alpha^2 l - M x l; \]  
(3.22)
this shows that the torsion constant $I_t$ can be calculated with

$$I_t = \sum_q \left[ \frac{1}{2} q^T \mathbf{k} q - q^T \mathbf{Q} + C \right].$$  

(3.23)

Moreover, the warping constant $\Gamma$ (2.18) can now be calculated, because the nodal warping values represent a continuous warping distribution [equation (3.9)] over the entire cross-section.

**Examples**

In order to get a first impression of the hybrid formulation in operation, a simple rectangular element was derived with four nodes.

A bi-linear distribution was chosen for both the stress and warping functions:

$$\Phi = \sum_{i=1}^{4} N_i(\xi, \eta) \varphi_i$$

$$\psi = \sum_{i=1}^{4} N_i(\xi, \eta) \psi_i.$$

Unlike the usual element formulations, the global coordinates $y$ and $z$ appeared explicitly in the functional (2.13), because the cross-section cannot be deformed in its plane. They were treated in the usual way:

$$y = \sum_{i=1}^{4} N_i(\xi, \eta) y_i$$

$$z = \sum_{i=1}^{4} N_i(\xi, \eta) z_i,$$

where $y_i$ and $z_i$ are the coordinates of the $i$th node. In order to avoid singularity of matrix $\mathbf{H}$ [equation (3.6)], the column $\mathbf{\beta}$ (3.2) was defined as:

$$\mathbf{\beta} = \begin{bmatrix} \varphi_1 - \varphi_4 \\ \varphi_2 - \varphi_4 \\ \varphi_3 - \varphi_4 \end{bmatrix}$$

and $\mathbf{N}_b$ (3.1) was adapted accordingly. Further formulation of the relevant matrices was straightforward. A test case will now be presented in which a square and a rectangle with aspect ratio 1:10 were involved (Fig. 3). Use was made of the double symmetry of these cross-sections. The results obtained for the torsion integral $I_t$ could be expressed with a factor $k$, defined by

$$I_t = k(2a)^3(2b).$$

These results were compared then with the results obtained from the classical compatible formulation based on a bi-linear warping for each element, and also with the exact values of $k$ in Table 1.

The results obtained for the warping constant are given in Table 2.

**Table 1**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a$</th>
<th>$n \times m$</th>
<th>Number of elements</th>
<th>$k$ Compatible</th>
<th>$k$ Hybrid</th>
<th>$k$ Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>$2 \times 2$</td>
<td>4</td>
<td>0.148</td>
<td>—</td>
<td>0.141</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$4 \times 4$</td>
<td>16</td>
<td>0.142</td>
<td>0.139</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$6 \times 6$</td>
<td>36</td>
<td>0.141</td>
<td>0.140</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
<td>$2 \times 10$</td>
<td>20</td>
<td>0.315</td>
<td>0.294</td>
<td>0.312</td>
</tr>
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<td>50</td>
<td>0.314</td>
<td>—</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>$5 \times 15$</td>
<td>65</td>
<td>—</td>
<td>0.309</td>
<td></td>
</tr>
</tbody>
</table>
4. CONCLUSIONS

A hybrid formulation can be derived for calculating the torsion and warping constants for prismatic bars with an arbitrary cross-section by means of finite elements. Unlike other hybrid formulations [10, 11], these constants can be calculated independent of and prior to the calculation of the response of the bar to a torque. The new procedure therefore complies with the beam and bar theory.

The classical displacement (potential energy) formulation provides separate analytical expressions for the two torsional constants, as well as the response of the bar to a torque. Discretization by means of finite elements then follows after splitting the—in fact—three-dimensional problem. In the hybrid formulation, separating the expression which is used for calculating the torsion constant from the response of the bar to torsion is only possible in the discretized formulation.

The classical analytical expression for the warping constant, which is relevant in cases of non-uniform torsion, still remains valid. Calculating this constant requires warping to be known at every point on the cross-section. In the functional that relates to the hybrid formulation, equation (2.15), warping only appears on the element boundaries. The development of this functional, however, shows that warping on the element boundaries arises from the original warping, defined at every point on the cross-section. Therefore, the warping constant can be calculated consistently.

In this paper, an emphasis has been placed on deriving the general hybrid variational formulation. The function (2.15) which is obtained, together with the general discretization procedure, can be used as a basis for developing arbitrary finite elements. Only a simple rectangular element is developed here, in order to illustrate the procedure.

A few simple examples that make use of this element will show that the warping constant is more accurate than the one calculated by the classical method. An evaluation of this hybrid formulation for analysing complicated cross-sections, by means of higher order elements, is at present under consideration.

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REFERENCES

Hybrid determination of torsion and warping constants