THE ACOUSTICS OF A LINED DUCT WITH FLOW

BY

S.W. RIENSTRA*
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SUMMARY

The present report describes a theory to calculate the sound propagation (including attenuation, reflection, radiation) through a three-sectioned cylindrical flow duct, modelling an aero-engine inlet. The flow is uniform apart from a thin boundary layer; two of the three sections are hard-walled, with an impedance-walled section in between. The modal amplitudes of the sound field in the duct are determined by an iterative technique allowing the modal expansions to include as many terms as required. The modal eigenvalues are found using a classification based on a distinction between acoustic modes and surface waves. Numerically, the main results (in the sense of practical applications) are contour plots of constant attenuation, in the complex impedance plane. One of the most striking observations is a dramatic effect of lining (via the occurrence of surface waves) on a sound field that is cut-off in a hard-walled duct.

In addition to the above problem with a lining of constant impedance, the problem of a (necessarily variable) non-reflective impedance is briefly considered. This problem is mainly relevant to an (acoustic) wind tunnel. It is shown that, for a given free field of the source, the solution, describing this impedance distribution, can be given analytically, in closed form.

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INTRODUCTION

After the introduction of the jet-engined civil aircraft in the early fifties, aircraft noise became rapidly a serious community problem. For the first engines, the main source of noise was the turbulent jet exhaust flow. However, the turbojet-engine with by-pass flow, introduced in the early sixties, combined a higher efficiency with a lower jet noise level by applying a lower jet speed (Lighthill [1]). Since then, although the jet noise component has never been unimportant, the fan noise, radiated out of the inlet, became of primary importance, and much research has been devoted to this and related subjects. More specifically, sound generation by a fan, its propagation through a duct and the attenuation by acoustic lining has been studied both theoretically and experimentally and the research still continues.

Also at NLR a research programme in this area is carried out (Refs. [2, 3, 4]) and the present report describes a new version of the duct-acoustical model, calculating the sound field in a cylindrical lined flow duct, for a given source and geometry.

After the pioneering work of Pridmore-Brown [5], who considered sound waves in lined ducts with shear flow, most of the duct acoustical models concentrated on cylindrical ducts, with a modal description of the field ([6]). This was and is particularly the case when high (dimensionless) frequencies are involved, which is the normal situation in an engine ([7, 8, 9]). Furthermore, it became clear that acoustically effective shear flow is found only in the boundary layer along the wall ([10, 11, 12, 13]), and most of the mean flow can be modelled appropriately with a uniform velocity profile. Furthermore, since the boundary layer thickness is usually much less than the typical acoustic wave length, the effect of the boundary layer can, to a large extent, be described analytically ([10, 11, 14]). A survey of the status in 1975 can be found in reference [15].

The theoretical model, to be presented here, will be based on these observations. We will add, however, some new aspects which resolve or clarify a number of problems encountered by other authors. These problems are the following.

A popular technique to calculate the modal amplitudes for the various sections in a duct is to construct in one operation a so-called transfer matrix ([8, 16, 9]), consisting of all sound transfer and reflection matrices at the section interfaces taken together by inversion, shifting, and multiplication into one matrix. However, if this transfer
matrix is to include exponentially decreasing modes (which is the case if we want a convergent modal expansion), all the section interfaces are coupled via exponentially small and large terms, making the problem of the transfer matrix very ill-conditioned and practically impossible to solve. So this approach is only useful if most of the acoustic energy is carried by the non- or weakly-attenuated modes, and we do not have to bother about the inaccuracies introduced by ignoring the other modal terms. However, this is not always the case. For example, the first harmonic of an engine, normally designed to be cut-off, cannot be studied this way. Our alternative will be an iterative procedure in a way that never a matrix is to be inverted with exponential terms.

Another problem involves finding all the modes, that is finding all the eigenvalues. Some of the eigenvalues were known to behave sometimes very irregularly, making the finding of them difficult and uncertain. One successful approach to this problem is proposed by Eversman [17, 18], and consists of rewriting the (algebraic) eigenvalue equation into a differential equation, which is then to be solved by standard techniques. We, however, will tackle the problem by using the new classification of modes (Rienstra, [19, 20]), which shows that these singular modes are physically of another type than the acoustic modes. They are surface waves, but with a behaviour just as orderly and predictable as the acoustic modes.

Finally, there is the problem of the direction of propagation (causality) of one of the modes, or: the presence of an instability ([11, 19, 18, 21]). Sometimes, one of the modes is to be interpreted as exponentially increasing, instead of decreasing. However, a number of aspects are still puzzling, and we have not attempted here to include this instability. On the other hand, knowing that this (potentially) unstable mode is one of the singular modes (surface waves), we are much more able to anticipate its occurrence and possible effects.

2 FORMULATION OF INLET MODEL

We consider a cylindrical duct (Fig. 1) with (dimensional) radius $a$, described in cylindrical coordinates $(x, r, \theta)$, dimensionless on $a$, as

$$0 \leq x \leq L, \quad r = 1, \quad 0 \leq \theta < 2\pi$$
The duct contains an inviscid fluid of mean density, pressure and sound speed $\rho_0$, $p_0$ and $c_0$, respectively. The fluid flows with velocity $U_0$ (Machnumber $M = U_0/c_0$) in negative $x$-direction (so $U_0$ and $M$ are negative) with an almost uniform profile. Only very close to the duct wall (in the boundary layer) the mean velocity is dependent on $r$ (only), and is assumed to reduce linearly to zero at $r = 1$ (Fig. 2): 

$$M(r) = \begin{cases} M & \text{if } 0 \leq r \leq 1-\delta \\ M(1-r)/\delta & \text{if } 1-\delta \leq r \leq 1 \end{cases}$$

where $\delta$ is the (dimensionless) small boundary layer thickness.

The flow is perturbed by acoustic waves, generated at the source plane $x = 0$, of frequency $f^*$. These sound waves propagate upstream in a hard-walled section $0 \leq x < x_1$, scatter at the downstream edge $x = x_1$, of an impedance-walled (= lined) section $x_1 \leq x \leq x_2$ (specific acoustic impedance $Z$), scatter again at the upstream edge $x = x_2$, continue further in a hard-walled section $x_2 \leq x \leq L$, and reflect at and radiate out of the open duct end at $x = L$. The source plane $x = 0$, at the other end, is assumed to be anechoic.

In the present report we will concentrate on the propagation and attenuation of sound along the duct including the effect of the open end reflection. A detailed analysis of the duct end acoustics is given in Rienstra [22], and the corresponding computer program ([23]) is integrated in the program system LINDA (= Lined Duct Acoustics), corresponding to the present work ([24]).

In the acoustic approximation of small perturbations, our problem is linear, and we may reduce the acoustic field by Fourier analysis (in space and time) to more elementary wave forms. This is particularly relevant to the time- and $\Theta$-coordinate, since the type of source we primarily consider is rotor-stator interaction on the rotor-bound pressure field, both generating (in the range of interest) only a limited, discrete, sequence of frequencies and circumferential wave numbers (Tyler and Sofrin, [7]). Therefore, we will assume hereafter for the acoustic perturbations a behaviour in time $t$ and $\Theta$ according to the factor (in the usual complex notation)

$$e^{i\omega t + im\Theta}$$

where $\omega = 2\pi f^* a/c_0$ denotes the dimensionless frequency, and the in-
Integer $m$ is the circumferential wave number. Since the geometry considered is stationary and axi-symmetric, we may suppress this exponential part in the formulas. In view of the symmetry of the problem it is sufficient to consider $m \geq 0$.

For the perturbations in pressure $p$, in density $\rho$, and in velocity $\tilde{v}$, dimensionless on $\rho_0 c_0^2$, $\rho_0$, and $c_0$ respectively, we have the linearised Euler equations (Pierce, [25])

\[
i\omega\rho + M^2 \rho \rho + \nabla \cdot \tilde{v} = 0
\]

\[
i\omega \tilde{v} + M^2 \tilde{v} + \tilde{\varepsilon} \cdot (\tilde{v} \cdot \tilde{e}_r) dM/dr + \nabla p = 0
\]

\[p = \rho
\]

(where, of course, any $\alpha$ denotes a factor $i\omega$)

with boundary condition

\[p = Z (\tilde{v} \cdot \tilde{e}_r) \quad \text{at } r = 1
\]

and a source

\[p = f(r) \quad \text{at } x = 0,
\]

which will in practice be described in terms of modal amplitudes.

It may be noted that in the uniform part of the mean flow, $0 \leq r \leq 1-\delta$, both the mean flow and the acoustic field are irrotational. Vorticity may be generated at the source plane $x = 0$ or, in the boundary layers near $r = 1$, but this vorticity is convected away by the mean flow without entering the inlet duct region of uniform flow.

In the next chapter we will solve the above equations by a modal expansion in the uniform flow region, and an asymptotic expansion for small $\delta$ in the boundary layer. Effectively, this boundary layer solution provides a modified boundary condition for the uniform flow solution, which is, evidently, the most important (in terms of energy) part of the sound field. The solution is similar to those presented by Eversman [10], Tester [11], or, slightly modified, Myers and Chuang [14].
3.1 Matched asymptotic expansion in the boundary layer

It is convenient (and indeed appropriate to the modal representation later) to introduce a Fourier transform to $x$

$$p(x,r) = \int_{-\infty}^{\infty} \hat{p}(r,k)e^{-ikx} dk,$$

and similarly for $v^+$ and $\rho$. Furthermore, we introduce the local variable

$$\eta = (1-r)/\delta$$

and an auxiliary function $M_0(\eta) = M(r)$. We assume the outer solution, for $1-r = O(1)$, to have an expansion

$$\hat{p} = P_0(r) + \delta P_1(r) + O(\delta^2)$$

which becomes for $r \rightarrow 1$ (in the region $\eta$ is large but $O(1)$)

$$\hat{p} = P_0 + \delta \{ P_1 - \eta P_0' \} + O(\delta^2),$$

where $P_0 = P_0(1)$, etc. Similar expressions may be found for $v^+ = (v_x^+, v_r^+, v_{\theta})$, with

$$\hat{v}_x = \hat{p} k/(\omega - Mk)$$

$$\hat{v}_r = i\hat{p} r/(\omega - Mk)$$

$$\hat{v}_{\theta} = -m\hat{p}/(\omega - Mk)r$$

Note that we assumed $k$, $\omega$, $m = O(1)$, and $\omega - Mk \neq 0$ (avoiding critical layers; see Swinbanks [26]). These outer expansions have to match (for $r \rightarrow 1$) to the inner expansions, to be constructed below. Written out in $\eta$ we have

$$i(\omega - kM_0)\hat{p} - ik\hat{v}_x - \hat{v}_r'/\delta + (\hat{v}_r + im\hat{v}_{\theta})/(1-\delta \eta) = 0$$

$$i(\omega - kM_0)\hat{v}_x - M \hat{v}_r'/\delta - ik\hat{p} = 0$$
where a prime denotes a derivative to \( n \). Since at \( n = 0 \) we have \( \hat{\rho} = 2\vartheta_r \), so (for \( Z \neq 0 \)) \( \hat{\rho}/\vartheta_r = O(1) \), and the large \( \delta^{-1} \) terms have to be balanced by \( v_x \) (some jet-like behaviour of the perturbations in the boundary layer). So we assume the expansion

\[
\hat{\rho} = \hat{\rho}_0 + \delta\hat{\rho}_1 + \ldots
\]

\[
\vartheta_x = \delta^{-1}\vartheta_{x0} + \vartheta_{x1} + \ldots
\]

\[
\vartheta_r = \vartheta_{r0} + \delta\vartheta_{r1} + \ldots
\]

\[
\vartheta_\theta = \vartheta_{\theta0} + \delta\vartheta_{\theta1} + \ldots
\]

Substitution in the above equations, equating like powers of \( \delta \), and solving with application of the matching conditions for \( y \to \infty \), yield

\[
\hat{\rho}_0 = P_0
\]

\[
\vartheta_{r0} = iP_0' (\omega-kM_b)/(\omega-kM)^2
\]

\[
\vartheta_{x0} = P_0' M_b^\prime/(\omega-kM)^2
\]

\[
\hat{\rho}_1 = P_1 + P_0' \left[K_{0-0}^{\prime n} (\omega-kM_b)^2dn'\right]/(\omega-kM)^2
\]

\[
\vartheta_{r1} = i(\omega-kM_b) \left[P_0n + (P_1' + P_0'n)/(\omega-kM)^2 + (m^2+k^2)P_0(K_{1-0}^{\prime n} (\omega-kM_b)^{-2}dn')\right]
\]

\[
\vartheta_{x1} = kP_0/(\omega-kM_b) +
\]

\[
+ M_b' \left[P_0n + (P_1' + P_0'n)/(\omega-kM)^2 + (m^2+k^2)P_0(K_{1-0}^{\prime n} (\omega-kM_b)^{-2}dn')\right]
\]

\[
\vartheta_{\theta0} = -mP_0/(\omega-kM_b)
\]
where the constants $K_0$ and $K_1$ are given by

$$K_0 = \int_0^\infty (\omega - kM)^2 - (\omega - kM)^2 \, d\eta,'$$

$$K_1 = \int_0^\infty (\omega - kM)^2 - (\omega - kM)^2 \, d\eta',$$

and $M$ denotes the mean flow Mach number. After application of the boundary condition $\hat{p} = 2q_0$ at $\eta = 0$, and some manipulation, we obtain the equivalent boundary condition for the outer field at $r + 1$:

$$(1-i\omega\delta(m^2+k^2)K_1)\hat{p}(1) = (i\omega Z - \delta K_0)\hat{p}(1)/(\omega - kM)^2$$

(3.2).

Note that the limit $Z + 0$, $\delta + 0$, $k + \omega/M$ is non-uniform. Therefore, the boundary layer corrections for the modes $k \approx \omega/M$ are not included by (3.2).

Up to now we have not yet used the assumed linear shape of the mean velocity profile. The constants $K_0$ and $K_1$ are still general. For a linear profile these constants are

$$K_0 = kM(\omega - 2kM/3)$$

$$K_1 = -kM/\omega(\omega - kM)^2$$

(3.3)

As shown in ref. [13], the detailed shape of the profile does not seem to be very important as long as we compare profiles of the same displacement thickness, so we may as well assume a linear profile. Other profiles are easily accounted for by the corresponding $K_0$ and $K_1$.

Equation (3.2) is the principal result of the present section. It is derived for a given, fixed, value of $k$. In the present context only discrete values of $k$ are possible, corresponding to the prevailing duct modes. These eigenvalues $k$ will be calculated as an asymptotic series by assuming $k = k_0 + \delta k_1 + \ldots$. For $k_0$ we will have to solve a nonlinear algebraic equation, but $k_1$ will be expressible in terms of $k_0$. We will work this out in section 4.1.
3.2 Modal solution in the uniform mean flow

By elementary manipulation of the equations we can eliminate $\dot{v}$ and $p$, use the fact that the mean flow is constant here, and derive the convective Helmholtz equation for $p$

$$(i \omega + M_3) \frac{\partial}{\partial x} p + \nu^2 p = 0 \quad (3.4)$$

with boundary condition (3.2). This equation may be solved by superposition of suitable elementary solutions. These solutions can be found via Fourier decomposition in $x$, or separation of variables, thus leading to modes, i.e. waves which retain their shape as they propagate along the duct. Their behaviour in the $x$-coordinate is exponential, and in $r$ according to a Bessel function. Mathematically, the problem is an eigenvalue problem with the (discrete) set of complex axial wavenumbers being the eigenvalues, and the corresponding modes the eigenvectors. These eigenvectors are, in general, not orthogonal.

The resulting modal series is

$$p(x,r) = \sum_{\mu=1}^{\infty} A_{m \mu} N^+_{m \mu} J_m(\gamma^+_{m \mu} r) e^{-ik^+_{m \mu} x} + B_{m \mu} N^-_{m \mu} J_m(\gamma^-_{m \mu} r) e^{-ik^-_{m \mu} x} \quad (3.5)$$

($J_m$ denoting Bessel's function; Watson [27])

where $A_{m \mu}$ and $B_{m \mu}$ are the amplitudes of right- and left running waves, and

$$\gamma^+_{m \mu} = \gamma(k^+_{m \mu}), \quad \gamma^-_{m \mu} = \gamma(k^-_{m \mu}) \quad \text{with}$$

$$\gamma(k) = (\omega^2 - k^2)^{\frac{1}{4}} \quad (\text{Im} \gamma \leq 0) \quad (3.6)$$

The positive real normalization constants $N^+_{m \mu}$ and $N^-_{m \mu}$ are introduced for convenience, and defined in a way that

$$\left( N^+_{m \mu} \right)^2 \int_0^1 r |J_m(\gamma^+_{m \mu} r)|^2 \, dr = 1 .$$

Using some well-known indefinite integrals of Bessel functions [27] we obtain

$$\gamma \text{ real:} \quad N^+_{m \mu} = 2^{\frac{1}{2}} \sqrt{\frac{1 - m^2 / \gamma^2_{m \mu}}{J_m(\gamma_{m \mu})^2 + J'_m(\gamma_{m \mu})^2}} \quad (3.7)$$
\[ N_{m\mu} = 2^{1/2}\left[ (1-m^2/y_m^2) \left| J_m(\gamma_{m\mu}) \right|^2 - \left| J'_m(\gamma_{m\mu}) \right|^2 \right]^{1/2} \]  

(3.8)

\[ N_{m\mu} = \left| \frac{\text{Im}(\gamma_{m\mu}^2) / \text{Im}(\gamma_{m\mu} J_m(\gamma_{m\mu}) J'_m(\gamma_{m\mu}))}{Z} \right|^{1/2} \]  

(3.9)

where superscript * denotes a complex conjugate.

In case of hard walls, the function \( N_{m\mu} J_m(\gamma_{m\mu} r) \) is equal to \( U_{m\mu}(r) \) of ref. 1.

The wave numbers \( k_{m\mu}^+ \) and \( k_{m\mu}^- \) are solutions of

\[ (1-i\omega Z\delta(m^2+k^2)k_1)J_m(\gamma) = (i\omega - \delta K_0)' \gamma J'_m(\gamma)/(\omega-km)^2 \]  

(3.10)

These solutions (an infinite denumerable set) are in general complex. Then, in the lower half plane we find the solutions \( k = k_{m\mu}^+ \), and in the upper \( k = k_{m\mu}^- \):

\[ \text{Im} (k_{m\mu}^+) < 0, \text{Im} (k_{m\mu}^-) > 0. \]

If the imaginary part equals zero, the appropriate limit in \( Z \) is taken, giving a condition for \( \delta k/\delta Z \).

It should be noted that the above classification may be not completely correct. For some values of the problem parameters \((\omega, m, M, Z, \text{the way } Z \text{ depends on } \omega)\), one of the otherwise decreasing upstream running modes with \( \text{Im} k < 0 \) is really an unstable downstream running one ([19]).

Although much about this mode has become clear lately ([19, 18, 21]), including its role with respect to a liner's leading edge Kutta condition, causal behaviour of the liner, and acoustic energy exchange with the mean flow, there are still some problems not well resolved (the appropriate liner's trailing edge condition, the unpractical exponential growth requiring some saturation mechanism, the correct causality criterium to be used). Therefore, we have chosen here, as a preliminary, but
practical simplification, not to try to recognize this mode as unstable
(which is, under some assumptions, possible but laborious), but to
always count it as an upstream running decreasing one. Physically,
this means that the Green's function built up by the present modes is
non-causal. For acoustic sources with a strong coupling with the mean
flow, like turbulence, the source and its field are unseparable, and
a non-causal, bounded (formal) solution seems indeed to be appropriate
([28]). However, this is not the case in the present problem, and
we leave the question here open for further research.

The above modal solution is not yet the finally required solution
of the complete problem, but rather a kind of framework in which the
description of the true solution is greatly simplified, namely, through
the amplitude vectors \( \hat{A} = (A_{m1}, A_{m2}, \ldots) \) and \( \hat{B} = (B_{m1}, B_{m2}, \ldots) \). A
particular modal solution is only valid in the duct section with the
corresponding wall impedance, so for the complete solution in a three­
sectioned duct we need three pairs of amplitude vectors. These amplitude
vectors are determined from the end conditions of the duct at \( x = 0 \)
(source) and \( x = L \) (open end) and continuity conditions for pressure and
axial velocity at the interfaces \( x = x_1 \) and \( x = x_2 \). This will be
considered in the next sections. As implied by the foregoing discussion
on the sometimes occurring instability mode, some smoothness condition at
\( x = x_2 \) and \( x = x_1 \) may be in order (determining the amount of shed
vorticity triggering - at \( x = x_2 \) - or being triggered by - at \( x = x_1 \) -
the instability wave), but as we already ignored the instability wave we
will not apply any special condition at the liner edges.

The end conditions at \( x = L \) will be given as a reflection matrix for
the open end radiation ([22, 23]) (by-passed in favour of an anechoic
termination, if required for reasons of study). The source
at \( x = 0 \) will usually be given by an amplitude vector \( A_{m1} \). If the source
is given as a function \( f(r) \), the modal amplitudes can be calculated;
this is, however, rather laborious for non-zero boundary layer thickness,
because of the non-orthogonal eigenfunctions. If \( \delta = 0 \) (or for the
moment may be taken so) the source amplitudes are found quite easily
as follows:
\[ A_{m \mu} = N_{m \mu}^+ \int_0^1 r J_m (j_{m \mu}^') (r) \, dr \]

where \( j_{m \mu}^+ = j_{m \mu}^' \), with \( J_m^' (j_{m \mu}^') = 0 \), and using (3.7).

3.3 Acoustic power

A very important quantity, to measure the acoustic field by a single number, is the power transmitted through a suitable surface. The physical way to define the net damping attained by a liner, is via the acoustic power. (The other common, more subjective, measure is EPNL, or, in the form of [24], ESPL.)

The power we will consider is the power transmitted through a cross-sectional surface in a hard-walled part of the duct, with a (temporarily assumed) vanishing boundary layer. For a complete description of the energy balance we would need the power through a full duct-cross section including the boundary layer. However, in the boundary layer, with its inherent vorticity field, the acoustic field may couple to the mean flow and no definitions for (conserved) energy exist ([29]). Therefore, we have left the boundary layer out, assuming that a possible boundary layer effect is practically always (c.f. Tester [11]) only important if it carries through to the uniform flow region. So, in general, the power will have a relative error of at most order \( \delta \), which is negligible on decibel scale.

Several definitions for acoustic energy in a moving medium are proposed. The one we will use here, for homentropic flow, was derived by Cantrell and Hart [30]; it is widely used ([22, 25, 31, 29]), and recently Myers [32] showed that this definition indeed follows by a regular perturbation scheme (for small amplitude) from the general principle of energy conservation in homentropic (inviscid) flow.

The acoustic energy equation ([32]) in general, dimensional form, is then
where subscript $0$ denotes mean flow values, $\xi = \nabla \times \vec{v}$ denotes vorticity, and the energy density $E$ and energy flux $\dot{W}$ are given by

$$E = c_0^2 \rho^2 / 2 \rho_0 + \rho_0 v^2 / 2 + \rho \dot{\vec{v}} \cdot \dot{\vec{v}}$$

$$\dot{W} = (c_0^2 \rho + \rho \dot{\vec{v}}_0 \cdot \dot{\vec{v}}) (\dot{\vec{v}} + \rho \dot{\vec{v}}_0 / \rho_0) .$$

Energy is conserved if the "source"-term of (3.11) vanishes. So this expression shows explicitly how vorticity of mean flow or acoustic order may act as a source or sink of acoustic energy.

For harmonic perturbations, as we have here, the acoustic power transmitted through a surface $S$ is then

$$P = \iint_S \tilde{I} \cdot \vec{n} \, dS$$

where the acoustic intensity $\tilde{I}$ is the time-averaged energy flux $\dot{W}$.

(Note that acoustic intensity is not defined the same by all authors. For example, we follow Goldstein [29], but Pierce [25] and others define intensity and energy flux as the same.)

For the present situation of an acoustic field (3.5) in a uniform mean flow, at a cross section of a hard-walled duct we can derive (using some orthogonality properties of the eigenfunctions).
\[ P = \rho_0^2 C_0^2 a^2 \pi \int_0^1 \text{Re}(p + M v_x)(v_x^* + M p^*) \, r \, dr = \]

\[ = \rho_0^2 C_0^2 a^2 \pi \omega \beta^4 \sum_{\mu=1}^{\mu_0-1} \Omega_{\mu} \left[ \left| \frac{A_{\mu}}{(\omega - M \Omega_{\mu})^2} \right|^2 - \frac{\left| B_{\mu} \right|^2}{(\omega + M \Omega_{\mu})^2} \right] \]

\[ + \rho_0^2 C_0^2 a^2 \pi \omega \beta^4 \sum_{\mu=\mu_0}^{\infty} \text{Re} \left[ \frac{\Omega_{\mu} A_{\mu} B_{\mu}^*}{(\omega - M \Omega_{\mu})^2} \right] \]  

(3.12)

where \( \beta = (1 - M^2)^{1/4} \), superscript * denotes a complex conjugate and \( \Omega_{\mu} \) is defined as

\[ \Omega_{\mu} = (\omega^2 - \beta^2 \gamma_{\mu}^2)^{1/4} \]

with \( \gamma_{\mu} = j' \) and \( \text{Im} \Omega_{\mu} < 0 \). The modes \( \mu = 1, \ldots, \mu_0-1 \) are cut-on (\( \Omega_{\mu} \) is real), and the others are cut-off (\( \Omega_{\mu} \) is negative imaginary). Expression (3.12) is the same as given in reference 22.

Finally, we can define the following quantities (in decibels)

- **Power level**   \[ \text{PWL} = 10 \log P + 120, \]

- **Insertion loss** \[ IL = \text{PWL} (Z = \infty) - \text{PWL} (Z) \text{ at } x = L, \]

- **Transmission loss** \[ TL = \text{PWL} (x = 0) - \text{PWL} (x = L). \]

\[ PWL \] measures the net level radiated from the duct, \( IL \) measures the "usefulness" of a liner compared to no liner (hard walls), and \( TL \) measures the net amount of energy dissipated by the liner, without the possible benefit due to reflections.
4 EIGENVALUES

4.1 General

Although the eigenvalue equation (3.10) is transcendent, and only very little explicit solutions are available, we can simplify the equation somewhat by utilizing the small parameter $\delta$, the boundary layer thickness. If we assume

$$k = k_0 + \delta k_1 + \ldots$$

(4.1)

and construct a sequence of equations for the various powers of $\delta$, we will find that $k_0$ still satisfies a complicated equation, but that the higher order terms can explicitly be expressed in $k_0$. The gain to solve for $k_0$ instead of $k$ is not very much, but at least it eases a systematic classification, and the breakdown of the small-$\delta$ approximation is easier recognized via the observation of $\delta k_1 = O(1)$.

After some relatively straightforward analysis, and using properties of the Bessel function, we obtain

$$\omega^2 J_m(\gamma_0) - i\omega Z J'_m(\gamma_0) = 0$$

(4.2)

$$k_1 = (\omega - M k_0)^2 \left[ k_0/\omega^2 Z^2 + (m^2 + k_0^2) k_1 \right] / \ldots$$

$$\left[ (\omega - M k_0)^2 (\omega M + \beta^2 k_0)/\omega^2 Z^2 \gamma_0^2 + 2i M (\omega - M k_0)/\omega \right.$$  

$$- (\omega M + \beta^2 k_0) (1 - m^2/\gamma_0^2) \left. \right]$$

(4.3)

where $\beta^2 = 1 - M^2$, $\gamma_0 = \gamma(k_0)$, and $m^2/\gamma_0^2 = 0$ if $m = 0$.

As discussed before, equation (3.10) is not valid if $Z = 0$ and $k \approx \omega/M$. However, since for $\delta = 0$ and $Z = 0$ the eigenvalue $k = \omega/M$ is a double eigenvalue, which we would like to avoid (see Sec. 4.6), it is convenient not to assume $\delta = 0$, but to have some expression available for $\delta > 0$, even though not completely correct:

if $Z = 0$ and $k_0 = \omega/M$, the assumption (4.1) breaks down, and instead we have $k = k_0 + \delta^{1/2} k_1 + \ldots$, where
\[ k_1 = \pm i \left[ K_0(\omega/M)I_m'(\omega/M)/I_m(\omega/M) \right]^{\frac{1}{2}} / M \] (4.4)

(I. denotes the modified Bessel function of the first kind, Watson [27], and \( K_0 \) is given in (3.3).) These two coalescing eigenvalues are (for high enough \( \omega \)) surface waves: one acoustic and one hydrodynamic (Rienstra [19, 20]). In the next section we will say some more about the surface waves.

The numerical routine to calculate the eigenvalues from (4.2) is based on this distinction between acoustic modes and surface waves. It consists of a Newton search routine in two forms: one easily converging to acoustic modes and another easily converging to the surface waves modes. The starting values for the acoustic modes are approximations for \( Z + 0 \) and \( Z + \infty \), which will be discussed in later sections.

4.2 Surface waves and hydrodynamic modes

As discussed in much more detail in Rienstra [19, 20], sometimes there are solutions (with a maximum of four per \( m \)) of (4.2) which behave spatially as surface waves: that is, they are only significant near the wall, and further away negligibly small in an exponential way.

If \( \omega \) is sufficiently large and \( k \) is not near the branch cuts of \( \gamma \) (the line \( \text{Re}(k) = -\omega M/(1-M^2) \) and the real interval \( -\omega/(1-M) < \text{Re}(k) < \omega/(1+M) \) where \( \text{Im} \gamma = 0 \), \( \gamma(k) \) has a large (negative) imaginary part which makes \( J_m(\gamma) \) and \( J'_m(\gamma) \) exponentially large in a way that

\[ J'_m(\gamma)/J_m(\gamma) + i. \]

Equation (4.2) then becomes

\[ (\omega-Mk)^2 + \omega Z \gamma(k) = 0 \] (4.5)

which is the impedance-wall surface-wave equation, studied extensively in [19]. It is clear that for smaller \( \omega \) (typically of the order of \( m \) or smaller) equation (4.2) degenerates less and less into (4.5), and the notion of surface wave becomes vague. Physically, one might say that the wave decreases away from the wall too slowly to be negligible in the centre-area. However, the trend in the complex \( k \)-plane of four solutions behaving more or less singularly is very long clearly notable, so in practice the distinction between acoustic and surface waves is usually evident and useful.
The numerical routine to calculate the roots of equation (4.2) (the eigenvalues) is based on the above observation that (i) for the surface waves the ratio $J'_m(\gamma)/J_m(\gamma)$ tends to a constant, and that (ii) the acoustic modes are not far away from the hard wall ($Z = \infty$) or soft wall ($Z = 0$) modes (corresponding to the zero's of $J'$ and $J$, respectively); at least, the eigenvalues are found near the branch cuts of $\gamma$ ([20]).

Therefore we tried a Newton search routine based on the function

$$F_A(k) = \left[ (\omega - Mk)^2 J_m(\gamma) - i\omega Z \gamma J'_m(\gamma) \right] (2/\gamma)^m = 0$$

(4.6)

for the acoustic modes (the factor $(2/\gamma)^m$ is to exclude the trivial solution $\gamma = 0$), and another one based on the function

$$F_S(k) = (\omega - Mk)^2 - i\omega Z \gamma J'_m(\gamma)/J_m(\gamma) = 0$$

(4.7)

for the surface-wave modes. This appeared to be indeed a successful approach, if we use as starting values for $F_A$ approximations for $Z = 0$ and $Z = \infty$, for $F_S$ approximations for $\omega = \infty$ (the solutions of equation (4.5)), and apply in addition some deflation to handle cases of eigenvalues grouped closely together (occurring only when surface waves are involved).

4.3 Behaviour for $Z + \infty$

The eigenvalues for hard walls ($Z = \infty$) are particularly important since two of the three sections of the duct we consider ($0 \leq x \leq x_1$, $x_2 \leq x \leq L$) are hard-walled. Furthermore, as said before, approximations based on perturbations around $Z = \infty$ are used as starting values for the finite-$Z$ eigenvalues required for the lined section. When studying the behaviour for $Z + \infty$ we will not include possible surface waves. Their occurrence depends on the way $Z$ tends to infinity, giving either no, one, three, or four surface waves ([19]). This is entirely covered by the previous section. (Note that the surface waves diverge to infinity as $Z + \infty$; in the hard-wall case there are no surface waves.)

To leading order the solutions of (4.2) come down to $\gamma = j'_m \mu$, the
The $\mu$-th non-trivial zero of $J_m$ ($\gamma = 0$ is a trivial zero, except for $m = 0$, $\mu = 1$). The next order is a perturbation on this.

The results obtained are (the notation $k_0$, $k_1$, is for short and has nothing to do with (4.1-4)).

\[ k = k_0 + Z^{-1} k_1 + \ldots \]

\[ k_0^\pm = ( - \omega M \pm \sqrt{\omega^2 - (\beta j_{m\mu}^{\prime})^2} ) / \beta^2 \]

\[ k_1 = -i (k_0^2 + (j_{m\mu}^{\prime})^2) / (\omega M + \beta^2 k_0) (1 - (m/j_{m\mu}^{\prime})^2) \]

where $k_0^+$ and $k_0^-$ correspond to right-running and left-running waves, respectively. The square root in $k_0$ is positive real (cut-on) or negative imaginary (cut-off).

Note that the sequence $\{j_{m\mu}^{\prime}\}$, $\mu=1,...$ diverges to infinity and therefore we have always a finite number of cut-on modes. Furthermore, since $j_{m\mu}^{\prime} > m$ (except for $m = 0$ where $j_{01}^{\prime} = 0$) we have for low enough frequency (typically $\omega < m$) only cut-off modes. This observation is used in practice to eliminate the first harmonic of the fan noise (Tyler and Sofrin [7]). This works, however, only for hard-walled ducts. We shall see below that the occurrence of surface waves for impedances near the negative imaginary axis may spoil this mechanism largely.

4.4 Behaviour for $Z \to 0$

The approximations of $k$ for $Z \to 0$ of (4.2) are based on perturbations around $\gamma = j_{m\mu}^{\prime}$, the $\mu$-th positive zero of $J_m^{\prime}$, corresponding to the solution for $Z = 0$, with, in addition, the solution $k = \omega / M$ (the surface waves; here two in number and coinciding for $Z = 0$).

The results obtained are

\[ k = k_0 + Z k_1 + \ldots \]
where \( k_0^+ \) corresponds to a right-running and \( k_0^- \) to a left-running wave. The square root in \( k_0 \) is either positive real or negative imaginary. We observe here a similar cut-on, cut-off behaviour at \( Z = 0 \) as at \( Z = \omega \).

A difference is that since \( m \leq j \mu < j \beta \mu \), there is for any \( m \) (including \( m = 0 \)) a positive frequency below which the mode is cut-off. However, (see section (4.5)) if \( \lambda = 0 \) we have now two surface waves cut-on for any \( m \) or \( \omega \). (Anticipating the section on double eigenvalues, we note here that for \( Z = 0 \) the eigenvalue \( k = \omega/M \) is a double eigenvalue for which only one eigenfunction is of the form \( J_m(\gamma r)\exp(\pm im\theta - ikx) \); the other is degenerated into a slightly different form).

We proceed now with the results for \( k \) near \( \omega/M \). Although these modes are surface waves, and more or less covered by section 4.2, we include them here in view of the relatively simple form. We recall, however, that the limit \( Z \rightarrow 0, \delta \rightarrow 0 \) near \( k = \omega/M \) is non-uniform, related to the critical layer at \( \omega - kM = 0 \), and the results are to be interpreted carefully. In any case we must have \( \delta \ll |Z| \).

\[
\begin{align*}
  k &= k_0 + Z^{\frac{1}{2}} k_1 + \ldots \\
  k_0 &= \omega/M \\
  k_1^\pm &= \pm (\omega/M)[iI_m'(\omega/M)/I_m(\omega/M)]^{\frac{1}{2}}
\end{align*}
\]

where the solution with \( |\text{Re} \ k| > |\omega/M| \) corresponds to the hydrodynamic surface wave (the instability), and the other one corresponds to the acoustic surface wave. As discussed before, the instability will not be treated as an unstable (left-running, if \( M < 0 \)) but evanescent (right-running) mode. Note that if \( Z \) is negative imaginary, \( k \) is just real, in which case we could have counted the instability (neutrally stable now) in the correct direction without too much numerical problems. This is not done, however, since then the transition of the solution for \( \text{Re} \ Z \)
from zero to nonzero would be discontinuous, making the damping contour plots in the Z-plane (see the sections below) unnecessarily irregular near the imaginary Z-axis.

4.5 Imaginary Z

Although not as much explicit results for (4.2) can be derived with general imaginary Z as with Z + 0 or Z + ∞, yet there are some interesting points to be noted.

If we write $Z = iX$ (X real), eq. (4.2) becomes

$$(\omega-Mk)^2 J_m(\gamma) + \omega X J'_m(\gamma) = 0$$

This equation allows real k, real $\gamma$ solutions (cut-on acoustic modes) if $\omega$ is not too small. More explicit results may be obtained using the equivalent equation

$$\gamma J_{m+1}(\gamma)/J_m(\gamma) = m + (\omega-Mk)^2/\omega X.$$ 

An exhaustive classification is possible if $M = 0$: there are real $k$, real $\gamma$ solutions if

- $X > 0$ or $X < -\omega/m$, (i) $\omega > j_{m,1}$
- $X = 0$, $\omega \geq j_{m,1}$
- $X = -\omega/m$: always at least the solutions

$$k = \pm \omega, \quad \gamma = 0 \text{ (the eigenfunction requires a limit process).}$$

- $-\omega/m < X < 0$: (i) $\omega > j_{m+1,1}$
- $\omega < j_{m+1,1}$ and $\omega J_{m+1}(\omega)/J_m(\omega) > m + \omega/X$.

(Use is made of the local monotony of $\gamma J_{m+1}(\gamma)/J_m(\gamma)$ and the ordering $j_{m,1} < j_{m+1,1} < \ldots$ ; [27].)
A similar analysis can be made for propagating surface waves: \( k \) real, \( \gamma \) negative imaginary \((-k(l-M) < \omega < k(l+M))\). If we write \( \gamma = -i\sigma \) (\( \sigma \) real) then

\[
\frac{\sigma I_{m+1}(\sigma)/I_m(\sigma)}{= -m - (\omega-Mk)^2/\omega X.}
\]

Since

\[
\frac{d}{d\sigma} \left( \frac{\sigma I_{m+1}(\sigma)/I_m(\sigma)}{= 2 \int_0^\infty I_m^2(x)dx/\sigma I_m^2(\sigma) > 0} \right)
\]

and

\[
\frac{\sigma I_{m+1}(\sigma)/I_m(\sigma)}{= \sigma^2/2(m+1) \text{ for } \sigma \to 0}
\]

\[
I_m(\sigma) = e^{\sigma/\sqrt{2\pi\sigma}} \text{ for } \sigma \to \infty
\]

is \( \sigma I_{m+1}(\sigma)/I_m(\sigma) \) a monotonically increasing function with linear behaviour for \( \sigma \to \infty \). So, taking into account the quadratic behaviour for \( \sigma \to \infty \) of the factor \((\omega-Mk)^2\), we see readily that there is at least one solution if

\[
X < -\frac{\omega/m}{(l+|M|)^2}
\]

and no solutions if \( X > 0 \). If \( M = 0 \) we have the sharper result that there are exactly two solutions if \(-\omega/m \leq X < 0\) and no solutions otherwise.

Up to now we have only considered the possibility of real \( k \) solutions. We shall now prove that real \( k \) solutions can only occur for imaginary \( \Omega \).

Consider

\[
X = -\frac{(\omega-Mk)^2 J_m(\gamma)}{\omega \gamma J_m^1(\gamma)}
\]

If \( k \) is real, \( \gamma = ((\omega-Mk)^2 - k^2)^{1/2} \) is either real or imaginary. If \( \gamma \) is real, \( X \) is evidently real too. If \( \gamma \) is imaginary, we have

\[
J_m(\gamma)/\gamma J_m^1(\gamma) = I_m(\gamma)/\gamma I_m^1(\gamma)
\]

real, and so \( X \) is real.
With this theorem we can now construct an easy criterium to distinguish left- and right-running modes if k is real. As noted before, we have left- and right-running waves if \( \text{Im} \ k > 0 \) and \( \text{Im} \ k < 0 \), respectively, but no distinction was made yet if \( \text{Im} \ k = 0 \). Since we now know that this only happens if \( Z \) is imaginary, we only have to determine from which side \( k \) approaches the real axis when \( Z \) becomes imaginary. This amounts to investigating the sign of \( \frac{dk}{dZ} \).

By differentiating the defining equation (4.2) to \( Z \), we obtain

\[
\frac{dk}{dz} = -i\omega \gamma^2 (\omega - Mk)^2 / \left( (i\omega Z)^2 (\gamma^2 - \beta^2) (M\omega + \beta^2 k) + 2i\omega Z \gamma^2 (\omega - Mk) \right) + (\omega - Mk)^4 (M\omega + \beta^2 k) \cdot 
\]

If we substitute \( Z = iX \) the criterium for left- and right-running becomes

\[
\text{Im}(\frac{dk}{dZ}) > 0 \quad \text{for left-running modes.} \\
\text{Im}(\frac{dk}{dZ}) < 0 \quad \text{for right-running modes.}
\]

If \( \frac{dk}{dZ} = 0 \), \( k \) is a double eigenvalue which we will avoid anyway (see the next section).

An interesting aspect of the present modes with imaginary \( Z \) is their radial energy dissipation into the soft wall (or possibly the boundary layer region along the wall). For convenience we will consider only single modes. (Since energy is non-linear the energy of a sum of modes is not (necessarily) the sum of their energy.) Since radially we have no convection terms, the acoustic intensity is just ([25])

\[
I_r = \frac{1}{2} \text{Re}(p^*v_r) = \frac{1}{2} \text{Re} \left[ i\gamma J_m(\gamma x) J_m^*(\gamma x) (\omega - Mk)^{-1} \exp(-i(k - k^*)x) \right] .
\]

which becomes at \( r = 1 \) (use eq. (4.2))

\[
I_r = -(M/\omega X) \text{Im}(k) |J_m(\gamma)|^2 \exp(2 \text{Im}(k)x)
\]

If the mode is cut-on (\( \text{Im}(k) = 0 \)), \( I_r = 0 \) and no energy is exchanged; if \( M = 0 \) this is true for all modes. Remarkable is, however, that for \( M \neq 0 \)
the cut-off modes (Im(k) = 0) may both dissipate energy into and extract energy from the impedance wall or boundary layer. Depending on the sign of $M/X$, the left-running modes dissipate and the right-running modes extract, or vice versa. Although this creation of acoustical energy is unusual, it is not in conflict with any law of energy conservation. It is known that acoustical energy in a mean flow is only conserved in irrotational and isentropic regions (for example, [32]), and regions of vorticity (here the boundary layer) may act as sources, with their energy being supplied by the mean flow (see eq. (3.11)). Other examples of acoustic energy generation are found in [31] and [21].

4.6 Double eigenvalues

Up to now we only considered equation (4.2) for given values of $Z$, yielding a discrete infinite sequence $\{k_m\}$ of solutions. For certain, more or less accidental values of $Z$, however, it is possible that two of these solutions coalesce, thus becoming a double eigenvalue. They may be found by requiring also the derivative to $k$ of (4.2) to vanish. Since we have more parameters available ($\omega, M$) it is even possible to construct triple or quadruple eigenvalues (Zorumski and Mason [33]).

At first sight, two eigensolution unite into one, and the dimension of the solution space decreases by one. This is, however, against very basic mathematical arguments of continuity, and, indeed, not the case.

What happens is that the second eigensolutions is present but with a different form, which may be obtained by taking some appropriate limit (for example, differentiation to $k$). The two eigensolutions corresponding to a double eigenvalue $k$ are then

$$p_1 = J_m(\gamma r) \exp(\pm i\omega k x)$$

$$p_2 = [- (\omega M + \delta^2 k) r J_m'(\gamma r) / \gamma - ix J_m(\gamma r)] \exp(\pm i\omega k x)$$

Note the linearly diverging behaviour in $x$, compensated by a decreasing exponential. In spite of the occurrence of this linear term in the eigensolution, the corresponding impedance is in some sense a local optimum (Cremer's optimum, [34, 33]), however, for a much more simplified model than ours here.
In view of their rare occurrence, and the complicated and exceptional behaviour of their eigensolutions, we will not consider these double eigenvalues here further, and assume that an arbitrarily picked $Z$ has chance zero of producing a double eigenvalue. (The exception is of course $Z = 0$ with $k = \omega / M$, but this solution was already excluded as such.)

TRANSMISSION THROUGH THE OPEN-ENDED, THREE-SECTIONED DUCT

5.1 General

In this chapter we describe a procedure to calculate the sound field in the various sections of the duct. The method is essentially iterative, and similar to the one described in reference 35 for varying ducts without mean flow. The approach takes advantage of the fact that usually the sound waves propagate down the duct with relatively little reflection. Mathematically one might say that, although the problem is, strictly speaking, elliptic (all field points are coupled), it is nearly parabolic in $x$-direction (the field points are only dependent on points closer to the source). The iteration then is to correct for the difference between the actual elliptic problem and the parabolic approximation (i.e., to take into account the reflection effects).

Globally, the strategy is as follows. We start at the source ($x = 0$). Then calculate for the given incident source field the reflected and transmitted waves at $x = x_1$. The reflected field is ignored, and the transmitted field is taken as the incident field at $x = x_2$. Of the resulting field after scattering at $x = x_2$, the reflected part is stored for later use, and the transmitted part is taken as incident field at $x = L$. Of this field the reflection at $x = L$ is then calculated. This completes the first iteration step. The next steps are the same, except for the incident fields taken at $x = x_1$ and $x = x_2$. Now also the left-running waves calculated in the previous iteration step are included as incident field at $x = x_1$ and $x = x_2$ (at $x = x_1$: the reflection from $x = x_2$, and at $x = x_2$: the reflection from $x = L$). The process is repeated until a converged solution is obtained, with the power at $x = L$ as the norm applied.

So the process consists of two parts, to be discussed in the next sections: (1) calculation of the scattered outgoing field due to a given incident field at $x = x_1', x_2', L$; (the reflection at $x = L$, the open end,
is described in [22]; (ii) the organization, including coordinate shifts
from interface to interface, for the entire iteration.

Some authors ([36, 8, 9]) construct a (so-called) transfer matrix,
to connect directly the modal amplitudes at one end of the duct with the
other. The matrices describing the transmission and reflection at \( x = x_1 \),
\( x_2 \), \( \ldots \) are taken together, to build one matrix for the combined effect of
all. In this way all the left- and right-running modes are linked, in both
directions. This, indeed, reflects the elliptic nature of the problem.
The NLR duct acoustic model previous to the present one ([2, 37]) was also
based on this approach.

Although this method may, at first sight, seem to be attractive (it
is compact and direct), it is actually inherently unstable, and inherently
unable to converge to the complete solution with increasing number of modes.
Formulated this way, the acoustic problem is ill-posed. The reason for
this poor behaviour is the following.

The sound field at one interface (say \( x = x_1 \)) is connected to the
field at another interface (say \( x = x_2 \)) via the modes valid in that
section. From \( x_1 \) to \( x_2 \), a mode changes, evidently, through
the exponential part \( \exp(-ik_{m\mu}x) \). Then, if the mode is attenuated or
cut-off, the effective amplitude varies exponentially. These exponential-
ly large and small terms appear in the system of matrix equations, to be
solved for the combined complete-duct matrix. As soon as we try to
increase the number of modes, and include cut-off or too strongly
attenuated modes, these matrices become very rapidly ill-conditioned,
preventing an accurate numerical solution.

After all this criticism, we can, however, report that in practice
things do not work out always that badly. If we have a reasonable number
of only mildly attenuated modes, the results are in general not very
different (at least, on a dB scale) from the converged solution. But if
the incident wave in the hard-walled section is already cut-off (rotor-
alone noise, the first harmonic of rotor-stator interaction), the transfer
matrix method is definitely useless; furthermore, "tuning" the method for
each impedance on its number of allowable modes is rather cumbersome, and
introduces irregularities in the results which may mask physical trends.
5.2 Mode matching at an interface

In this section we determine, formally, the resulting field of a given sound field incident to a duct cross section at \( x = x_1 \) or \( x = x_2 \), where the boundary condition discontinuously changes, and the modes take another form. The method we follow comes down to the construction of a matrix equation, by using the condition of continuity of pressure and axial acoustic velocity at the interface, and projecting the solutions at both sides of the interface to the same, suitable, basis of test functions; in this case the eigenfunctions of the hard-walled ducts. These eigenfunctions have the advantage of forming an almost orthogonal set, making some of the matrices diagonal dominant. (If the boundary layer thickness is zero, they are exactly orthogonal).

We start with \( x = x_1 \); the other transition at \( x = x_2 \), will be similar. Consider the amplitude vectors \((A^I), (B^I)\) of the solution in \( x \leq x_1 \), and \((A^{II}), (B^{II})\) for \( x \geq x_1 \). Assume the origin shifted to \( x = x_1 \), so, for example, \( A^I_m = A_m \exp(-ik^I_m x_1) \). Incident modes \((A^I)\) and \((B^I)\) are given, and \((A^{II})\) and \((B^{II})\) are to be found.

Continuity of pressure and axial velocity yields

\[
\sum_{\mu=1}^{\infty} A^{II}_{m \mu} M^+_{m \mu} J_m (\gamma^+_{m \mu} r) - B^{II}_{m \mu} N^-_{m \mu} J_m (\gamma^-_{m \mu} r) =
\]

\[
\sum_{\mu=1}^{\infty} A^I_{m \mu} N^+_{m \mu} J_m (\gamma^+_{m \mu} r) - B^I_{m \mu} M^-_{m \mu} J_m (\gamma^-_{m \mu} r) =
\]

\[
\sum_{\mu=1}^{\infty} A^{II}_{m \mu} \lambda^+_{m \mu} M^+_{m \mu} J_m (\gamma^+_{m \mu} r) - B^{II}_{m \mu} \kappa^-_{m \mu} N^-_{m \mu} J_m (\gamma^-_{m \mu} r) =
\]

\[
\sum_{\mu=1}^{\infty} A^I_{m \mu} K^+_{m \mu} N^+_{m \mu} J_m (\gamma^+_{m \mu} r) - B^I_{m \mu} \lambda^-_{m \mu} M^-_{m \mu} J_m (\gamma^-_{m \mu} r)
\]

where \( \kappa_{m \mu} = k_{m \mu}/(\omega - M_k_{m \mu}) \), and \( M_{m \mu}, \gamma_{m \mu}, \) and \( \lambda_{m \mu} \) correspond to \( N_{m \mu}, \gamma_{m \mu}, \) and \( \kappa_{m \mu} \) in \( x > x_1 \).

Now multiply left- and right-hand side with \( r M^+_{m \mu} J_m (\gamma^+_{m \mu} r) \), and integrate from \( r = 0 \) to \( 1 \). This gives the linear system of equations

\[
C \begin{pmatrix} A^{II} \\ B^{II} \end{pmatrix} = D \begin{pmatrix} A^I \\ B^I \end{pmatrix}
\]
which is (formally) solved by inversion of $C$, to give the scattering matrix $T_1 = C^{-1}D$. The matrices $C$ and $D$ are built up by four submatrices.

$$
C = \begin{pmatrix}
c^+ & -a^-
d^+ & -b^-
\end{pmatrix}
$$

$$
D = \begin{pmatrix}
a^+ & -c^-
b^+ & -d^-
\end{pmatrix}
$$

The elements of these submatrices are given by

$$
a_{\mu\nu} = N_{\mu\nu}^+ N_{\nu\mu} \left[ \gamma_{\mu\nu} J_m(\gamma_{\mu\nu}^+) J_m'(\gamma_{\mu\nu}) - \gamma_{\mu\nu}^+ J_m(\gamma_{\mu\nu}) J_m'(\gamma_{\mu\nu}^+) \right] \left[ (\gamma_{\mu\nu}^+) - (\gamma_{\mu\nu})^2 \right]
$$

if $\gamma_{\mu\nu} = \gamma_{\mu\nu}^+$

$$
= \frac{1}{2} \left( N_{\mu\nu}^+ \right)^2 \left[ (1-(m/\gamma_{\mu\nu})^2 J_m(\gamma_{\mu\nu})^2 + J_m'(\gamma_{\mu\nu})^2 \right]
$$

if $\gamma_{\mu\nu} = \gamma_{\mu\nu}^+$

$$
c_{\mu\nu} = N_{\mu\nu}^+ M_{\mu\nu} \left[ r_{\mu\nu} J_m(\gamma_{\mu\nu}^+) J_m'(\gamma_{\mu\nu}) - \gamma_{\mu\nu}^+ J_m(\gamma_{\mu\nu}) J_m'(\gamma_{\mu\nu}^+) \right] \left[ (\gamma_{\mu\nu}^+)^2 - (r_{\mu\nu})^2 \right]
$$

$$
b_{\mu\nu} = \kappa_{\mu\nu} a_{\mu\nu},
$$

$$
d_{\mu\nu} = \lambda_{\mu\nu} c_{\mu\nu}.
$$

(For brevity, the superscripts $+$ or $-$ are suppressed where possible.)

A similar operation can be performed for the transmission problem at $x = x_2$. If we assume the boundary layer thickness $\delta$ the same in $(0, x_1)$ as in $(x_2, L)$, the problem is, apart from symmetry, the same. If we call the solution amplitude vector at $x = x_2$ for $x \leq x_2$ $(A^{III})$ and $(B^{III})$, and for $x \geq x_2$ $(A^{IV})$ and $(B^{IV})$, we arrive at the equation

$$
D \begin{pmatrix} A^{IV} \\ B^{III} \end{pmatrix} = C \begin{pmatrix} A^{III} \\ B^{IV} \end{pmatrix}
$$
with C and D as given before, and yielding the scattering matrix \( T_2 = D^{-1}C \).

For completeness we recall, that in [22] we described the reflection at the open end \( x = L \), giving for a solution amplitude vector at \( x = L \) \((A^V)\) and \((B^V)\) the equation

\[
(B^V) = R(A^V)
\]

where \( R \) is the open-end reflection matrix.

### 5.3 Iterative procedure through the sections

In this section we give some details of the iterative procedure as described in section 5.1. These details are of minor importance theoretically, but, of course, crucial for a correct computer program. Therefore, we describe the various steps here briefly.

To start with, we introduce some notation. At each interface we deal with effective amplitudes, that is, the amplitudes as if the origin were shifted to that interface position. In addition, these amplitudes belong to a modal expansion at the left- or at the right-hand side of the interface. The following variables are fixed through the iteration:

- \( \hat{A} \) the amplitude vector of the source at \( x = 0 \)
- \( (k_{m\mu}^+) \) the set modal wave numbers in the hard-walled sections
- \( (\lambda_{m\mu}^+) \) the set modal wave numbers in the soft-walled section.

Apart from these, we have the quantities which are determined during the iteration:

- \( \hat{B}_{10} \) the (left-running) amplitudes at \( x = 0 \)
- \( \hat{A}_{11}, \hat{B}_{11} \) the shifted amplitudes at \( x = x_1^- \)
- \( \hat{A}_{21}, \hat{B}_{21} \) \( x = x_1^+ \)
- \( \hat{A}_{22}, \hat{B}_{22} \) \( x = x_2^- \)
- \( \hat{A}_{32}, \hat{B}_{32} \) \( x = x_2^+ \)
- \( \hat{E}^+, \hat{E}^- \) \( x = L \)

Step by step, the procedure is as follows:
(1) \( \hat{A}_{11} = \hat{A} \exp(-ikx_1) \)
\( \hat{B}_{22} = \hat{B} \)
\( \hat{E}^- = \hat{E} \)

(2) \( \hat{B}_{21} = \hat{B}_{22} \exp(-i\lambda^-(x_1 - x_2)) \)
\[ \begin{pmatrix} \hat{A}_{21} \\ \hat{B}_{11} \end{pmatrix} = T_1 \begin{pmatrix} \hat{A}_{11} \\ \hat{B}_{21} \end{pmatrix} \]

(3) \( \hat{A}_{22} = \hat{A}_{21} \exp(-i\lambda^+(x_2 - x_1)) \)
\( \hat{B}_{32} = \hat{E}^- \exp(-ik^-(x_2 - L)) \)
\[ \begin{pmatrix} \hat{A}_{32} \\ \hat{B}_{22} \end{pmatrix} = T_2 \begin{pmatrix} \hat{A}_{22} \\ \hat{B}_{32} \end{pmatrix} \]

(4) \( \hat{E}^+ = \hat{A}_{32} \exp(-ik^+(L-x_2)) \)
\( \hat{E}^- = \hat{R} \hat{E}^+ \)

(5) return to (2) unless the power at \( x = L \) does not change anymore.

Step (1) sets the starting values, while the iteration goes from (2) to (5). It is seen that \( \hat{A}_{21} \), calculated in (2), is carried over to (3). The same goes, for \( \hat{A}_{32} \), calculated in (3) and used in (4). The reflected parts \( \hat{B}_{22} \) and \( \hat{E}^- \) are stored, and used always one iteration cycle later.

6 NUMERICAL EXAMPLES

Based on the present theory a computer program has been developed (Hofstra [24]), calculating eigenvalues (axial wavenumbers), the entire sound field in the duct, and the obtained damping per impedance \( Z \) for given source and duct geometry. Some test results and examples are given below.

The practical motivation for the present study and its associated computer program is to have available a tool to calculate the impedance of the lined section \((x_1, x_2)\) with the highest attenuation. What is to be meant with "highest attenuation" is of course yet to be defined, and indeed dependent on the application. In the examples to be given below we
will consider optimum impedances based on minimum radiated power (~I), but maximum dissipated power (~TL) and minimum ESPL (a simplified EPNL; see [24]) are criteria possible as well. In these cases the duct geometry and source are assumed to be fixed and given. If, in practice, we do not know the source in every detail, and mean flow and geometry differ to some extent from reality, we will have to apply some statistics, and consider a certain variety of possible cases. For example, one could take an average on randomly chosen sources with all an equal energy distribution over the modes. More realistic, of course, is to vary over sources originating from rotor-stator interaction calculations ([3]).

In figure 3 we have the (very severe) consistency test of transmission loss (TL) for \( M = 0 \) and imaginary impedances, versus increasing number of included modes \( (\mu_{\text{max}}) \). In this case there is no energy dissipation at all, and so \( TL = 0 \). We see indeed for increasing \( \mu_{\text{max}} \) a TL rapidly converging to \( 0 \).

Of the same configuration the insertion loss \( I_L \) is plotted in figure 4. It clearly converges for \( \mu_{\text{max}} \to \infty \) but no longer to zero now. This is because the IL includes reflection effects which occur at any discontinuity. Interesting is the possibility of a negative IL for some impedances. Probably this is due to the surface waves, increasing the effective number of cut-on modes. If the hard-wall sound field is cut-off, this effect of a negative \( I_L \) is very strong, as we will see.

Similar convergence tests are given in figure 5 (TL) and figure 6 (IL) for \( M = -0.3 \). The lack of energy conservation with mean flow is clear. All sequences converge for increasing \( \mu_{\text{max}} \), although sometimes at least 16 modes are required, which is considerably more than the two modes (cut-on in the hard-walled section), used in a transfer matrix approach. It is interesting that now for almost all impedances the TL is negative and on the other hand the IL is positive. Apparently, the generation of sound (\( TL < 0 \)) is not sufficient to compensate for the large(r) reflections at the liner ends.

A check on the continuity of the field at the interfaces \( x = x_1 \) and \( x = x_2 \) is provided by figures 7.a-b where the sound pressure level (SPL = \( 20 \log(|p| - \text{reference}) \)) is plotted just in front of and just behind the interface. The matching is seen to be very good. Only near the wall (\( r = 1 \)) a slight difference is notable, but this is just to be expected in view of the singular behaviour of the (exact) solution near the impedance discontinuity ([38]). As a reference, figures 8.a-b show the field at the
interfaces in the absence of a liner (all is hard-walled). The field with liner is not a small modification of the field without liner, and hence the found continuity of the field is genuine, and not a result of the smallness of the effect.

After these examples, testing the internal consistency of the modelling, we compare in figures 9.a-c, 10.a-b with other work. No examples are available in the literature for the complete problem, as we modelled it. However, there are experiments and calculations by Wyerman [39] for a three-sectioned circular duct (hard-soft-hard) with an anechoic termination, and without mean flow. In figure 9 the radial shape of the sound pressure level is given for some sources and impedances, and the agreement is unquestionable. Where our results deviate from Wyerman's calculations (which is reasonable since he used a transfer matrix with a limited number of modes), ours always tend to the measured values. The insertion loss (IL) and transmission loss (TL) comparison in figure 10 also show an almost perfect agreement.

After these test cases we finish this chapter with a discussion of a series of typical examples. First, we consider the behaviour of the axial wavenumbers \( k_{m\mu} \) in the complex plane for variations of impedance \( Z \) along paths \( \text{Re}(Z) = \text{constant} \). Then, we present some plots of constant-power contours in the \( Z \)-plane (for a given duct geometry and sound source), displaying this way readily the location and size of the area in the \( Z \)-plane with highest damping (IL), and the corresponding attenuation there attainable.

In figures 11.a-d we have axial wavenumbers with \( \omega = 10, m = 0, M = 0 \) for \( \text{Re}Z = 0, 1, 2 \) and 3. In figure 11.a, where \( \text{Re}Z = 0 \) we see that for increasing \( \text{Im}Z \) all eigenvalues move up one position, in a way that the imaginary part decreases or the real part increases (absolutely), where the first modes \( (\mu = 1) \) become unattenuated surface waves, disappearing \( (k + \infty) \) for \( \text{Im}Z > 0 \). For \( \text{Re}Z > 0 \) (Fig. 11.b, c, d) these surface waves return to finite values for \( \text{Im}Z + \infty \), and now only a finite number of the eigenvalues move up, and the others return to their own (hard wall) start value. When \( \text{Re}Z \) increases, the loops become smaller, until finally all the eigenvalues return to their own start value.

In figures 12.a-b we have a similar behaviour for the case \( \omega = 10, m = 10, M = 0 \), with all modes cut-off at \( Z = \infty \). Now, the first mode starts as cut-off, and, when we follow it along \( \text{Re}Z = 0 \), it turns into cut-on \( (k = 0) \) when \( \text{Im}Z \) passes the value \( -J_m(\omega)/J'_m(\omega) = -2.459 \). Then, this
mode becomes a surface wave when $\text{Im } Z$ passes $-\omega/m = -1$, and disappears ($k = \infty$) when $\text{Im } Z$ passes 0.

Figures 13.a-f show some mean flow effects ($M = -0.5$; $\delta$ is taken zero for clarity). Figure 13.a ($\text{Re } Z = 0$) shows a number of interesting points. Instead of only two for $M = 0$ we see now four surface waves, the acoustic (quadrant II and IV) and the hydrodynamic (quadrant I and III). Furthermore, the surface wave eigenvalues are now not always real, but are also found on an egg-shaped contour, similar to the one in [19]. When $\text{Im } Z$ increases from $-\infty$ to $\infty$, the eigenvalues of the hydrodynamic surface waves come, along the real axis, from infinity inward, whereas (above some negative value of $\text{Im } Z$) the acoustic surface wave eigenvalues go outward along the real axis. When hydrodynamic and acoustic eigenvalues meet, one moves up and one down (following the egg-shaped contour) and return, for some finite value of $\text{Im } Z$, to the "normal" acoustic modes, somewhere near $\text{Im } k = \pm \omega/M$. Like with $M = 0$, all the eigenvalues move up one position, but in a more complicated way. All the eigenvalues inside the egg tend to move up/right ($\text{Im } k < 0$) or down/left ($\text{Im } k > 0$), as for $M = 0$, but the more cut-off modes outside the egg (approx. $|\text{Im } k| > \omega/|M|$) move away from the real axis.

Another interesting point in figure 13.a is the wiggling behaviour of the cut-off eigenvalues, whereas the cut-on modes follow a straight path. Finally, we added for the surface waves their boundary layer correction for $\delta = 0.01$. For these modes the effect is rather strong. This may be explained from the fact that a relatively bigger part of the surface wave is immersed in the boundary layer than is the case for the other modes.

In the other figures (13.b-f) the real part of $Z$ is stepwise increased, and we see that the hydrodynamic surface wave loops diverge to infinity, while the acoustic surface wave loops decrease and finally collapse.

A sequence, similar to figures 13, is given in figures 14.a-d, but now with $\omega = 10$, $m = 10$, $M = 0.5$. So for $Z = \infty$ we start with all modes cut-off. Although the frequency $\omega$ is not high compared to $m$, yet the basic behaviour of the four surface waves remains clearly present.

Figure 15.a-b show a result of the complete model, for $\omega = 40$, $m = 20$. Contours of constant inlet power level are plotted in the impedance plane, for a source with all amplitudes of constant modulus and a random phase, without (15.a) and with (15.b) mean flow. (Note that, owing to the linearity of the problem, the given power levels have only a relative
meaning.) All impedancies give a reduction, and a clear optimum is found at \( Z = 2.5 \ (M = 0) \) and at \( Z = 4.3 + 0.3i \ (M = -0.3) \) with in both cases a considerable reduction of about 18 dB. This behaviour (an optimum around 3, a considerable reduction) seems to be typical for high-frequency sources with \( \omega/m \approx 2 \). For higher values of \( \omega/m \) the attainable attenuation appears to become poor, and the optimum is then sensitive to details of the source and geometry. If we decrease this parameter \( \omega/m \) to a value of one, the hard-wall sound field is cut-off (for example: rotor alone noise), and then the application of a liner has much more dramatic consequences.

This case is shown in figure 16.a-b (\( \omega = 20, \ m = 20 \)). Centred at the point along the imaginary axis where (abruptly) propagating surface waves appear (\( Z \approx 0 \) and \( Z \approx -1 \)), the contours form a fan-shaped bundle with which the attenuation decreases from high values for \( \text{Im} \ Z > 0 \) to negative values along \( \text{Im} \ Z < 0 \). The application of a liner may both reduce the power level with more than 120 dB, as well as increase it by a similar amount. Responsible for this increase are obviously the surface waves, carrying the sound energy unattenuated along the liner for \( \text{Im} \ Z < 0 \), \( \text{Re} \ Z = 0 \).

The most important conclusion to be drawn from figures 16.a-b is probably that the common reasoning, that, if the first harmonic is cut-off, a liner has to be designed for the second harmonic only, is not always correct. If the liner, optimized for the second harmonic, yields an impedance, for the first harmonic, in a region of \( \text{Im} \ Z < 0 \) too close to the imaginary axis, the anticipated benefit of the cut-off mechanism may be completely spoiled, and another liner that attenuates both first and second harmonic, should be chosen.

Finally, we end with two practical examples (Figs. 17, 18), displaying generally the behaviour shown in figure 15.b and figure 16.b.

DISCUSSION AND CONCLUSIONS

In the present report we have described a model for the calculation of sound transmission through a three-sectioned cylindrical duct with mean flow. The three sections consist of two with hard walls, and one in between with impedance type walls. Together with the solution of the open-end radiation problem, described elsewhere [22, 23], the model applies to the sound radiation problem of aircraft turbo-engine inlets.
Furthermore, the model is meant to be used in combination with the source model, described in [3], although any source may be considered as input.

The model assumes a cylindrical duct with an almost uniform mean flow. These are widely used assumptions, modelling the first order effects correctly ([36, 9, 2, 37, 39, 16]). The present approach displays a number of deep problems very well. (i) The highly singular behaviour of the elliptic equations for high frequencies (or rather, high wave number: $k_{m}$, $\gamma_{m}$, or $m$), giving a strong coupling between spatially well separated points. This is solved here by taking the back reaction always one iteration step behind. (ii) The completeness of our basis of eigensolutions, which amounts to finding all the eigenvalues. This problem (often mentioned as crucial in the literature) is solved here by a sharp classification between acoustic modes and surface wave-type modes, bringing order in apparent disorder, and making a numerical search easier. (iii) Causality of the solution. Sometimes, depending on the problem parameters, the type of liner ($Z = Z(\omega)$), and the applied causality condition (!), one of the modes is really a downstream running instability, rather than an upstream running decaying mode. At the same time an additional condition at the liner's edges is required (Kutta condition, or likewise) ([19]). If this instability is really present, some sort of saturation mechanism should be considered to prevent this mode of exploding exponentially and spoiling the rest of the field ([18]). In view of the intricate and puzzling aspects of this last problem (iii) both physically and mathematically, we have postponed the solution (possible in principle, but still complicated to implement). So we assume for the moment the role of the instability only important in certain special cases (for example, if $Z = 0$), and accept an, in certain cases, non-causal solution, with a liner edge singularity less smooth than in reality present.

It is clear that these phenomena will pop up in any approach (numerical or otherwise) of the problem, in a more or less disguised form. So a further understanding is important. Areas of further research, with a clear prospect on progress, are a study of the double limit $\delta \to 0$, $Z \to 0$, the type of singularity at a liner edge in case of a finite boundary layer thickness, the critical layers in the boundary layer, exact Wiener-
Hopf solutions to further clarify the issue of causality, instability and edge condition, the importance of (parts of) a sound field with radially a typical length scale of the order of (or smaller than) the boundary layer.

Apart from these more local aspects of the problem, considerable progress can be made with respect to the, more global, aspect of the geometry of the duct, along the present lines of approach. Following Nayfeh c.s. [40, 41, 42], who used, however, incorrect equations, we have found recently [43] the (correct) solution of sound propagation in slowly varying ducts and this solution appears to be expressible in explicit, analytic form. This solution is thus relatively easily incorporated into the present model.

Furthermore, one of the most striking effects, shown by the numerical examples, is the role of the surface waves in spoiling a cut-off sound field when the liner impedance is suitably chosen in the lower complex half plane. An experimental confirmation of this effect is important, verifying the presence of the surface waves, and giving a practical guide to which extent the first harmonic of a properly designed engine may contribute to the radiated sound field.

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Fig. 1 Duct geometry

Fig. 2 Mean flow profile ($M < 0$)
Fig. 3 Transmission loss TL for increasing number of modes \( \mu_{\text{max}} \) with imaginary impedancies. 

\( \omega = 6, \ m = 1, \ M = 0, \ L = 1.4, \ x_1 = 0.4, \ x_2 = 1.3, \ A_1 = 1, \ A_2 = 1, \ A_3 = \ldots = 0. \)
Fig. 4  Insertion loss IL for increasing number of modes. See fig. 3
Fig. 5 Transmission loss TL for increasing number of modes $\nu_{\text{max}}$, with imaginary impedancies.

$\omega = 6$, $m = 1$, $M = -0.3$, $L = 1.4$, $x_1 = 0.4$, $x_2 = 1.0$, $\delta^* = 0.0$, $A_1 = 1.0$, $A_2 = 1.0$, $A_3 = \ldots = 0.0$. 

$\nu_{\text{max}}$
Fig. 6 Insertion loss IL for increasing number of modes. See fig. 5
Fig. 7a Radial acoustic pressure distribution (SPL) at interface between hard-walled and lined section \((x = x_1)\), verifying continuity of the field.

\(\omega = 6, \, m = 1, \, M = -0.3, \, L = 1.4, \, x_1 = 0.4, \, x_2 = 1.3, \\delta^* = 0., \, \nu_{\text{max}} = 28, \, Z = 0.01, \, A_1 = 1., \, A_2 = 1., \, A_3 = \ldots = 0.\)
Fig. 7b The same as fig. 7a at $x = x_2$. 
Fig. 8a  The same as fig. 7a with $Z = \infty$ (hard wall)
Fig. 8b The same as fig. 7b with $Z = \infty$ (hard wall)
Fig. 9a Comparison with Wyerman's fig. 4.13. Radial pressure distribution at station 1,...,4 for incident (0.1) mode

station 1  △  - - - -  △
station 2  ▼  - - - -  ▼
station 3  ○  - - - -  ○
station 4  □  - - - -  □

reference: station 1
ω = 3.46 (1250 Hz), m = 0, M = 0., L = 33.6, x₁ = 16.8,
x₂ = 21.6, Z = 1.0-11.7, ν₀ = 10, A₁ = 1., A² = . . . = 0.,
anechonic termination, dimensional duct radius 0.1524 m
Fig. 9b  See fig. 9a. Wyerman's fig. 4.15, (0.2) mode.  
\( \omega = 4.4 \) (1600 Hz), \( Z = 1.0 - 1.1 \), \( A_1 = 0 \),  
\( A_2 = 1 \), \( A_3 = 0 \).

Fig. 9c  See fig. 9a. Wyerman's fig. 4.20, (1,1) mode.  
\( \omega = 4.15 \) (1500 Hz), \( Z = 1.0 - 1.3 \), \( m = 1 \)  
\( A_1 = 1 \), \( A_2 = 0 \).
Fig. 10a  Comparison with Wyerman’s fig. 4.48. Transmission loss and insertion loss for (1,1) and (0,2) mode

Wyerman   present theory

TL   ○   ●
IL   □   ■

For duct geometry, see fig. 9a.
ω = 1 corresponds to 360 Hz.
At 500 Hz is Z = 1.0-14.8
1000 Hz is Z = 1.0-12.2
1500 Hz is Z = 1.0-11.3
2000 Hz is Z = 1.0-10.8
2500 Hz is Z = 1.0-10.3

Fig. 10b  Comparison with Wyerman’s fig. 4.49 and fig. 4.50. Insertion loss for (0,1) and (0,2) mode.

Wyerman   △
present theory   ▲

For duct geometry, see fig. 9a and fig. 10a
Fig. 11a  Trajectories of complex axial wave numbers $k$ for variations of impedance $Z$ along $\text{Re } Z = 0$.\textsuperscript{m\textsubscript{1}}

-\textup{v} hard wall value

$\omega = 10, m = 0, M = 0.$

Fig. 11b  See fig. 11a; $\text{Re } Z = 1.$
Fig. 11c  See fig. 11a; Re Z = 2.

Fig. 11d  See fig. 11a; Re Z = 3.
Fig. 12a Trajectories of $k_{\mu \nu}$ for $Re Z = 0$.
\n\n- hard wall value
\n- $\omega = 10$, $m = 10$, $M = 0$.

Fig. 12b See fig. 12a; $Re Z = 1$. 
Fig. 13a Trajectories of $k_{_{m\mu}}$ for Re $Z = 0$.

- Hard wall value; $\delta^* = 0$;
- $\delta^* = 0.001$

$\omega = 10$, $m = 0$, $M = -0.5$

Fig. 13b See fig. 13a; Re $Z = 1$. 
Fig. 13c  See fig. 13a; Re Z = 2.

Fig. 13d  See fig. 13a; Re Z = 3.
Fig. 13e  See fig. 13a; Re Z = 4.

Fig. 13f  See fig. 13a; Re Z = 5.
Fig. 14a Trajectories of $k_{m\mu}$ for $\text{Re} \, Z = 0$.

- Hard wall value
- $\omega = 10$, $m = 10$, $M = -0.5$, $\delta^* = 0$.

Fig. 14b See fig. 14a; $\text{Re} \, Z = 1$. 
Fig. 14c  See fig. 14a; Re Z = 2.

Fig. 14d  See fig. 14a; Re Z = 3.
Fig. 15a  Contours of constant radiated power in Z-plane.
\(\omega = 40, \, m = 20, \, M = 0, \, \mu_{\text{max}} = 20,\)
\(|A_j| = 1, \, \text{phase } (A_j) \text{ random}\)
hard wall value 201 dB

Fig. 15b  See fig. 15a; \(M = -0.3, \, \delta^* = 0.\)
hard wall value 199 dB
Fig. 16a Contours of constant radiated power in Z-plane
$\omega = 20, m = 20, M = 0, \nu_{\text{max}} = 20,$
$|A_x| = 1, \text{phase } (A_x) \text{ random}$
hard wall value 15 dB

Fig. 16b See fig. 16a; $M = -0.3, \delta^* = 0.$
hard wall value 50 dB
Fig. 17 Contours of constant radiated power in Z-plane.
\( \omega = 44, \ m = 26, \ M = -0.32, \ \delta^* = 0.001, \ \mu_{\text{max}} = 18 \)
hard wall value 174 dB

Fig. 18 Contours of constant radiated power in Z-plane.
\( \omega = 22, \ m = 22, \ M = -0.32, \ \delta^* = 0.001, \ \mu_{\text{max}} = 24 \)
hard wall value 86 dB
Fig. 19a Reflectionless impedance (real) for plane waves incident from direction $\theta_o$, $\varphi_o$.

$\varphi_o = 0^\circ$

Fig. 19b See fig. 19a; $\varphi_o = 45^\circ$
APPENDIX A
NON-REFLECTIVE IMPEDANCES; WITH AN APPLICATION TO WIND TUNNELS

A1 Introduction

One restriction, built in the present model, is that the liner should be of constant impedance. The numerical examples discussed in section 6 show impedances, optimum under this condition. Obviously, more attenuation may be obtained if we assume a pointwise varying liner. It is, however, also clear that then the number of possibilities becomes so large that finding the optimum impedance distribution is difficult.

On the other hand, an approach leading to an impedance probably close to the optimum is to construct an (of course source dependent) impedance which produces just no reflection, and therefore absorbs all the incident acoustic energy. The only energy not captured is carried by waves radiating directly out of the open end. As we will show, this non-reflective impedance distribution can be determined in closed form (analytically) if the free field of the source is given.

In the present section we will discuss this approach. Although probably not relevant to the engine inlet problem (where a constant liner is preferred, and the source is not exactly known) it may very well be useful for applications as wind tunnels with acoustic treatment ([44]. In this case the absence of reflections is exactly what is required, more than any other dissipation property.

Two cases will be considered. One is that of relatively high frequency, where we can model the radiated sound waves as (locally one-dimensional) rays, reflecting at a plane surface. Especially when the radiation of the source is highly beamed, and the directivity pattern is concentrated in a few direction, this case is relevant. The other case is the general problem of an arbitrary sound field, given as it were in a free field. Of course, the first (ray) problem is a special case of the second one.

For simplicity we assume a vanishing boundary layer thickness. This is, in principle, not essential if we are able to calculate through the boundary layer, but we have for the moment left it out of the analysis in
view of the already more complicated effective boundary condition for an arbitrarily varying impedance (or wall geometry, or mean flow). The complete condition (for $\delta = 0$) is given by Myers [45], and will be used below.

A2 Ray-acoustical approximation

Consider a local coordinate system ($x$, $y$, $z$) at the wall, with in $y > 0$ the mean flow in positive $x$-direction along the wall at $y = 0$.

From the direction $\hat{x}_0$ (so, with propagation direction $-\hat{x}_0$) we have a plane wave ([31]).

$$p(\hat{x}) = \exp(i\omega(r'D + Mx'/\beta)/\beta)$$

where transformed spherical coordinates ($r'$, $\theta'$, $\phi'$) are given by

$$x = \beta r'\cos\theta' \cos\phi'$$
$$y = r'\sin\theta'\cos\phi'$$
$$z = r'\sin\phi',$$

and

$$D = \cos(\theta'-\theta_0')\cos\phi'\cos\phi_0' + \sin\phi'\sin\phi_0'$$

With the boundary condition

$$(i\omega + M\omega^2)\rho = i\omega Z(\hat{v}\cdot\hat{n})$$

at $y = 0$ (the normal $\hat{n}$ pointing away from the flow region in $-y$-direction), solved for $Z$, we obtain the impedance without reflection

$$Z = (1 + M \cos\theta_0' \cos\phi_0')^2/\beta^3 \sin\theta_0' \cos\phi_0'$$

(A.1)

This result is plotted in figures 19.a-b, for $M = 0$ to 1 with steps of 0.1, as a function of the physical angles $\theta_0$ and $\phi_0$, given by
\begin{align*}
\chi_0 &= r_0 \cos \theta_0 \cos \phi_0, \\
\gamma_0 &= r_0 \sin \theta_0 \cos \phi_0, \\
\tilde{z}_0 &= r_0 \sin \phi_0
\end{align*}

We see that the impedance obtained is always real (pressure and velocity are always in phase), and usually larger than 1.

A3 The general problem

For a given (free field) incident field \((p, \hat{v})\) and a known mean flow field \(\tilde{v}\), the boundary condition (along an arbitrarily shaped impedance boundary; see (see [45]))

\[ i\omega (\hat{v} \cdot \hat{n}) = \left[ i\omega + \tilde{v} \cdot v - \hat{n} \cdot (\hat{n} \cdot \tilde{v} v) \right] (p/Z) \]

is to be considered as a differential equation for \(Z\) as a function of arclength \(\tau\), measured along a streamline at the surface. This is easily seen from the fact that \(\tilde{v} = V \ddot{\tau} \hat{e}_\tau\) at the surface \((V = |\tilde{\tau}|, \hat{e}_\tau\) is the unit vector along a streamline), so \(\tilde{v} \cdot v = V \frac{d}{d\tau} \).

The impedance without reflection is then (\(\sigma\) parametrizes the streamlines)

\[ Z(\tau, \sigma) = \frac{p}{\exp\left( -\int_{\tau}^{\tau'} \left[ \frac{i\omega \cdot \hat{n} \cdot (\hat{n} \cdot \tilde{v} \tilde{v} / V) / V_{\tau} \right] d\tau \right)} \times \int_{\tau}^{\tau'} \left[ \frac{i\omega \cdot \hat{n} \cdot (\hat{n} \cdot \tilde{v} \tilde{v} / V)}{V_{\tau}} \right] \exp\left( \int_{\tau}^{\tau'} \left[ \frac{i\omega \cdot \hat{n} \cdot (\hat{n} \cdot \tilde{v} \tilde{v} / V)}{V_{\tau}} \right] d\tau' \right) d\tau \]  \quad (A.2)

This result simplifies considerably if \(V_{\tau}\) is constant along a straight wall:

\[ Z = \frac{p}{(i\omega / V) \int_{\tau}^{\tau'} (\hat{v} \cdot \hat{n}) \exp\left( -\frac{i\omega / V}{(\tau - \tau')} \right) d\tau'}. \]