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Product recovery policies in stochastic remanufacturing systems with multiple reuse options

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Abstract

In many product recovery situations returned products can be reused in multiple ways. Under these circumstances the problem arises in which quantities reusable items should be allocated to the different remanufacturing options, especially in case of insufficient stock of returns. For this problem a periodic review model is formulated which also includes a disposal option and incorporates uncertainties in returns and demands for the different serviceable options. The structure of the optimal policy is analyzed, and it is shown that under specific allocation rules a near-optimal policy with a simple structure exists. An efficient computational procedure for determination of the optimal policy parameters is presented. This procedure is applied in a numerical investigation that gives interesting managerial insights in important product recovery issues.
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1 Introduction

Recently, more and more manufacturers reuse old products and incorporate product recovery activities in their regular production environment. The reason for this is two-fold. Firstly, reuse of old products, in the form of components, subassemblies or even completely after cleaning and upgrading, can be highly profitable. This is especially the case for complex, high tech products with a reasonably long product life cycle, like copiers and medical equipment. Secondly, legislation aimed at environment-benign production forces manufacturers to take back their products from end-users after they discard them.

Reuse of old products raises a lot of Operations Management problems for manufacturers, starting with collection and ending with sales of products with used components. In this paper we focus on one of these new Operations Management problems. In many situations an old product can be used in different ways, each yielding different costs and profits. E.g. a returned copier can be sold as new after cleaning and software upgrades. Often besides different ways of upgrading also downgrading alternatives or pure restoring are possible options when used products are remanufactured. Additional recovery options like disassembly and reuse of major modules may be reasonable ways to exploit most of the value a used product still contains. These options are mutually exclusive and differ with respect to the costs incurred during the remanufacturing process. The selection of these options is one of the main Operations Management problems associated with remanufacturing.

One main question in this context is how to allocate a possibly limited amount of reusable products to the different remanufacturing options, taking into consideration all relevant cost and demand aspects. It has to be decided which stock level for the different serviceable options should in general be achieved and how the desired level of reusables stock should be fixed. The latter inventory can not only be controlled by remanufacturing decisions, but also by determining to which extent used products should be disposed of in case of excessive returns. The complexity of this Operations Management problem is caused by the fact that the disposal, remanufacturing and allocation decisions have to be made simultaneously and that from both the demand
and the returns side major uncertainties can come into the problem.

In this paper we consider a special class of product recovery problems with multiple remanufacturing options. It is our objective to determine the structure of the optimal periodic review policy for both discounted costs over finite and infinite horizon, and long-term average costs. It is shown that the optimal policy is of a fairly simple structure: Returns are discarded if the echelon stock of returns exceeds a control limit. The echelon stock of returns equals the sum of the stock of returns and the inventory positions of the serviceable options. The stocks of serviceable options are replenished according to order-up-to-policies. In case insufficient returns are available, an allocation policy determines how much is remanufactured of each option. In order to compute near-optimal policies, we apply the so-called linear allocation rules introduced in Diks et al. (1996). Note that we assume that serviceable options cannot be procured from outside suppliers.

To our knowledge this paper is the first to address the stochastic remanufacturing problem with multiple reuse options. Several models exist in literature that describe the single option situation. In Inderfurth (1997) the optimal policy is determined for a periodic review model with remanufacturing and procurement. It turns out that in case procurement lead times and remanufacturing lead times are identical, the structure of the policy is simple. In case the lead times differ the structure of the optimal policy is not obvious. Van der Laan (1997) considers a number of continuous review single option models which include lotsizing and leadtime differences. Assuming different variants of simply structured policies Van der Laan derives expressions for costs and service levels, from which cost-optimal policies within the class of policies considered are derived. For a general overview of Operations Management problems associated with product recovery we refer to Thierry et al. (1995). A review of quantitative models in the field of remanufacturing is given in Fleischmann et al. (1997).

The paper is organized as follows. In Section 2 we define the model in detail and derive the optimization problem to be solved. In Section 3 we derive the structure of the optimal policy for discounted and average cost criteria, both for finite and infinite horizon. In Section 4 we concentrate on the infinite horizon, average cost case, for which we derive near-optimal policies. An approximation algorithm is presented that computes these near-optimal policies. This algorithm is applied in Section 5 to various situations in order to gain insight in different aspects of interest for managerial decisions. Conclusions and topics for further research are presented in Section 6.
2 The Multiple Remanufacturing Options Problem

2.1 Problem Description and Notation

The specific problem addressed in this paper can be described as follows. In a periodic review system we face stochastic returns $R_t$ of used products in each time period $t$. These products can either be disposed of or remanufactured using $n$ alternative modes from a set $N$ of different options. A remanufacturing option $i$ takes a processing leadtime of $\lambda_i$ periods and has an outcome which is needed for satisfying stochastic demands $D_{it}$. Additional sources of procuring serviceable items of option $i$ do not exist. Demands are assumed to be independent in time and across options.

Returned products ($RP$) as well as serviceable items of each options $i$ ($SP_i$) can be stocked so that the structure of the multiple remanufacturing options systems can be depicted as in Figure 2.1.

![Figure 2.1: System with Multiple Remanufacturing Options](image)

Stocks on hand are charged with a per unit cost $h_R$ for returned products and $h_{Si}$ for serviceable items of option $i$, respectively. Demand for an item $i$ which is not satisfied immediately is backordered at a cost of $v_i$. Different amounts of the backorder costs also reflect differences in the selling prices of the respective serviceable options. Disposal and remanufacturing costs are proportional per unit, described by cost parameters $c_B$ and $c_{Ri}$, respectively. For sake of simplicity all model parameters are assumed to be time-independent. Only for the steady-state analysis in the sequel of this paper, this assumption is absolutely necessary.

The objective is to choose the quantities $r_{it}$ of items to be remanufactured for option $i$ and the amount $b_t$ of units for disposal in each period $t$ of a planning horizon $T$ in such
a way that the expectation of the total discounted costs of remanufacturing, disposal, stock holding and backordering is minimized. We also consider the case where the horizon $T$ tends to infinity.

The sequence of state observations and decisions in each period is the following one. At the beginning of a period $t$ (before arrival of returns and previous remanufacturing orders) we observe both the inventory on hand of returned products ($x_{Rt}$) and the net inventories (on hand minus backorders) of serviceables with option $i(x_{Si}^N)$. Based on these stock levels and on the information about in-process remanufacturing orders we decide upon the number of returned items which are disposed of and remanufactured in period $t$. After these decisions are made the stochastic returns and demands during the period are taken into account. Holding and backordering costs inventories at the end of the period. As aggregated stock information for decision making we additionally define the inventory position $x_{Si}$, for each serviceable option $i$ (net inventory plus in-process remanufacturing orders) and the systems echelon inventory position $x_{Et} = x_{Rt} + \sum_{i\in N} x_{Si}$. Summarizing we use the following notation:

\begin{align*}
N & : \text{set of remanufacturing options} \\
n & = |N| : \text{number of remanufacturing options} \\
T & : \text{number of planning periods} \\
r_{it} & : \text{remanufacturing quantity w.r.t. option } i (i \in N) \text{ in period } t \\
b_i & : \text{disposal quantity in period } t \\
x_{Rt} & : \text{on-hand inventory of returned products in period } t \\
x_{Si}^N & : \text{net inventory of serviceable option } i \text{ in period } t \\
x_{Si} & : \text{inventory position of serviceable option } i \text{ in period } t \\
x_{Et} & : \text{echelon inventory position in period } t \\
c_{Ri} & : \text{remanufacturing cost per unit w.r.t. option } i \\
c_B & : \text{disposal cost per unit} \\
h_R & : \text{inventory holding cost of returned products per unit and period} \\
h_{Si} & : \text{inventory holding cost of serviceables option } i \text{ per unit and period} \\
v_i & : \text{shortage cost of backordered serv. option } i \text{ per unit and period} \\
\alpha & : \text{discount factor} (0 \leq \alpha \leq 1) \\
\lambda_i & : \text{remanufacturing leadtime w.r.t. option } i \\
D_i & : \text{stochastic demand per period w.r.t. option } i \text{ (i.i.d.)} \\
R & : \text{stochastic returns per period (i.i.d.)} \\
S = R - \sum_i D_i & : \text{stochastic net supply of remanufacturable products} \\
\varphi_{Di}(.) & : \text{continuous density function of demand } D_i \\
\varphi_R(.) & : \text{continuous density function of returns } R \\
\varphi_S(.) & : \text{continuous density function of net supply } S
\end{align*}
2.2 The Optimization Model

Following the above notation we can formulate the problem as a stochastic dynamic optimization model. The objective function is given by

\[
\text{Min}_{\{R_t, d_t, \ldots, d_{nt}\}} \left\{ \sum_{t=1}^{T} \alpha^{t-1} \cdot \left[ c_B \cdot b_t + \sum_{i \in N} c_{R_i} \cdot r_{it} + h_R \cdot (x_{R_t} - b_t - \sum_{i \in N} r_{it} + R_t) \right] \\
+ \sum_{i \in N} h_{S_i} \cdot [x_{S_{it}} + r_{i,t-\lambda_i} - D_{it}]^+ + \sum_{i \in N} v_i \cdot [D_{it} - x_{S_{it}}^N - r_{i,t-\lambda_i}]^+ \right\}
\]

where we use \([x]^+ := \max\{x, 0\}\).

We find dynamic restrictions in the form of inventory balance equations for each period \(t\). These restrictions refer to the inventory of returns

\[
x_{R,t+1} = x_{R_t} - b_t - \sum_{i \in N} r_{it} + R_t
\]

and to the serviceable options inventories

\[
x_{S_i,t+1}^N = x_{S_{it}}^N + r_{i,t-\lambda_i} - D_{it} \quad \forall i \in N.
\]

A static restriction in each period is given by the condition that disposal and remanufacturing must not exceed the number of units in the inventory of returns

\[
b_t + \sum_{i \in N} r_{it} \leq x_{R_t}.
\]

Finally we face non-negativity of all decision variables

\[
b_t, r_{it} \geq 0.
\]

The value of any stock at the end of the planning horizon is assumed to be equal to zero. In the sequel the problem formulated above will be called the Multiple Remanufacturing Options Problem (MROP)
3 Policy Analysis for the MROP

3.1 The Optimal Disposal, Remanufacturing, and Allocation Policy

For a dynamic stochastic product recovery system as described in Section 2 it is obvious that the state information which is necessary to control such a system in an optimal way consists of the returns' stock on hand $x_{R_t}$ and the inventory positions of serviceables $x_{S_{it}}$ at the beginning of each period. The latter information is used to describe how the remanufacturing decisions of a period affect the expected holding and backorder costs of the next controllable time period, which is $\lambda_i$ periods ahead.

The balance equations with respect to the serviceables inventory positions are given by

$$ x_{S_{i,t+1}} = x_{S_it} + r_{it} - D_{it} \quad \forall i \in N $$.  

Thus for the echelon inventory position $x_{E_t} = x_{R_t} + \sum_i x_{S_it}$ we find

$$ x_{E_t,t+1} = x_{E_t} - b_t + R_t - \sum_{i \in N} D_{it} $$

As expected holding costs of returns in period $t$ we can write

$$ h_R \cdot \int_0^\infty (x_{Rt} - b_t - \sum_{i \in N} r_{it} + R) \cdot \varphi_R(R) \cdot dR = $$

$$ = h_R \cdot (x_{Rt} - b_t - \sum_{i \in N} r_{it} + E[R]) $$

Because holding costs $h_R \cdot E[R]$ can not be influenced by the decisions to make, this fixed part of the costs will be omitted in the sequel. The expected holding and backorder costs for a serviceable option $i$ as a function of the inventory position after $y_i$ remanufacturing can be described by

$$ L_i(y_i) = h_{Si} \cdot \int_0^{y_i} (x - D_i) \cdot \varphi_{D_i}^{\lambda_i+1}(D_i) \cdot dD_i + v_i \int_{y_i}^\infty (D_i - y_i) \cdot \varphi_{D_i}^{\lambda_i+1}(D_i) \cdot dD_i $$

where $\varphi_{D_i}(.)$ represents the $\lambda_i$-fold convolution of $\varphi_{D_i}(.)$.  

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It can easily be shown and is well-known from standard inventory theory that \( L_i(\cdot) \) are convex functions.

Using the above formulations and following Bellman’s optimality principle we can express the functional equations of dynamic programming for the MROP in the following way (where for sake of simplicity we omit time index \( t \) for the variables).

\[
\begin{align*}
 f_{t}(x_R, x_{S1}, \ldots, x_{Sn}) = \\
 \min_{b+\sum_{i} r_i \leq x_R} \left\{ c_B \cdot b + \sum_{i} c_{R_i} \cdot r_i + h_R \cdot (x_R - b - \sum_{i} r_i) + \sum_{i} L_i(x_{S_i} + r_i) + \right. \\
 \left. + \alpha \cdot \int_{0}^{\infty} \ldots \int_{0}^{\infty} f_{t+1}(x_R - b - \sum_{i} r_i + R, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \cdot \\
 \cdot \varphi_R(R) \cdot \varphi_{D_1}(D_1) \ldots \cdot \varphi_{D_n}(D_n) \cdot dR \cdot dD_1 \ldots \cdot dD_n \right\}
\end{align*}
\]

(1)

for \( t \geq 1 \)

and with \( f_{T+1}(x_R, x_{S1}, \ldots, x_{Sn}) \equiv 0 \).

It is evident that the value functions \( f_t(\cdot, \ldots, \cdot) \) depend on \( n + 1 \) state variables. The way of analyzing the structure of the optimal policy for these kinds of problems is regularly by induction. First, the policy structure is evaluated for a single-period problem. Then it is checked if this kind of policy also holds for a \((n + 1)\)-period problem, given that it is assumed to be valid in a \( n \)-period situation.

So we start our investigation of the optimal MROP policy with considering a single-period model, i.e. setting \( T = 1 \). The respective minimization problem can be written as

\[
\begin{align*}
 f_{1}(x_R, x_{S1}, \ldots, x_{Sn}) = \min_{b, r_1, \ldots, r_n} g_{1}(x_R, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) \\
 \text{w.r.t.} \quad b + \sum_{i} r_i \leq x_R \\
 b, r_1, \ldots, r_n \geq 0
\end{align*}
\]
Since $g_i(.,\ldots.,.)$ can be shown to be convex in the decision variables, we can exploit the Kuhn-Tucker conditions of optimality to find the optimal policy (for proofs and details see Appendix A).

Before we give a formal description of the optimal policy, we describe and interpret how the optimal disposal and remanufacturing decisions are linked together. Since in the single-period case disposing one unit results in an increase of disposal costs by $c_B$ and in a reduction of returns holding costs by $h_R$, we face a simple rule with respect to the disposal decision. If $c_B < h_R$, stockholding of returns is not economical, and all returned items, which are not used for remanufacturing, will be disposed of. If $c_B > h_R$, disposal will never be advantageous yielding $b^* = 0$ (where an asterisk denotes the optimal decision).

Due to the convexity of the cost functions $L_i(.)$, in principle the remanufacturing decisions follow an order-up-to-$M_i$ policy, where the constant $M_i$ corresponds to the level to which the respective serviceables inventory position should be brought up, if it is below this position. Otherwise no remanufacturing order should be placed. For each option $i$ the critical level $M_i$ has to be chosen in such a way that the marginal costs of remanufacturing equal the marginal cost advantage by decreasing the number of items in the returns stock and by increasing the serviceables inventory position by one unit.

$$c_R = \min \{c_B, h_R\} - L'_i(M_i) \quad ,$$

where $L'_i(.)$ stands for the first-order derivative of function $L_i(.)$.

This policy can only be carried through if a sufficient amount of returned items is in the respective stock. If not, i.e. if $x_R \leq \sum_{i \in N} [M_i - x_{Si}]^+$, an optimal allocation of scarce returns to the different options has to be made. This results in a decrease of the original remanufacture-up-to-levels $M_i$, which in this situation become dependent on all single inventory state variables

$$M_i = \bar{M}_i(x_R, x_{S1}, \ldots, x_{Sn}) \quad .$$

The modified parameters $\bar{M}_i$ are determined such that the net marginal cost advantages of increasing all single serviceables inventory positions by one unit are equal. Additionally, the

where

$$g_i(x_R, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) =$$

$$c_B \cdot b + \sum_i c_{R_i} \cdot r_i + h_R \cdot (x_R - b - \sum_i r_i) + \sum_i L_i(x_{Si} + r_i) \quad (2)$$
total number of returns allocated to the different remanufacturing options has to equal the number of available returns:

\[
L'_1(M_1) + c_{R1} = L'_2(M_2) + c_{R2} = \ldots = L'_n(M_n) + c_{Rn}
\]  

(4)

and

\[
\sum_{i \in N} [M_i - x_{Si}]^+ = x_R.
\]

The optimal disposal quantity is either equal to zero (if \(CB > h_R\)), or it equals the number of returned items which are not needed for remanufacturing purposes, i.e.

\[
b^* = x_R - \sum_{i \in N} [M_i - x_{Si}]^+ \quad \text{if} \quad CB \leq h_R.
\]

Thus, using \(\bar{U}\) as a critical disposal level, where

\[
\bar{U} = \bar{U}(x_{S1}, \ldots, x_{Sn}) = \begin{cases} 
\infty & \text{for } CB > h_R \\
\sum_{i} [M_i - x_{Si}]^+ & \text{for } CB \leq h_R
\end{cases}
\]

we can summarize the combination of optimal disposal and remanufacturing decisions as follows

\[
\begin{align*}
&x_R < \sum_i [M_i - x_{Si}]^+ \Rightarrow b^* = 0, \quad r_i^* = [M_i(x_{R1}, x_{S1}, \ldots, x_{Sn}) - x_{Si}]^+ \\
&\sum_i [M_i - x_{Si}]^+ \leq x_R \leq \bar{U}(x_{S1}, \ldots, x_{Sn}) \Rightarrow b^* = 0, \quad r_i^* = [M_i - x_{Si}]^+ \\
x_R > \bar{U}(x_{S1}, \ldots, x_{Sn}) \Rightarrow b^* = x_R - \bar{U}(x_{S1}, \ldots, x_{Sn}), \quad r_i^* = [M_i - x_{Si}]^+
\end{align*}
\]

Obviously, this policy has a highly complicated structure, mainly because all decisions are depending on every single state variable in a complex way.
Unfortunately, such kind of policy is not suitable to be used in practical decision making because of its structural complexity and its prohibitive amount of numerical computation for optimizing the respective policy parameters.

In the multi-period case the underlying optimization problem becomes even more complex, since an additional \((n + 1)\)-dimensional cost function appears in the functional equations. Thus, also the optimal policy will at best be of the same complexity as shown in the single-period situation. For that reason we will not go into the problem of analyzing the optimal policy for the multi-period problem, but turn to a reasonable approximation, which considerably simplifies the policy for controlling the recovery management system. Since the main complexity comes into the problem by determining the optimal allocation of returns in case of insufficient stock, we make two simplifying assumptions which refer to the way of allocating and to its consequences.

### 3.2 The Optimal Policy under Linear Allocation Rules

The first assumption we use, concerns the allocation rule which is employed whenever the stock of returned items is not sufficient to bring all serviceables inventory positions up to their desired levels \(M_i\). In this case we assume that the scarce amount \(x_R\) of remanufacturables is distributed among the different options according to a so-called linear allocation rule as follows

\[
\rho_i^0 = \rho_i(x_E, x_{S_i}) = M_i - q_i \cdot \left[ \sum_{j \in N} (M_j - x_E)^+ - x_{S_i} \right],
\]

where \(q_i\) are given allocation fractions which satisfy the conditions \(q_i \geq 0 \quad \forall i \in N\) and \(\sum_{i \in N} q_i = 1\). Superindex "0" is used to denote decisions under the conditions of linear allocation.

In case of insufficient supply this rule divides the amount allocated to each option proportional to the overall remanufacturables deficit \(\left[ \sum_{j \in N} (M_j - x_{S_j}) - x_R \right]^+\) just using the constant fractions \(q_i\), which have to be determined in an appropriate way. Such a policy seems promising since it is also successfully employed in multi-echelon divergent inventory systems, where similar allocation problems arise (cf. Diks et al, 1996).

A second assumption which is useful to facilitate the evaluation of a tractable policy for the MROP refers to the distribution of serviceables inventory positions. As often used as supposition in related multi-echelon inventory systems we assume that the inventory positions of all
serviceable options are properly balanced. This so-called balance assumption means that the echelon stock in the system is always distributed in such a way that under application of the remanufacturing policy with linear allocation rule we never face a situation where negative remanufacturing orders occur i.e.

\[ r_i^0 \geq 0 \quad \forall i \in N \quad . \tag{7} \]

Under these assumptions it can be shown that the resulting (only approximately) optimal policy has a very simple structure, and that the associated value function of dynamic programming has nice properties (for proofs and details see Appendix B).

### 3.2.1 The Finite Horizon Problem

#### The Single-period Case

For studying the MROP policy for finite planning horizons we start with the single-period case. It turns out that the optimal disposal and remanufacturing decisions in this case, different from the situation in (5), are solely dependent on \( n + 1 \) constant inventory levels \( M_i (i = 1, \ldots, n) \) and \( U \), if a linear allocation rule is applied and the balance assumption is made.

\[
\begin{align*}
\sum_j M_j < x_E \Rightarrow b^0 &= 0 \quad , \quad r_i^0 = M_i - q_i \cdot (\sum_j M_j - x_E) - x_{Si} \\
\sum_j M_j \leq x_E \leq U \Rightarrow b^0 &= 0 \quad , \quad r_i^0 = M_i - x_{Si} \\
x_E > U \Rightarrow b^0 &= x_E - U \quad , \quad r_i^0 = M_i - x_{Si}
\end{align*}
\tag{8}
\]

This kind of policy is easy to implement in practice. In the sequel it will be called \((nM, U)\)-policy. The remanufacture-up-to-levels \( M_i \) are given by (3). For the critical inventory \( U \), which can be interpreted as dispose-down-to level, we have

\[
U = \begin{cases} 
\infty & \text{for } c_B > h_R \\
\sum_i M_i & \text{for } c_B \leq h_R 
\end{cases} 
\]

From (8) it can be seen that the way of reacting with disposal and remanufacturing decisions mainly depends on the size of the echelon inventory positions in relation to the dispose-down-to and to the remanufacture-up-to levels. This holds for each feasible combination of
allocation fractions. Using the \((nM, U)\)-policy it can be shown that the resulting minimum single-period costs \(f_t^0(x_R, x_{S1}, \ldots, x_{Sn})\) perform a convex function. Furthermore, it can be proved that this value function can be expressed as being dependent on the echelon inventory position \(x_E\) instead of being affected by the stock level \(x_R\) of returns

\[
f_t^0(x_R, x_{S1}, \ldots, x_{Sn}) = f_1(x_E, x_{S1}, \ldots, x_{Sn}) .
\]

Eventually it turns out that this transformed value function is a sum of separate convex functions in the single state variables

\[
f_1(x_E, x_{S1}, \ldots, x_{Sn}) = f_{10}(x_E) + \sum_{i \in N} f_{1i}(x_{Si}) .
\]

The Multi-period Case

By induction it can be proved that the properties of the policy structure and value function evaluated in the single-period case also hold for each period of a multi-period problem with finite horizon \(T\), given the linear allocation policy and the balance assumption. From the functional equations in (1) we can derive the minimization problem in the value function determination for any period \(t < T\) to be

\[
\hat{f}_t(x_E, x_{S1}, \ldots, x_{Sn}) = \min_{b, r_1, \ldots, r_n} g_t(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n)
\]

w.r.t. \(b + \sum_{i \in N} r_i \leq x_E - \sum_{i \in N} x_{Si}\)

\(b, r_1, \ldots, r_n \geq 0\)

The function to be minimized is

\[
g_t(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) = c_B \cdot b + \sum_i c_{RI} \cdot r_i + G_{10}(x_E - b) + \sum_i G_{ti}(x_{Si} + r_i)
\]

where

\[
G_{10}(y_E) = h_R \cdot y_E + \alpha \cdot \int_{-\infty}^{\infty} f_{t+1,0}(y_E + S) \cdot \varphi_S(S) \cdot dS
\]
\[ G_{ti}(y_t) = -h_R \cdot y_t + L_i(y_t) + \alpha \cdot \int_0^\infty f_{t+1,i}(y_t - D_t) \cdot \varphi_{D_t}(D_t) \cdot dD_t. \]

Solving this optimization problem under the assumption of Section 3.2 we find a policy to be optimal which has the same \((nM,U)\) structure as in (8) for the single-period case. The policy parameters \(U_t\) and \(M_{ti}\) follow from

\[ c_B = G'_{ti0}(U_t) \quad \text{and} \quad c_{Ri} = -G_{ti}(M_{ti}) \quad \forall i \in N, \tag{11} \]

as long as these conditions result in \(U_t \geq \sum_{i \in N} M_{ti}\). If not, the optimal parameters are developed from the following set of equations:

\[ c_B - G'_{ti0}(U_t) = c_{R1} + G'_{t1}(M_{t1}) = \ldots = c_{Rn} + G'_{tn}(M_{tn}) \tag{12} \]

and \(U_t = \sum_{i \in N} M_{ti}\).

This result means that in principle the dispose-down-to-level \(U_t\) has to be chosen such that the costs of disposing one unit equals the marginal cost advantage of decreasing the echelon stock by one item. The remanufacture-up-to-levels \(M_{ti}\) are optimally determined if the cost of recovering one unit with respect to option \(i\) equals the marginal cost advantage of increasing the respective inventory positions. This holds as long as the cumulated \(M_{ti}\)-values do not exceed the disposal level \(U_t\). Otherwise these levels have to be equalized, and the single parameters have to be determined in such a way that the net marginal cost effects are the same for increasing or decreasing by one unit, respectively.

### 3.2.2 The Infinite Horizon Problem

In stationary situations it can be useful to consider problems with an unlimited planning horizon. For the product recovery problem under consideration this can only make sense, if it is assured that the expected returns per period exceed the sum of expected demands for the serviceable options, i.e.
\[ E[R] > \sum_{i \in N} E[D_i] \]  \hspace{1cm} (13)

Otherwise, total costs would tend to infinity since expected backorders would go on accumulating over time.

**The Discounted Cost Case**

In case of strictly discounted costs, i.e. \( \alpha < 1 \), it can be shown under the returns excess condition in (13) that in the infinite horizon case (i.e., \( T \to \infty \)) the value function \( f_t(x_E, x_{S1}, \ldots, x_{Sn}) \) converges to a stationary function (for a proof and details see Appendix C). So the functional equations in the limiting case can be written as

\[
\begin{align*}
&\min_{b + \sum_{i} r_i \leq x_E - \sum_{i} x_{Si}} \left\{ c_B \cdot b + \sum_{i} c_{Ri} \cdot r_i + h_R \cdot (x_E - b) + \sum_{i} L_i (x_{Si} + r_i) + \\
&\quad + \alpha \cdot \int_{-\infty}^{\infty} \cdots \int_{0}^{\infty} f_t(x_E - b + S, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \cdot \\
&\quad \cdot \varphi_S(S) \cdot \varphi_{D1}(D_1) \cdots \varphi_{Dn}(D_n) \cdot dS \cdot dD_1 \cdots dD_n \right\}
\end{align*}
\]  \hspace{1cm} (14)

For \( f_t(x_{E1}, x_{S1}, \ldots, x_{Sn}) \) the properties of separability and convexity hold. So in the infinite horizon case under application of a linear allocation rule the optimal disposal and remanufacturing decisions still follows a \( (nM, U) \)-policy as defined in (8). However, the policy parameters \( U \) and \( M_i \) are no longer time-dependent, but are the same for each period. They will depend on all cost parameters and the additional problem data. In order to compute these \( n + 1 \) parameters numerically, given a prespecified set of allocation fractions \( q_i \), we can transform the problem into a discrete one and apply the policy improvement technique or the method of successive approximations of dynamic programming (see e.g. Hillier and Lieberman, pp. 778-792). Because of the multi-dimensionality of the state space numerical solution procedures of dynamic programming are connected with a prohibitive amount of computation. Therefore it is desirable to find a practically acceptable way of computing the optimal values of the policy parameters \( U \) and \( M_i \). Such a procedure is available in the case of average cost minimization.
The Average Cost Case

The average cost case is equivalent to a problem setting where a discount factor equal to one is chosen, i.e. $\alpha = 1$. Unfortunately, for this case we were not able to prove formally the convergence property of the respective multi-period value functions of the costs per period if $T \to \infty$. However, we have the strong conjecture that also for the average cost criterion this property holds (as it does in many stochastic inventory models), so that also in this case a stationary $(nM,U)$-policy would be optimal under the linear allocation rule and balance assumption. By all means, this is valid for any $\alpha$-value arbitrarily near to one.

In the next section we will present a very effective procedure to determine the policy parameters $U$ and $M_i$ for an average cost situation. This approach additionally includes an optimization of the allocation fractions $q_i$ which are used for the linear allocation rules. It shall be pointed out that different from the discounted cost case under the average cost criterion the disposal cost $c_D$ and remanufacturing costs $c_{RI}$ will not affect the optimal policy parameter values. This is simply due to the fact that, because each demand finally has to be fulfilled, the average remanufacturing quantity of each option equals the respective average demand, whereas the average disposal amount is equal to the expected excess of return over cumulated demands. So the average remanufacturing and disposal costs per period are fixed and can not be affected by choosing the policy parameters, which only will depend on the different holding and backorder costs. However, if holding costs for returns ($h_R$) and for serviceables ($h_S$) also reflect the interest on capital tied up in the inventories, remanufacturing and disposal costs may influence these values so that they implicitly may affect the policy parameters.

3.3 Discussion of the $(nM,U)$-Policy

The $(nM,U)$-policy is highly attractive for practical application, because it is simply structured, directly to understand, and easy to calculate. Nevertheless, in Section 3.1 we have shown that the optimal policy is far more complex than this simple rule. For assessing the value of the $(nM,U)$-policy we have to discuss the significance of both the linear allocation rule and balance assumption, because given these conditions the $(nM,U)$-policy is optimal. As mentioned above the decision problem under consideration is similar to a two-echelon inventory control problem where a good is ordered at a central stockpoint from an outside supplier and shipped to various decentral stockpoints from where stochastic demands have to be satisfied. In the MROP the supply side of the problem is significantly different, because the inflow of goods is stochastic. But the demand side is almost identical, so that the procedure of providing downstream stockpoints with goods are comparable. For the divergent inventory control case with strictly proportional costs it is well-known that order-up-to-policies comparable to the remanufacture-up-to-rules are near-optimal (cf. Van Houtum et al., 1996). Additionally, it is known that a linear allocation rule with optimized allocation fractions is very well performing for the task of allocating insufficient central stock (cf. Van der Heijden et al., 1997). In fact, Diks (1997) shows for a two-echelon two-product model that the optimal allocation function and the optimal linear allocation function non-critical under a wide range of circumstances which also include lot-for-lot-ordering which is equivalent to a order-up-to-policy (cf. Diks et al., 1996). So we can expect the remanufacturing and allocation part of the $(nM,U)$-policy
to be very reasonable for tackling the product recovery problem under consideration.

On the other hand, the disposal part of the \((nM, U)\)-policy is equally well justified by the results from analyzing remanufacturing problems with a single option for product recovery. For these kinds of problems it is shown that under proportional costs the optimal disposal policy follows a dispose-down-to-rule as included in the \((nM, U)\)-policy (cf. Inderfurth, 1997). Altogether, this makes us very confident, that the proposed \((nM, U)\)-policy is an ideal candidate for being applied in making coordinated disposal, remanufacturing and allocation decisions in the MROP. Additionally, it turns out that in the special case of only one single remanufacturing option an \((nM, U)\)-policy is optimal, because in this situation the resulting \((1M, U)\)-policy is a degenerated version of an \((L, M, U)\)-type policy which can be shown to be optimal when an additional regular production option is available for procurement (cf. Inderfurth, 1997). For the case of exceptionally high procurement costs, which is equivalent to non-existence of a procurement option, the order-up-to-level \(L\) tends to minus infinity \((L \to -\infty)\) and both policies coincide. Thus the combination of a remanufacture-up-to and a dispose-down-to policy can be shown to be optimal, if an allocation problem cannot occur.
4 Parameter optimisation for the \((nM, U)\)-policy

In Section 3 we proved the optimality of the \((nM, U)\)-policy for the discounted cost case for both finite and infinite horizon. In this section we concentrate on the average cost case under the assumption of stationary demand for serviceable options and stationary returns of used products. We propose a computational scheme to derive near-optimal policies within the class of \((nM, U)\)-policies with linear allocation rules.

4.1 The stationary analysis

For the steady-state analysis in the sequel we will use the following random variables which are needed to determine the dynamics of relevant stocks when policy parameters \(U\) and \(\{M_i\}_{i=1}^n\) are applied.

\[
\begin{align*}
R_t &: \text{returns of reusable products during } (t-1, t] \\
D_{it} &: \text{demand for serviceable options } i \text{ during } (t-1, t], \ i \in N \\
D_{iT}^{\lambda_i+1} &: \text{demand for serviceable option } i \text{ during } (t-1, t + \lambda_i], \ i \in N \\
D_{ot} &: \text{total demand for serviceable options during } (t-1, t] \\
B_t &: \text{amount disposed of at time } t \\
Y_t &: \text{echelon system stock at time } t \text{ immediately after disposal} \\
Z_t &: U - Y_t \\
x_{Si}^{N} &: \text{net stock of serviceable option } i \text{ at time } t \text{ immediately after satisfying all demand in } (t-1, t] \text{ and immediately before input of newly available serviceable options, } i \in N \\
x_{Ra}^{A} &: \text{net stock of products available for remanufacturing at time } t \text{ immediately after the disposal decision and immediately after the allocation decision} \\
x_{Sa}^{A} &: \text{echelon inventory position of serviceable option } i \text{ at time } t \text{ immediately after the allocation decision, } i \in N
\end{align*}
\]

In the sequel we omit the index \(t\) for the above random variables to indicate their stationary equivalent. It readily follows that the long-term average costs under a given \((nM, U)\) policy, which are influenced by the policy parameters, are given by

\[
C(U, M) = \sum_{i \in N} v_i \cdot E[x_{Si}^{-N}] + \sum_{i \in N} h_{Si} \cdot E[x_{Si}^{N+}] + h_R \cdot E[x_{Ra}^{A}] 
\]

where \(M\) stands for the vector of parameters \(\{M_i\}_{i=1}^n\). As mentioned above, in the average cost case the expected remanufacturing and disposal costs per period are not affected by the choice of policy parameters since on the long run remanufacturing has to be equal to demand (due to the backorder assumption) and the disposal quantity per period has to be equal in
expectation to the excess of returns over total demands in a period.

Note that in (15) we suppressed the dependence of the above random variables on the policy parameters $U$ and $\{M_i\}_{i=1}^n$. To compute an expression for $C(U, M)$ we need to find expressions for $E[x_R^i], E[x_S^i], \forall i \in N$ and $E[x^i], \forall i \in N$. These random variables depend on the dynamics of the echelon stock of the system and on the dynamics of the echelon stocks of the serviceable options.

Let us first consider the dynamics of the echelon stock of the system. It is easy to see that the following equations hold,

$$Y_{t+1} = \min(U, Y_t - D_{0t} + R_t)$$

$$B_{t+1} = \max(0, Y_t - D_{0t} + R_t - U)$$

Substitution of the definition of $Z_t$ in the above equations yields,

$$Z_{t+1} = \max(0, Z_t + D_{0t} - R_t) \quad (16)$$

$$B_{t+1} = \max(0, R_t - D_{0t} - Z_t) \quad (17)$$

Now we note that equation (16) is identical to Lindley's integral equation for the $G/G/1$ queue, where the interarrival times are $R_t$ and the service times are $D_{0t}$, respectively. The waiting times are given by $Z_t$. In De Kok (1989) a moment-iteration method is developed to compute the first two moments of the waiting time distribution. This method can be applied here as well to compute the first two moments of $Z$. The essence of the method is to compute $E[Z^k_t]$ for $k = 1, 2$ from

$$E[Z_{t+1}^k] = E[\max(0, Z_t + D_{0t} - R_t)^k] \quad , \quad (18)$$

where we fit mixtures of Erlang distributions to the first two moments of $Z_t + D_{0t}$ and $R_t$, respectively (cf. Tijms, 1994). The expressions resulting from equation (18) under this fit assumption are easy to compute. By subsequent substitution of the newly found $E[Z^k_t]$ in the rhs. of (18) we obtain a series $E[Z_t^k]$, which converges fast, provided $E[D_0]/E[R] < 0.95$. From now on we assume that we have accurate approximations for $E[Z]$ and $E[Z^2]$. The next relevant operating characteristic is $E[x^A_R]$. It follows from the allocation policy that products are left at the stock point iff. the system echelon stock exceeds $\sum_{i \in N} M_i$. The amount left is the difference between the echelon stock and $\sum_{i \in N} M_i$. Defining $\delta$ as
\[ \delta = U - \sum_{i \in N} M_i \]

we thus find that

\[ E[x_A^i] = E[(\delta - Z)^+] \]

Under the linear allocation policies defined in Section 3 we have that in the stationary situation

\[ x^A_{Si} = M_i - q_i \cdot (Z - \delta)^+ \]

From standard inventory theory arguments we then find the following expression for \( x^N_{Si} \),

\[ x^N_{Si} = x^A_{Si} - D_i^{\lambda_i+1} \]

Hence we find for \( i = 1, 2, ..., n \),

\[ E[x^N_{Si+}] = E[(M_i - q_i \cdot (Z - \delta)^+ - D_i^{\lambda_i+1})^+] \]

\[ E[x^N_{Si-}] = E[(q_i \cdot (Z - \delta)^+ + D_i^{\lambda_i+1} - M_i)^+] \]

The above expected values for \( E[x_A^i], E[x^N_{Si+}], \forall i \in N \) and \( E[X^N_{Si}], \forall i \in N \), can be easily computed when we subsequently fit mixtures of Erlang distribution to the first two moments of \( Z \) and fit a mixture of Erlang distributions to the first two moments of \( q_i \cdot (Z - \delta)^+ + D_i^{\lambda_i+1} \).

### 4.2 Algorithm for parameter determination

Now we want to solve the following problem

\[ \min_{U, \{M_i\}_{i=1}^n, \{q_i\}_{i=1}^n} C(U, \{M_i\}_{i=1}^n, \{q_i\}_{i=1}^n) \]

To solve this problem we compare it with the problem of finding cost-optimal policies for two-echelon divergent inventory system as studied in Diks (1997). A close look at the cost equations derived in Diks (1997) shows that these equations are identical when \( Z \) is replaced by a random variable that denotes the total demand for options during a fixed lead time preceding the allocation moment. In Diks (1997) it is proven that the optimal order-up-to-policy with linear allocation rules can be found by recursively solving Newsboy equations.
Detailed study of his proof shows that it applies also to our multiple options model. The key condition to be satisfied is that $Z$, which replaces the total demand during a fixed lead time preceding time $t$, is independent of the demand for each option during another fixed lead time following time $t$. This clearly holds. In Dong et al. (1994) a similar approach based on substitution of random variables shows the equivalence of a multi-stage planned lead times model with a serial N-echelon inventory model studied by Clark and Scarf (1960). Thus the following theorem holds.

Theorem 4.1

The optimal policy $(U^*, \{M^*_i\}_{i=1}^n, \{q^*_i\}_{i=1}^n)$ satisfies the following Newsboy equations.

$$P\{D_i^{\lambda, +1} \leq M_i^*\} = \frac{v_i + h_R}{v_i + h_R + h_{si}} \quad \forall i \in N$$

$$P\{D_i^{\lambda, +1} + q_i^* \cdot (Z - \delta^*)^+ \leq M_i^*\} = \frac{v_i}{v_i + h_R + h_{si}} \quad \forall i \in N$$

$$\sum_{i \in N} q_i = 1$$

In order to solve these equations we apply the algorithm proposed in De Kok (1990) according to the refinements in Diks (1997) (see also Dong et al., 1994).

Algorithm 4.1

Step 1:

a. Fit a mixture of Erlang distributions $\tilde{F}_i$ to the first two moments of $D_i^{\lambda, +1}$ and use a bisection procedure to solve

$$\tilde{F}_i(M_i^*) = \frac{v_i + h_R}{v_i + h_R + h_{si}} \quad \forall i \in N$$

b. Set $\delta = 0$.

Step 2:

a. Fit a mixture of Erlang distributions $\tilde{F}_Z$ to $Z$ and compute the first moments of $E[(Z - \delta)^+]$ under this distribution fit.
For $i = 1, 2, \ldots, n$ set $q_{i,\text{min}} = 0$ and $q_{i,\text{max}} = 1$.

b. $q_i = \frac{q_{i,\text{min}} + q_{i,\text{max}}}{2}$

Fit a mixture of Erlang distributions $\hat{F}_i(\cdot, q_i)$ to the first two moments of $D_i^{\lambda_{i+1}} + q_i^* \cdot (Z - \delta)^+$. 

If
\[
\hat{F}_i(M_i^*, q_i) \leq \frac{v_i}{v_i + h_R + h_{Si}}
\]
then $q_{i,\text{min}} := q_i$ ,
otherwise $q_{i,\text{max}} := q_i$

c. If
\[
|\hat{F}_i(M_i^*, q_i) - \frac{v_i}{v_i + h_R + h_{Si}}| \leq \epsilon
\]
then goto step 3 else goto step 2b.

Step 3:

If $\sum_{i \in N} q_i \geq 1 + \epsilon$ then decrease $\delta$, goto step 2.

If $\sum_{i \in N} q_i \leq 1 + \epsilon$ then increase $\delta$, goto step 2.

Otherwise goto step 4.

Step 4:

\[
U = \sum_{i \in N} M_i^* + \delta
\]

If $\delta < 0$ then
\[
M_i^* := M_i^* + q_i \cdot \delta \quad \forall i \in N
\]
\[
\delta := 0
\]

Stop.
In the implementation of the above procedure the main difficulty occurs when the difference in holding costs $h_R$ and $h_S$ is small. In that case it may happen that the optimal policy does not maintain any stock of input material. This results into a non-positive value of $\delta$. To detect this situation one should solve for $(q_i)_{i=1}^n$ with $\delta = 0$ and check if $\sum_{i \in N} q_i > 1$. If so, one should search for negative $\delta$, otherwise one should search for positive $\delta$. Step 4 of Algorithm 4.1 updates the order-up-to-levels $M_i^*$ so that the final policy satisfies $\delta > 0$. The validity of the update equations follows from Lemma 4.1.

Lemma 4.1:

Given a policy $(U, \{M_i\}, \{q_i\})$. Let $\delta = U - \sum_{i \in N} M_i$. If $\delta < 0$, then the sample paths of the inventory positions $\{x_{Si}\}$ under this policy are identical to the sample paths under a $(U, \{M_i + q_i \cdot \delta\}, \{q_i\})$ policy.

Proof:

First note that $Z_t$ is independent of $\delta$. At time $t$ $x_{Sit}$ is given by

$$x_{Sit}^A = M_i - q_i \cdot (Z_t - \delta)^+$$

Since $\delta$ is negative this is equivalent to

$$x_{Sit}^A = M_i - q_i \cdot Z_t + q_i \cdot \delta$$

Under the $(U, \{M_i + q_i \cdot \delta\}, \{q_i\})$-policy we find similarly

$$x_{Sit} = M_i + q_i \cdot \delta - q_i \cdot (Z_t - (U - \sum_{i \in N} (M_i + q_i \cdot \delta)))^+$$

Since

$$\sum_{i \in N} (M_i + q_i \cdot \delta) = \sum_{i \in N} M_i + \sum_{i \in N} q_i \cdot \delta = \sum_{i \in N} M_i + \delta = U$$

we find

$$x_{Sit}^A = M_i + q_i \cdot \delta - q_i \cdot Z_t,$$

which is identical to (19).

In Diks (1997) and De Kok (1989) the accuracy of the two-moment fits is demonstrated for the two major building blocks of our approach. From De Kok (1989) we may conclude that
our approximation for the first two moments of $Z$ is accurate. From Diks (1997) we may conclude that Algorithm 4.1 yields excellent approximations for the optimal policies within the class of $(nM, U)$-policies with linear allocation rules. In Section 5 we apply Algorithm 4.1 to gain insight into a number of issues concerning the Multiple Remanufacturing Options Problem.
5 Managerial insights

In this section we provide some insight into the impact of different input characteristics on the average system costs consisting of disposal costs, holding costs and penalty costs. Due to the assumption of backordering demand for serviceable options we do not consider the remanufacturing costs. Although the disposal costs are independent of the policy chosen, we do consider disposal costs in some comparisons since the disposal costs depend on the demand/return ratio $\rho$, with

$$\rho = \frac{E[D_0]}{E[R]}, \quad (20)$$

which under certain circumstances can be considered to be depending on managerial activities. Under long-term conditions finite costs can only be assured if $\rho < 1$. For a fixed ratio $\rho$ the optimization procedure under a $(nM, U)$-policy is restricted to determine the amount of the overall echelon stock in the system and to balance this stock between the remanufacturable products and the different serviceable options in order to keep expected holding and backorder costs at the lowest possible level.

In the numerical experiments discussed below we depart from a base case: we assume that $n = 3, E[D_i] = 100 \ (i = 1, 2, 3), \sigma(D_i) = 100 \ (i = 1, 2, 3), h_R = 1, h_{Si} = 1.5 \ (i = 1, 2, 3)$. The backorder costs are determined indirectly as follows. From the Newsboy equation in Theorem 4.1 it follows that the optimal policy satisfies a particular non-stockout probability for each option determined by holding and shortage costs. Since stockout probabilities are more intuitive than shortage costs we conversely choose a non-stockout probability and determine the backorder costs from the Newsboy equations. In all cases considered we assume a non-stockout probability of 0.98 for all serviceable options.

In Figure 5.1 below we show the impact of $c_R^2$, i.e. the squared coefficient of variation of the number of returns per period, on the minimum of the relevant costs in (15). It follows that the costs increase almost linearly with the squared coefficient of variation of the number of returns for both values of $\rho$ considered. This behaviour is similar to the behaviour of the average waiting time in an $M/G/1$ queue as a function of the squared coefficient of variation of the service times, expressed in the famous Pollaczek-Khintchine formula (e.g. Tijms, 1994).
In Figure 5.2 we show the impact of $\rho$ on the sum of relevant costs. Similar to the standard behaviour of waiting times in queues we find that the costs explode when $\rho$ approaches 1. For low values of $\rho$ the costs seem independent of $c^2_R$.

The results shown in Figure 5.2 motivate us to consider the impact of the disposal costs on the total costs. Disposal cost effects are particularly considered for two reasons. Firstly, total disposal costs reflect the level of disposal activities which is a critical issue from an environmental perspective and is usually desired to be as low as possible from a global point of view. Secondly, although the latter may not be valid under managerial considerations, disposal costs per unit often determined by governmental price setting are relevant cost inputs which together with other cost parameters affect the product recovery policy of a firm. In our context managerial reaction on disposal cost parameters variations is only reasonable, if the demand/returns ratio $\rho$ can be influenced. By increasing $\rho$ the fraction of returns dispo-
sed of can be decreased. In principle this can be done by two different types of managerial actions. On the one hand, the amount of products returned per period may be reduced, e.g. by extending the life cycle of the products. On the other hand, it may be possible to increase demand for serviceable options, e.g. by additional marketing activities in traditional or by penetration of new markets.

In the sequel we analyze the influence of the disposal costs in combination with a possible variation of the ratio \( \rho \). Towards this end we define a base case with the demand and cost parameters as above. Furthermore we assume that \( \rho_0 = 0.75 \) and \( c_R^2 = 0.5 \). For this base case we determine the disposal cost per unit \( c_B \), such that the disposal costs \( c_B \cdot (E[R] - E[D_0]) \) equal 20% of the overall holding and backordering costs in (15). For this parameter constellation (where we have \( c_B = 13.36 \)) total costs including expenditures for disposal (but excluding remanufacturing costs) can be shown to follow an U-shaped function as depicted in Figure 5.3. There exists an optimal value for \( \rho \) at a medium demand/returns level. This value characterizes the optimal trade-off between holding and backorder costs which are monotonically increasing and disposal costs which are monotonically decreasing as \( \rho \) increases. Figure 5.3 shows that it makes sense to influence the flow of returns to prevent that the system is in a suboptimal situation.

![Figure 5.3](image)

To pursue this point somewhat further we determine the optimal \( \rho \) as a function of \( \phi \), which is defined as the average disposal costs as a percentage of total costs (excluding remanufacturing costs). It follows from the above that we must first define a base case from which we derive the disposal cost per unit that is associated with the given value of \( \phi \). In Figure 5.4 below we consider three base cases defined by \( \rho_0 = 0.5, 0.75, 0.9 \). Next for each value of \( \phi \) we determine the optimal value of \( \rho \).
It follows from the results in Figure 5.4 that the optimal $\rho$ increases with $\phi$. This is intuitively clear, since the higher $\phi$, the higher the contribution of disposal cost to the total costs, and therefore we want to reduce the disposal costs by increasing $\rho$. The fact that the optimal values of $\rho$ increase with $\rho_0$ follows from a similar argument.

To gain insight into the sensitivity of the total cost for changes in $c_B$ we again start from a particular base case and determined $c_B$ for a given value of $\phi$. Next we determine for $\rho = 0.1, 0.2, \ldots, 0.9$ the relative change in $c_B$ that would result in exactly the same cost as in the base case. As base cases we assume $\rho_0 = 0.5, 0.8$. From Tables 5.5a we find that in case $\rho_0 = 0.5$ the value of $c_B$ may increase substantially for high values of $\rho$ in case $\phi$ is high, i.e. disposal costs are a large fraction of total cost in the base case. Hence decreasing the fraction of input disposed gives a high reduction in cost when $c_B$ remains unchanged or gets only slightly higher. In case $\rho_0 = 0.8$ table 5.5b shows that an increase in disposal costs per unit increases costs in almost all cases. In fact, a substantial decrease in disposal costs per unit is required if $\rho$ gets lower, i.e. if the amount disposed increases.
Finally we study the impact of $p$ on the division of system stocks and holding costs between the stock of returns and the stocks of serviceable options. Again we assume the base case data given above for demand and cost parameters. It follows from Figure 5.6 below that the fraction of system stock in serviceable options reduces as $p$ increases. This is caused by the fact that with higher $p$, i.e. average returns closer to average demand, we need a higher system stock to prevent high penalty costs. Since the cost of holding returns is lower than the cost of holding serviceable options, we need to keep relatively higher return stocks.

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<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
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<td>0%</td>
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<td>23%</td>
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Figure 5.5a: $p_o=0.5$

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Figure 5.5b: $p_o=0.8$
This concludes our discussion of the impact of model parameters on costs and control aspects.
6 Conclusions

In this paper a recovery management problem is investigated which often occurs in practice, but has not been addressed in literature up to now. It refers to developing optimal and near-optimal control rules for a stochastic product recovery problem where multiple remanufacturing options exist which yield different serviceable products satisfying specific demands. It is demonstrated that the structure of the optimal policy in this case is extremely complicated, due to the inherent allocation problem in case of scarce remanufacturables. However, employing so-called linear allocation rules it turned out that a fairly simple near-optimal control policy exists which is characterized by a single dispose-down-to level and a specific remanufacture-up-to level for each reuse option. For this policy which can easily be used in practical decision making, an efficient algorithm is developed for numerical evaluation of the policy parameters. Using this algorithm a numerical study was carried out, which helps to gain insight into the influence of disposal costs and the demand/returns ratio on the performance of such a remanufacturing system.

The results presented in this paper are developed under some restrictive assumptions which partly can be relaxed without difficulty. Due to the similarity of the serviceable options correlations of demands across these options may be present. For this case of correlated demands a straightforward extension of the policy analysis demonstrated above is possible. Furthermore, it is easy to take into account stochastic remanufacturing leadtimes for the different reuse options. On the other hand, including fixed remanufacturing costs or capacity constraints for the remanufacturing facilities makes an analysis of the optimal policy extremely difficult. From problems with a single reuse option we know that in these cases simple policies cannot be expected to be optimal. Under these circumstances plausible, but not necessarily optimal control rules which are known from the single-option case (cf. Van der Laan and Salomon, 1997) should be adapted.

Necessary extensions of our model which deserve further research should include the following additional aspects. In some instances demands for remanufactured products can also be fulfilled by newly manufactured items, usually at a considerably higher cost. Then manufacturing of serviceables, possibly in a specific mode for each different option, can be an additional source of satisfying demands. In this situation a combined optimization of disposal, remanufacturing and manufacturing is necessary. A further possible extension incorporates different quality levels of the returned products which may lead to different remanufacturing costs and leadtimes. Under these conditions multiple stocks of returned items have to be taken into account. Forthcoming research is devoted to investigating these extended multiple remanufacturing options problems basing on the results developed in this paper.
References


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Appendix
Appendix A

Analysis of the optimal MROP policy

A.1 Functional Equations for the General Optimization Problem

Different from the description of the functional equations in (1) we choose a backwards numbering of time periods. This is done for simplifying formulations, especially in case of \( T \to \infty \). Thus, with using index \( m \) to describe the number of periods remaining up to the end of the planning horizon, the functional equations can be written as

\[
\begin{align*}
  f_{m+1}(x_R, x_{S1}, \ldots, x_{Sn}) = \\
  \min_{b+\sum_{i} r_i \leq x_R, \ b, r_1, \ldots, r_n \geq 0} \left\{ c_B \cdot b + \sum_{i} c_{Ri} \cdot r_i + h_R \cdot (x_R - b - \sum_{i} r_i) + \sum_{i} L_i(x_{Si} + r_i) + \\
  + \alpha \cdot \int_{0}^{\infty} \ldots \int_{0}^{\infty} f_{m}(x_R - b - \sum_{i} r_i + R, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \cdot \\
  \cdot \varphi_R(R) \cdot \varphi_{D_1}(D_1) \cdot \ldots \cdot \varphi_{D_n}(D_n) \cdot dR \cdot dD_1 \cdot \ldots \cdot dD_n \right\}
\end{align*}
\]

\( \text{(A.1)} \)

for \( m \geq 1 \)

with \( f_0(x_R, x_{S1}, \ldots, x_{Sn}) \equiv 0 \)

In later considerations we also will use modified functional equations with value functions \( \bar{f}_m(x_E, x_{S1}, \ldots, x_{Sn}) \) where the returns stock variable \( x_R \) is replaced by the state variable \( x_E \).
for the echelon inventory position.

A.2 Optimal Policy for the Single-period Problem

From (A.1) we get as value function and optimization problem for \( m = 1 \)

\[
f_1(x_R, x_{S_1}, \ldots, x_{S_n}) = \min_{b, r_1, \ldots, r_n} g_1(x_R, x_{S_1}, \ldots, x_{S_n}, b, r_1, \ldots, r_n)
\]

w.r.t

\[
b + \sum_i r_i \leq x_R
\]

\[
b, r_1, \ldots, r_n \geq 0
\]

(A.2)

and with

\[
g_1(x_R, x_{S_1}, \ldots, x_{S_n}, b, r_1, \ldots, r_n) =
\]

\[
= c_B \cdot b + \sum_i c_{R_i} \cdot r_i + h_R \cdot (x_R - b - \sum_i r_i) + \sum_i L_i(x_{S_i} + r_i)
\]

Lemma 1: \( g_1(x_R, x_{S_1}, \ldots, x_{S_n}, b, r_1, \ldots, r_n) \) is convex in \( (b, r_1, \ldots, r_n) \)

Proof: Since \( L_i(x) \) is twice-differentiable and convex we find for the first-order and second-order partial derivatives

\[
g_1^{'}[b] = c_B - h_R
\]

\[
g_1^{'}[r_i] = c_{R_i} - h_R + L_i^{'}(x_{S_i} + r_i) \quad \forall i \in N
\]

\[
g_1^{''}[b, b] = 0
\]

\[
g_1^{''}[r_i, r_i] = L_i^{''}(x_{S_i} + r_i) \geq 0
\]

\[
g_1^{''}[r_i, r_j] = g_1^{''}[r_i, b] = 0 \quad \forall i \in N
\]

\[
g_1^{''}[r_i, r_j] = 0 \quad \forall i, j \in N \quad \text{and} \quad i \neq j
\]

where we use the notation:
\[ g'_1[y] = \frac{\partial g_1}{\partial y} \quad , \quad g''_{1[y_1, y_2]} = \frac{\partial^2 g_1}{\partial y_1 \cdot \partial y_2} \quad , \quad L'_i(y) = \frac{dL_i}{dy} \quad , \quad L''_i(y) = \frac{d^2 L_i}{dy^2} \]

For the \((n + 1) \times (n + 1)\) Hessian matrix we get

\[
H = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & L_1'' & 0 & \ldots & 0 \\
0 & 0 & L_2'' & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & L_n''
\end{bmatrix}
\]

Thus we find

\[
\det H = 0
\]

so that \(H\) is positive semidefinite.

\[
\square
\]

Since \(g_1(\cdot, \ldots, \cdot)\) and the restrictions in (A.2) are convex we can analyze the optimal policy by evaluating the Kuhn-Tucker optimality conditions.

With \(\kappa\) as a Lagrangian multiplier we can write the Lagrangian function \(g^*_k\) for problem (2)

\[
g^*_k(x_R, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n, \kappa) =
\]

\[
= c_B \cdot b + \sum_i c_R \cdot r_i + h_R \cdot (x_R - b - \sum_i r_i) + \sum_i L_i \cdot (x_{Si} + r_i) + \kappa \cdot (b + \sum_i r_i - x_R)
\]

So as Kuhn-Tucker conditions we find
(a) \[ g_{i[b]}^{\sigma'} = c_B - h_R + \kappa \geq 0 \]

(b.i) \[ g_{i[r_i]}^{\sigma'} = c_{R_i} - h_R + L'_i(x_{S_i} + r_i) + \kappa \geq 0 \quad \forall i \in N \]

(c) \[ g_{i[k]}^{\sigma'} = b + \sum_i r_i - x_R \leq 0 \]

(d) \[ b \cdot g_{i[b]}^{\sigma'} = 0 \]

(e.i) \[ r_i \cdot g_{i[r_i]}^{\sigma'} = 0 \quad \forall i \in N \]

(f) \[ \kappa \cdot g_{i[k]}^{\sigma'} = 0 \]

**Derivation of the optimal policy**

**Case 1:** \( \kappa = 0 \quad \rightarrow \quad b + \sum_i r_i \leq x_R \)

**Case 1.1:** \( b > 0 \)

From (d) and (a): \( c_B - h_R + \kappa = c_B - h_R = 0 \)

\( \rightarrow b > 0 \) is only feasible in the special case \( c_B = h_R \) (zero marginal cost!)

**Case 1.1.1:** \( r_i > 0 \) for any \( i \in N \)

From (e.i) and (b.i): \( c_{R_i} - h_R + L'_i(x_{S_i} + r_i) = 0 \)

\( \rightarrow x_{S_i} + r_i = M_i^{(1)} \) with

\[ M_i^{(1)} \text{ from: } L'_i(M_i^{(1)}) = h_R - c_{R_i} \quad (A.3) \]

\( \rightarrow r_i = M_i^{(1)} - x_{S_i} > 0 \)

\( \rightarrow x_{S_i} < M_i^{(1)} \)

**Case 1.1.2.:** \( r_i = 0 \) for any \( i \in N \)
From (e.i) and (b.i): \( c_{R_i} - h_R + L'_i(x_{S_i} + r_i) \geq 0 \)

\[ x_{S_i} + r_i = x_{S_i} \geq M_i^{(1)}, \text{ because of convexity of } L_i(\cdot) \]

Results in Case 1.1:

Cost condition: \( c_B = h_R \)

\[ r_i = [M_i^{(1)} - x_{S_i}]^+ \quad \forall i \in N \]

\[ \sum_i [M_i^{(1)} - x_{S_i}]^+ < x_R \]

\[ b \in (0, x_R - \sum_i [M_i^{(1)} - x_{S_i}]^+) \]

Case 1.2: \( b = 0 \)

From (d) and (a): \( c_B - h_R \geq 0 \)

Cases 1.2.1 and 1.2.2 are identical with Cases 1.1.1 and 1.1.2
Results:

Cost condition: $c_B \geq h_R$

$r_i = [M_i^{(1)} - x_{Si}]^+ \quad \forall i \in N$

$\sum_i [M_i^{(1)} - x_{Si}]^+ \leq x_R$

$b = 0$

Case 2.: $\kappa > 0 \quad \rightarrow \quad b + \sum_i r_i = z_R$

Case 2.1.: $b > 0 \quad \rightarrow \quad \sum_i r_i < z_R$

From (d) and (a):

$c_B - h_R + \kappa = 0 \quad \rightarrow \quad c_B < h_R$

$\quad \rightarrow \quad \kappa = h_R - c_B$

Case 2.1.1: $r > 0$ for any $i \in N$

From (e.i) and (b.i):

$c_{Ri} - h_R + L_i'(x_{Si} + r_i) + h_R - c_B = 0$

$\rightarrow \quad x_{Si} + r_i = M_i^{(2)}$ with

$M_i^{(2)}$ from: $L_i'(M_i^{(2)}) = c_B - c_{Ri}$

$\rightarrow \quad r_i = M_i^{(2)} - x_{Si} > 0$

$\rightarrow \quad x_{Si} < M_i^{(2)}$

Case 2.1.2: $r_i = 0$ for any $i \in N$

From (e.i) and (b.i):

$c_{Ri} - c_B + L_i'(x_{Si} + r_i) \geq 0$

$\rightarrow \quad x_{Si} + r_i = x_{Si} \geq M_i^{(2)}$

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Results in Case 2.1:

Cost condition: \( c_B < h_R \)

\[
\begin{align*}
\forall i \in N \quad r_i &= [M_i^{(2)} - x_{Si}]^+ \\
\sum_i [M_i^{(2)} - x_{Si}]^+ &< x_R \\
b &= x_R - \sum_i [M_i^{(2)} - x_{Si}]^+
\end{align*}
\]

Case 2.2.: \( b = 0\) \( \implies \sum_i r_i = x_R \)

From (d) and (a): \( c_B - h_R + \kappa \geq 0 \)

Case 2.2.1.: \( r_i > 0\) for any \( i \in N \)

From (e.i) and (b.i): \( c_{RI} - h_R + L_i(x_{Si} + r_i) + \kappa = 0 \)

\( \rightarrow \) \( x_{Si} + r_i = M_i(\kappa) \) with

\[
M_i(\kappa) \text{ from: } L_i(M_i(\kappa)) = h_R - c_{RI} - \kappa \quad \text{(A.5)}
\]

\( \rightarrow \) \( r_i = M_i(\kappa) - x_{Si} > 0 \)

\( \rightarrow \) \( x_{Si} < M_i(\kappa) \)

Case 2.2.2.: \( r_i = 0\) for any \( i \in N \)

From (e.i) and (b.i): \( c_{RI} - h_R + L_i(x_{Si} + r_i) + \kappa \geq 0 \)

\( \rightarrow \) \( x_{Si} + r_i = x_{Si} \geq M_i(\kappa) \)
Results in Case 2.2:

Cost condition: \( c_B + \kappa \geq h_R \)

\( b = 0 \)

\( r_i = [\bar{M}_i - x_{S_i}]^+ \quad \forall i \in N \)

where \( \bar{M}_i \) and \( \kappa \) can be evaluated from (A.5) and from \( \sum r_i = x_R \)

\[
L'_i(\bar{M}_i) = h_R - c_{Ri} - \kappa \quad \forall i \in N
\]

\[
\sum_i [\bar{M}_i - x_{S_i}]^+ = x_R \quad \text{(A.6)}
\]

So in this case the policy parameters \( \bar{M}_i \) depend on all state variables:

\( \bar{M}_i = \bar{M}_i(x_R, x_{S_1}, \ldots, x_{S_n}) \quad \forall i \in N \).

From (A.6) we also see that the following condition holds for \( \bar{M}_i \):

\[
L'_1(\bar{M}_1) + c_{R1} = L'_2(\bar{M}_2) + c_{R2} = \ldots = L'_n(\bar{M}_n) + c_{Rn}
\]

Summary of the Optimal Policy

Using (A.3) and (A.4) we define

\[
M_i = \begin{cases} 
M_i^{(1)} & \text{for } c_B > h_R \\
M_i^{(2)} & \text{for } c_B \leq h_R 
\end{cases}
\]

and

\[
\bar{U}(x_{S_1}, \ldots, x_{S_n}) = \begin{cases} 
\infty & \text{for } c_B > h_R \\
\sum_i [M_i - x_{S_i}]^+ & \text{for } c_B \leq h_R 
\end{cases}
\]
Then we can summarize

\[
x_R < \sum_i (M_i - x_{S_i})^+ \Rightarrow b^* = 0, \quad r_i^* = [M_i(x_R, x_{S_1}, \ldots, x_{S_n}) - x_{S_i}]^+
\]

\[
\sum_i (M_i - x_{S_i})^+ \leq x_R \leq \bar{U}(x_{S_1}, \ldots, x_{S_n}) \Rightarrow b^* = 0, \quad r_i^* = [M_i - x_{S_i}]^+
\]

\[
x_R > \bar{U}(x_{S_1}, \ldots, x_{S_n}) \Rightarrow b^* = x_R - \bar{U}(x_{S_1}, \ldots, x_{S_n}), \quad r_i^* = [M_i - x_{S_i}]^+
\]

(A.7)
Appendix B

Optimal Policy under Linear Allocation Rules

B.1 The Single-period Problem

Applying the linear allocation rules in (6)

\[ r_i^0 = M_i - q_i \cdot \left[ \sum_j M_j - x_E \right]^+ - x_{Si} \quad \forall i \in N \]

and using the balance assumption according to (7)

\[ r_i^0 \geq 0 \quad \forall i \in N \]

we experience

\[ x_{Si} \leq \begin{cases} M_i & \text{if } \sum_j M_j \leq x_E \\ M_i - q_i \cdot \left( \sum_j M_j - x_E \right) & \text{if } \sum_j M_j > x_E \end{cases} \]

Thus we have

\[ \sum_i [M_i - x_{Si}]^+ = \sum_i M_i - \sum_i x_{Si} \quad \text{(B.1)} \]
From (B.1) the comparisons of the optimal single-stage policy in (A.7) simplify to

$$x_R < \sum_i [M_i - x_{Si}]^+ \Leftrightarrow x_R < \sum_i M_i - \sum_i x_{Si}$$

$$\Leftrightarrow x_R + \sum_i x_{Si} = x_E < \sum_i M_i$$

and

$$x_R \leq \bar{U}(x_{S1}, \ldots, x_{Sn}) \Leftrightarrow x_R \leq \begin{cases} \infty & \text{for } c_B > h_R \\ \sum_i M_i - \sum_i x_{Si} & \text{for } c_B \leq h_R \end{cases}$$

$$\Leftrightarrow x_R + \sum_i x_{Si} = x_E \leq \begin{cases} \infty & \text{for } c_B > h_R \\ \sum_i M_i & \text{for } c_B \leq h_R \end{cases}$$

Defining

$$U = \begin{cases} \infty & \text{for } c_B > h_R \\ \sum_i M_i & \text{for } c_B \leq h_R \end{cases}$$

and using (6) and (7) the optimal policy in (A.7) can be specified as

\[
\begin{align*}
 x_E < \sum_j M_j & \Rightarrow b^0 = 0, \quad r_i^0 = M_i - q_i \cdot (\sum_j M_j - x_E) - x_{Si} \\
 \sum_j M_j \leq x_E \leq U & \Rightarrow b^0 = 0, \quad r_i^0 = M_i - x_{Si} \\
x_E > U & \Rightarrow b^0 = x_E - U, \quad r_i^0 = M_i - x_{Si}
\end{align*}
\]  

(B.2)

Inserting the \((nM, U)\)-policy from (B.2) in cost function \(g_1(x_R, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n)\) we find as minimum cost under linear allocation rules a function \(\bar{f}_1(x_R, x_{S1}, \ldots, x_{Sn})\) which depends on the echelon stock \(x_E\) and the single inventory positions \(x_{Si}\) as follows
\[
\hat{f}_1(x_E, x_{S_1}, \ldots, x_{S_n}) = \sum_i c_{R_i} \cdot (M_i - x_{S_i}) +
\]

\[
\left\{ \begin{array}{l}
-c_R \cdot (\sum_i M_i - x_E) + \sum_i L_i(\hat{M}_i(x_E)) \quad \text{for} \quad x_E < \sum M_i \\
h_R \cdot (x_E - \sum_i M_i) + \sum_i L_i(M_i) \quad \sum M_i \leq x_E \leq U \\
c_B \cdot (x_E - \sum_i M_i) + \sum_i L_i(M_i) \quad x_E > U
\end{array} \right. \]  

(B.3)

with the definitions

\[
\hat{M}_i(x_E) = M_i - q_i \cdot (\sum_j M_j - x_E) \quad \forall i \in N \]  

(B.4)

and

\[
c_R = \sum_i c_{R_i} \cdot q_i \]  

(B.5)

Thus under the linear allocation rule and balance assumption we find for the respective value function in (A.2)

\[
f_1^0(x_{R_1}, x_{S_1}, \ldots, x_{S_n}) = \hat{f}_1(x_E, x_{S_1}, \ldots, x_{S_n}) \]  

Properties of Cost Function $\hat{f}_1(x_E, x_{S_1}, \ldots, x_{S_n})$

Lemma 2: Separability

$\hat{f}_1(x_E, x_{S_1}, \ldots, x_{S_n})$ can be expressed as a sum of separate functions in its respective variables.

Proof:

From (B.3) we immediately see that
\[ f_1(x_E, x_{S1}, \ldots, x_{Sn}) = f_{10}(x_E) + \sum_i f_{1i}(x_{Si}) \]

where

\[
f_{10}(x_E) = \begin{cases} 
  -c_R \cdot \sum_i (M_i - x_E) + \sum_i L_i(\hat{M}_i(x_E)) & \text{for } x_E < \sum_i M_i \\
  h_R \cdot (x_E - \sum_i M_i) + \sum_i L_i(M_i) & \sum_i M_i \leq x_E \leq U \\
  c_B \cdot (x_E - \sum_i M_i) + \sum_i L_i(M_i) & x_E > U 
\end{cases}
\]

(B.6)

and

\[
f_{1i}(x_{Si}) = c_{Ri} \cdot (M_i - x_{Si}) \quad \forall i \in N \quad \Box
\]

\(f_{10}(x_E)\) is continuous since \(\hat{M}_i(x_E) = M_i\) for \(x_E = \sum_i M_i\) and \(h_R = c_B\) for \(x_E = U\). Thus, also \(f_1(x_E, x_{S1}, \ldots, x_{Sn})\) is a continuous function.

**Lemma 3: Convexity**

\(f_1(x_E, x_{S1}, \ldots, x_{Sn})\) is a convex function.

**Proof:**

For proof of convexity we analyse the Hessian matrix of second-order derivatives. Hereby we have to note that: \(x_E = x_E(x_{S1}, \ldots, x_{Sn}) = x_R + \sum_i x_{Si}\)

First order derivatives:

\[
f_{1i}^{'}(x_E) = \begin{cases} 
  c_R + \sum_i q_i \cdot L_i'(M_i(x_E)) & \text{for } x_E < \sum_i M_i \\
  h_R & \sum_i M_i \leq x_E \leq U \\
  c_B & x_E > U 
\end{cases}
\]

\[
f_{1i}^{'}(x_{Si}) = f_{1i}^{'}(x_E) - c_{Ri} \quad \forall i \in N
\]
From the definition of $U$ and $M_i$ $(i \in N)$ it can be seen that $\tilde{f}_{1[x_E]}$ is continuous in $x_E$.

Second-order derivatives:

$$\tilde{f}_{1[x_E,x_E]}'' = \begin{cases} 
\sum_i q_i^2 \cdot L_i''(M_i(x_E)) \geq 0 & x_E < \sum_i M_i \\
0 & x_E \geq \sum_i M_i 
\end{cases}$$

Furtheron we find

$$\tilde{f}_{1[x_S,x_S]}'' = \tilde{f}_{1[x_E,x_S]}'' = \tilde{f}_{1[x_S,x_E]}'' = f'' \quad \forall i \in N$$

where $f'' = \tilde{f}_{1[x_E,x_E]}''$.

Due to convexity of $L_i(x)$ $\forall i \in N$ it holds that $f'' > 0$.

Thus for the $(n + 1) \times (n + 1)$ Hessian Matrix we find

$$H = \begin{bmatrix} 
f'' & f'' & \cdots & f'' \\
f'' & f'' & \cdots & f'' \\
\vdots & \vdots & \ddots & \vdots \\
f'' & f'' & \cdots & f'' 
\end{bmatrix} \quad \text{and} \quad \det H = 0$$

This shows that $H$ is positive semidefinite, which completes the proof. □
B.2 The Multi-Period Problem

It will be shown by induction that, if an \((nM, U)\)-policy holds for an \(m\)-period problem and if for the value function we have \(f_m^0(x_R, x_S_1, \ldots, x_S_n) = f_m(x_E, x_S_1, \ldots, x_S_n)\) with separability and convexity property, the same holds for a \((m + 1)\)-period problem.

We assume that

\[
f_m^0(x_R, x_S_1, \ldots, x_S_n) = f_m(x_E, x_S_1, \ldots, x_S_n) = f_m^0(x_E) + \sum_i f_m^i(x_S_i) \quad (B.7)
\]

with convex functions \(f_m^0(x)\) and \(f_m^i(x)\) \(\forall i \in N.\)

Using the assumption in (B.7) as well as the balance equations for \(x_E\) and \(x_S_i\), the \(m\)-period value function in the general recursive equation (A.1) can be replaced as follows

\[
f_m(x_R - b - \sum_i r_i + R, x_S_1 + r_1 - D_1, \ldots, x_S_n + r_n - D_n) =
\]

\[
= f_m^0(x_E - b + R - \sum_i D_i) + \sum_i f_m^i(x_S_i + r_i - D_i).
\]

Furtheron, substituting the remaining stock variables \(x_R\) in (A.1) by \(x_E - \sum_i x_S_i\) leads to the minimization problem which is formulated in below (B.8) and lets the value function depend on a state vector \((x_E, x_S_1, \ldots, x_S_n)\) instead of \((x_R, x_S_1, \ldots, x_S_n)\)

\[
\bar{f}_{m+1}(x_E, x_S_1, \ldots, x_S_n) =
\]

\[
\min_{x_E, x_S_1, \ldots, x_S_n \geq 0} \left\{ c_B \cdot b + \sum_i c_{R_i} \cdot r_i + h_R \cdot (x_E - b - \sum_i(x_S_i + r_i)) + \sum_i L_i(x_S_i + r_i) +
\right.
\]

\[
+ \alpha \cdot \int_0^\infty \ldots \int_0^\infty [f_m^0(x_E - b + R - \sum_i D_i) + \sum_i f_m^i(x_S_i + r_i - D_i)] \cdot
\]

\[
\varphi_R(R) \cdot \varphi_{D_1}(D_1) \cdot \ldots \cdot \varphi_{D_n}(D_n) \cdot dR \cdot dD_1 \cdot \ldots \cdot dD_n \right\}
\]

(B.8)

Using the stochastic net supply variable \(S = R - \sum_i D_i\) and defining
\[ G_{m+1,0}(x) = h_R \cdot x + \alpha \cdot \int_{-\infty}^{\infty} f_{m0}(x + S) \cdot \varphi_S(S) \cdot dS \quad \text{and} \]
\[ G_{m+1,i}(x) = -h_R \cdot x + L_i(x) + \alpha \cdot \int_{0}^{\infty} f_{mi}(x - D_i) \cdot \varphi_{D_i}(D_i) \cdot dD_i \]

we can rewrite (B.8) as follows
\[ f_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) = \min_{b, r_1, \ldots, r_n} g_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) \]

w.r.t. \( b, r_1, \ldots, r_n \geq 0 \)
\[ b + \sum_i r_i \leq x_E - \sum_i x_{Si} \]

where
\[ g_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) = \]
\[ c_B \cdot b + \sum_i c_{Ri} \cdot r_i + G_{m+1,0}(x_E - b) + \sum_i G_{m+1,i}(x_{Si} + r_i) \]  

(B.9)

Lemma 4:
\[ g_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) \] is convex in \( (b, r_1, \ldots, r_n) \).

Proof:

Since additional to \( L_i(x) \) due to assumption also \( f_{m0}(x) \) and \( f_{mi}(x) \) are convex functions, we find the same for \( G_{m+1,0}(x) \) and \( G_{m+1,i}(x) \) for \( i \in N \).

Furthermore, \( G_{m+1,0}(x) \) and \( G_{m+1,i}(x) \) are twice-differentiable.

So we have the following first-order and second-order partial derivatives
Thus, for the \((m + 1) \times (m + 1)\) Hessian matrix we get
\[
H = \begin{bmatrix}
G''_{m+1,0} & 0 & 0 & \cdots & 0 \\
0 & G''_{m+1,1} & 0 & \cdots & 0 \\
0 & 0 & G''_{m+1,2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & G''_{m+1,n}
\end{bmatrix}
\]
resulting in \(\det H \geq 0\) and positive semidefiniteness of \(H\).

As in Appendix A.2 we formulate the Lagrange function
\[
g^\kappa_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n, \kappa)
\]
\[
= g_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_n) + \kappa \cdot (b + \sum_i r_i - x_E + \sum_i x_{Si})
\]
and apply the Kuhn-Tucker conditions

\((\alpha)\) \quad g_{m+1,[b]}^\prime = c_B - G'_{m+1,0}(x_E - b) + \kappa \geq 0

\((\beta.i)\) \quad g_{m+1,[r_i]}^\prime = c_{Ri} + G'_{m+1,i}(x_{Si} + r_i) + \kappa \geq 0 \quad \forall i \in N

\((\gamma)\) \quad g_{m+1,[\kappa]}^\prime = b + \sum_i r_i - x_E + \sum_i x_{Si} \leq 0

\((\delta)\) \quad b \cdot g_{m+1,[b]}^\prime = 0

\((\varepsilon.i)\) \quad r_i \cdot g_{m+1,[r_i]}^\prime = 0 \quad \forall i \in N

\((\zeta)\) \quad \kappa \cdot g_{m+1,[\kappa]}^\prime = 0

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Derivation of the Optimal Policy

The policy derivation follows exactly the lines shown in Appendix A.2 for Case 1 and Case 2, taking into account the analogy of the Kuhn-Tucker conditions.

Case 1: \( \kappa = 0 \) → \( b + \sum_i r_i \leq x_E - \sum_i x_{Si} \)

Case 1.1: \( b > 0 \)

From (5) and (6): \( c_B - G'_{m+1,0}(x_E - b) = 0 \)

→ \( x_E - b = U^{(1)} \) with

\[
U^{(1)} \text{ from: } G'_{m+1,0}(U^{(1)}) = c_B \quad (B.10)
\]

→ \( b = x_E - U^{(1)} > 0 \)

→ \( x_E > U^{(1)} \)

Case 1.2: \( b = 0 \)

From (5) and (6): \( c_B - G'_{m+1,0}(x_E - b) \geq 0 \)

→ \( x_E - b = x_E \leq 0 \)

So the result in Case 1 is

\[ b = [x_E - U^{(1)}]^+ \]

Analogously to Appendix A.2 we find

\( r_i = [M_i^{(1)} - x_{Si}]^+ \) \( \forall i \in N \) with

\[
M_i^{(1)} \text{ from: } G'_{m+1,i}(M_i^{(1)}) = -c_{Ri} \quad \forall i \in N \quad (B.11)
\]
Due to $\kappa = 0$ we additionally find

$$[x_E - U^{(1)}]^+ + \sum_i [M_i^{(1)} - x_{Si}]^+ \leq x_E - \sum_i x_{Si}$$

Under the balance assumption this yields

$$\sum_i [M_i^{(1)} - x_{Si}]^+ = \sum_i M_i^{(1)} - \sum x_{Si},$$

leading to $[x_E - U^{(1)}]^+ = \max\{x_E - U^{(1)}, 0\} = x_E - \sum_i M_i^{(1)}$

Thus as parameters restriction we get:

$$\sum_i M_i^{(1)} \leq U^{(1)}$$

Case 2: $\kappa > 0 \rightarrow b + \sum_i r_i = x_E - \sum_i x_{Si}$

Case 2.1: $b > 0$

From $(\delta)$ and $(\alpha)$: $c_B - G'_{m+1,0}(x_E - b) + \kappa = 0$

$\rightarrow x_E - b = U(\kappa)$ with

$$U(\kappa) \text{ from: } G'_{m+1,0}(U) = c_B + \kappa$$

and $U(\kappa) > U^{(1)}$ due to $\kappa > 0$

$\rightarrow b = x_E - U(\kappa) > 0$

$\rightarrow x_E > U(\kappa)$

From $(\varepsilon.i)$ and $(\beta.i)$ we find in extension to the results of Appendix A.2

$r_i = [M_i(\kappa) - x_{Si}]^+ \ \forall i \in N \ \text{with}$

$M_i(\kappa) \text{ from: } G'_{m+1,i}(M_i) = -c_{R_i} - \kappa \ \forall i \in N$

where $M_i(\kappa) < M_i^{(1)}$ due to $\kappa > 0$. 

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Together with \( b + \sum_i r_i = x_E - \sum_i x_{Si} \) (because of \( \kappa > 0 \)) we have a system of \( n + 2 \) equations which determines \( \kappa \) as well as the respective policy parameters \( U^{(2)} \) and \( M^{(2)}_i \) \( (\forall i \in N) \)

\[
G'_{m+1,0}(U^{(2)}) = c_B + \kappa \tag{B.12}
\]

\[
G_{m+1,i}(M^{(2)}_i) = -c_{Ri} - \kappa \quad \forall i \in N \tag{B.13}
\]

\[
x_E - U^{(2)} + \sum_i [M^{(2)}_i - x_{Si}]^+ = x_E - \sum_i x_{Si} \tag{B.14}
\]

Applying the balance assumption we can use \( \sum_i [M^{(2)}_i - x_{Si}]^+ = \sum_i M^{(2)}_i - \sum_i x_{Si} \), so that equation (B.14) yields

\[
U^{(2)} = \sum_i M^{(2)}_i \tag{B.15}
\]

Note that from (B.12) and (B.13) we also find that

\[
c_B - G'_{m+1,0}(U^{(2)}) = c_{R1} + G'_{m+1,1}(M^{(2)}_1) = \ldots = c_{Rn} + G'_{m+1,n}(M^{(2)}_n)
\]

From \( U^{(1)} < U^{(2)} = \sum_i M^{(2)}_i < \sum_i M^{(1)}_i \) we see that in case of \( b > 0 \) and \( \kappa > 0 \) the following relation between the policy parameters must hold

\[
U^{(1)} < \sum_i M^{(1)}_i
\]

Case 2.2: \( b = 0 \)

From \( (\delta) \) and \( (\alpha) \) : \( c_B - G'_{m+1,0}(x_E - b) + \kappa \geq 0 \)

\[
\Rightarrow \quad x_E - b = x_E < U^{(2)}
\]
Because of $b = 0$ and $\kappa > 0$, here $\tilde{M}_i = M_i(\kappa)$ and $\kappa$ are solutions of the following system of $n + 1$ equations

\[ G'_{m+1,i}(\tilde{M}_i) = -c_{Ri} - \kappa \quad \forall i \in N \quad (B.16) \]

\[ \sum_i [\tilde{M}_i - x_{Si}]^+ = x_E - \sum_i x_{Si} \quad (B.17) \]

Under the balance assumption we find

\[ \sum_i \tilde{M}_i = x_E \quad (B.18) \]

For summarizing the optimal policy under the linear allocation rule and balance assumption we define

\[ M_i = \min \{ M_i^{(1)}, M_i^{(2)} \} \quad \forall i \in N \]

\[ U = \max \{ U^{(1)}, U^{(2)} \} \]

with $M_i^{(1)}$ from (B.11), $U^{(1)}$ from (B.10) and $\{ M_i^{(2)}, U^{(2)} | i \in N \}$ from (B.12), (B.13) and (B.15).

Using the findings of Case 2.1. and Case 2.2 we get

\[
\begin{align*}
    x_E < \sum_j M_j & \Rightarrow b^0 = 0 \quad , \quad r_i^* = [\tilde{M}_i - x_{Si}]^+ \\
    \sum_j M_j \leq x_E \leq U & \Rightarrow b^0 = 0 \quad , \quad r_i^0 = M_i - x_{Si} \\
    x_E > U & \Rightarrow b^0 = x_E - U \quad , \quad r_i^0 = M_i - x_{Si}
\end{align*}
\quad (B.19)
\]

When we apply the linear allocation rule as defined in (6) we have to replace $r_i^* = [\tilde{M}_i - x_{Si}]^+$ by
\[ r_i^0 = M_i - q_i \cdot (\sum_j M_j - x_E) - x_{Si} \]

and find that the resulting policy is completely identical with the \((nM, U)\)-policy we derived for the single-period problem in (B.2).

**Derivation of the minimal costs**

Inserting the decisions from the \((nM, U)\)-policy into cost function \( g_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}, b, r_1, \ldots, r_2) \) in (B.9) we receive as state-dependent minimal costs

\[
\tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) = \sum_i c_{Ri} \cdot (M_i - x_{Si}) + \left\{
\begin{array}{ll}
-c_R \cdot (\sum_i M_i - x_E) + G_{m+1,0}(x_E) + \sum_i G_{m+1,i}(\tilde{M}_i(x_E)) & \text{for } x_R < \sum_i M_i \\
G_{m+1,0}(x_E) + \sum_i G_{m+1,i}(M_i) & \sum_i M_i \leq x_E \leq U \\
c_B \cdot (x_E - U) + G_{m+1,0}(U) + \sum_i G_{m+1,i}(M_i) & x_E > U \\
\end{array}
\right.
\]

with \( \tilde{M}_i(x_E) \) and \( c_R \) as defined in (B.4) and (B.5). So we see that also for the value function of the \((m+1)\)-period problem we have \( f_{m+1}^0(x_R, x_{S1}, \ldots, x_{Sn}) = \tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) \).

To complete the proof of optimality of a \((nM, U)\)-policy for any multi-period problem, the separability and convexity property of \( \tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) \) will be shown.

**Lemma 5: Separability**

\( \tilde{f}_{m+1}(x_R, x_{S1}, \ldots, x_{Sn}) \) can be expressed as a sum of separate functions in its respective variables.

**Proof:**

From (B.20) we directly see that
\[ \hat{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) = f_{m+1,0}(x_E) + \sum_i f_{m+1,i}(x_{Si}) \]

where

\[ f_{m+1,0}(x_E) = \begin{cases} 
-c_R \cdot (\sum_i M_i - x_E) + G_{m+1,0}(x_E) + \sum_i G_{m+1,i}(\hat{M}_i(x_E)) & \text{for } x_E < \sum M_i \\
G_{m+1,0}(x_E) + \sum_i G_{m+1,i}(M_i) & \sum M_i \leq x - E \leq U \\
c_B \cdot (x_E - U) + G_{m+1,0}(U) + \sum_i G_{m+1,i}(M_i) & x_E > U
\end{cases} \]

and

\[ f_{m+1,i}(x_{Si}) = c_{Ri} \cdot (M_i - x_{Si}) \quad \forall i \in \mathbb{N} \]

\[ f_{m+1,0}(x_E), \text{ and thus also } \hat{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}), \text{ is continuous since } \]

\[ \hat{M}_i(x_E) = M_i \quad \text{for } x_E = \sum M_i. \]

**Lemma 6: Convexity**

\[ \hat{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) \text{ is a convex function.} \]

**Proof:**

For proof of convexity we analyse the Hessian matrix of second-order derivatives of the function in (B.20). Hereby we have to not that \( x_E = x_E(x_{S1}, \ldots, x_{Sn}) = x_R + \sum x_{Si} \).

First-order derivatives:

\[ \hat{f}'_{m+1,0}(x_E) = \begin{cases} 
c_R + G'_{m+1,0}(x_E) + \sum_i q_i \cdot G'_{m+1,i}(\hat{M}_i(x_E)) & \text{for } x_E < \sum M_i \\
G'_{m+1,0}(x_E) & \sum M_i \leq x_E \leq U \\
c_B & x_E > U
\end{cases} \]
\[ \tilde{f}_{m+1, \{x_{Si}\}} = f_{m+1, \{x_E\}} - c_{Ri} \quad \forall i \in N \]

From the definition of \( U \) and \( M_i \) it can be seen that \( \tilde{f}_{m+1, \{x_E\}} \) is continuous in \( x_E \).

Second-order derivatives:
\[
\tilde{f}''_{m+1, \{x_E, x_E\}} = \begin{cases} 
G''_{m+1, 0}(x_E) + \sum_i q_i^2 \cdot G''_{m+1, i}(M_i(x_E)) & \text{for } x_E < \sum_i M_i \\
G''_{m+1, 0}(x_E) & \sum_i M_i \leq x_E \leq U \\
0 & x_E > U 
\end{cases}
\]

Further we find
\[
\tilde{f}''_{m+1, \{x_{Si}, x_{Si}\}} = \tilde{f}''_{m+1, \{x_E, x_E\}} = \tilde{f}''_{m+1, \{x_{Si}, x_E\}} = f''
\]

where \( f'' = \tilde{f}''_{m+1, \{x_E, x_E\}} \).

Due to convexity of \( G_{m+1, 0}(x) \) and \( G_{m+1, i}(x) \) \( \forall i \in N \) it holds that \( f'' \geq 0 \).

Thus for the \((n+1) \times (n+1)\) Hessian matrix we find
\[
H = \begin{bmatrix} 
f'' & f'' & \cdots & f'' \\
f'' & f'' & \cdots & f'' \\
\vdots & \vdots & \ddots & \vdots \\
f'' & f'' & \cdots & f'' 
\end{bmatrix} \quad \text{and } \det H = 0
\]

This shows that \( H \) is positive semidefinite, which completes the proof. \( \square \)

Alltogether we observe that for each \((m+1)\)-period problem under the linear allocation rule and balance assumption a modified version of the functional equations in (A.1) holds, which according to (B.8) can be expressed by
\begin{equation}
\tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) = \min_{b + \sum_{i} \tau_i \leq x_E - \sum_{i} x_S_i, \ \ b, \tau_1, \ldots, \tau_n \geq 0} \left\{ c_B \cdot b + \sum_{i} c_{R_i} \cdot r_i + h_R \cdot (x_E - b - \sum_{i} (x_S_i + r_i)) + \sum_{i} L_i (x_S_i + r_i) + \alpha \cdot \int_{-\infty}^{\infty} \cdots \int_{0}^{\infty} \tilde{f}_m (x_E - b + S, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \cdot \varphi_0(S) \cdot \varphi_{D_1}(D_1) \cdot \cdots \cdot \varphi_{D_n}(D_n) \cdot dS \cdot dD_1 \cdot \cdots \cdot dD_n \right\}
\end{equation}

\text{(B.21)}

for \( m \geq 1 \)

with \( f_0(x_E, x_{S1}, \ldots, x_{Sn}) \equiv 0 \),

where all functions \( \tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) \) are separable and convex.
Appendix C

The Infinite Horizon Problem under Linear Allocation Rules

In order to show that the value function for \( f_m(x_E, x_{S1}, \ldots, x_{Sn}) \) in (B.21) converges to a finite function \( f(x_E, x_{S1}, \ldots, x_{Sn}) \) when the number of periods tend to infinity \( (m \to \infty) \), we prove that \( f_m(x_E, x_{S1}, \ldots, x_{Sn}) \) is monotonously increasing and bounded from above.

Lemma 7: \( f_m(x_E, x_{S1}, \ldots, x_{Sn}) \) is monotonously increasing in \( m \).

Proof:

The Lemma will be proved by induction.

From (B.21) we see that

\[
\tilde{f}_0(x_E, x_{S1}, \ldots, x_{Sn}) = 0 \quad \text{and}
\]

\[
\tilde{f}_1(x_E, x_{S1}, \ldots, x_{Sn}) = \min_{b + \sum_{i} r_i \leq x_E - \sum_{i} x_{si}} \left\{ c_B \cdot b + \sum_{i} c_{R_i} \cdot r_i + \right.
\]

\[
+ h_R \cdot (x_E - \sum_{i} x_{si} - b - \sum_{i} r_i) + \sum_{i} L_i (x_{Si} + r_i) \right\}
\]

If all cost parameters (including \( c_B \)) are non-negative, obviously \( \tilde{f}_1(x_E, x_{S1}, \ldots, x_{Sn}) \geq 0 \), so that as start of induction we find

\[
\tilde{f}_1(x_E, x_{S1}, \ldots, x_{Sn}) \geq \tilde{f}_0(x_E, x_{S1}, \ldots, x_{Sn})
\]
If the disposal costs are allowed to be negative \((c_B < 0 \Rightarrow \text{disposal revenues})\), this result still holds when we consequently assume that at the end of the planning horizon a surplus of returned items is disposed at the same (negative) costs: \(\tilde{f}_0(x_E, x_{S1}, \ldots, x_{Sn}) = c_B \cdot (x_E - \sum x_{Si})\).

Next we assume that

\[
\tilde{f}_m(x_E, x_{S1}, \ldots, x_{Sn}) \geq \tilde{f}_{m-1}(x_E, x_{S1}, \ldots, x_{Sn}).
\]

Then from (B.21) we find due to non-negativity of the density functions

\[
\tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) = \min \left\{ c_B \cdot b + \sum_i c_{Ri} \cdot r_i + h_R \cdot (x_E - b - \sum_i (x_{Si} + r_i)) + \sum_i L_i(x_{Si} + r_i) \right. \\
+ \alpha \cdot \int_{-\infty}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \tilde{f}_m(x_E - b + S, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \\
\left. \cdot \varphi_S(S) \cdot \varphi_{D1}(D_1) \ldots \varphi_{Dn}(D_n) \cdot dS \cdot dD_1 \ldots dD_n \right\}
\]

\[
\geq \min \left\{ c_B \cdot b + \sum_i c_{Ri} \cdot r_i + h_R \cdot (x_E - b - \sum_i (x_{Si} + r_i)) + \sum_i L_i(x_{Si} + r_i) \right. \\
+ \alpha \cdot \int_{-\infty}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \tilde{f}_{m-1}(x_E - b + S, x_{S1} + r_1 - D_1, \ldots, x_{Sn} + r_n - D_n) \\
\left. \cdot \varphi_S(S) \cdot \varphi_{D1}(D_1) \ldots \varphi_{Dn}(D_n) \cdot dS \cdot dD_1 \ldots dD_n \right\}
\]

\[
= f_m(x_E, x_{S1}, \ldots, x_{Sn})
\]

This yields

\[
\tilde{f}_{m+1}(x_E, x_{S1}, \ldots, x_{Sn}) \geq \tilde{f}_m(x_E, x_{S1}, \ldots, x_{Sn})
\]

which completes the proof by induction.

**Lemma 8:** \(\tilde{f}_m(x_E, x_{S1}, \ldots, x_{Sn})\) is bounded from above for \(m = 1, 2, \ldots\)
Proof:

To prove this Lemma we show that for a specific feasible policy with linear allocation rules which is applied in each period an upper bound for $f_m(x_E, x_{S1}, \ldots, x_{Sn})$ exists, so that this is also true for the respective optimal policy.

This specific policy is characterized by the following decisions $\tilde{b}$ and $\tilde{r}_i$,

$$\tilde{b} = \max\{x_E - \frac{0}{M}, 0\} \tag{C.1}$$

and

$$\tilde{r}_i = \frac{1}{n} \cdot \min\{x_E, \frac{0}{M}\} - x_{Si} \quad \forall i \in N \tag{C.2}$$

where $M$ is an arbitrarily fixed value with $\sum_i x_{Si} \leq M < \infty$.

These decisions rules, denoted by $M$-policy, mean that the same inventory position is aimed to reach by allocation of returns to each remanufacturing option. Furthermore, returns which let the echelon inventory position exceed an upper level $M$ are disposed of. This $M$-policy turns out to be a specific linear allocation rule with $q_i = \frac{1}{n}$ and $M_i = \frac{0}{M}$, because $\tilde{r}_i$ in (C.2) can also written be as

$$\tilde{r}_i = \frac{1}{n} \cdot M - \frac{1}{n} \cdot \max\{M - x_E, 0\} - x_{Si}$$

Under the regular return and demand conditions in (13), i.e. $\bar{R} > \sum \tilde{D}_i$, with using this policy the values of the inventory position $x_{Si}$ ($\forall i \in N$) are bounded from above and a stationary (joint) probability distribution for $x_{Si}$ will exist with finite expected values.

From this it follows that the expected serviceables' inventory in each period has a finite value. Under the policy in (C.1) and (C.2) the same holds for the expected inventory of returned products as well as for the remanufacturing and disposal quantities. Thus the expected costs per period under the $M$-policy, denoted by $C(M)$, are finite: $C(M) < \infty$.
The discounted expected costs over \( m \) periods sum up to

\[
C_m^{0}(M) = \sum_{t=0}^{m-1} \alpha^{t} \cdot C^{0}(M).
\]

For \( \alpha < 1 \) and \( m \to \infty \) we find as expression for the total discounted costs

\[
C_{\infty}^{0}(M) = \frac{C^{0}(M)}{1 - \alpha} < \infty
\]

which turns out to be bounded from above because of the finiteness of the period costs \( C^{0}(M) \).

Lemma 7 and Lemma 8 guarantee the convergence of \( f_{m}^{0}(x_{E}, x_{S1}, \ldots, x_{Sn}) \) for \( m \to \infty \)

\[
\tilde{f}(x_{E}, x_{S1}, \ldots, x_{Sn}) = \lim_{m \to \infty} f_{m}(x_{E}, x_{S1}, \ldots, x_{Sn})
\]

so that the following stationary functional equation holds

\[
\tilde{f}(x_{E}, x_{S1}, \ldots, x_{Sn}) =
\]

\[
= \min_{b + \sum_{i} c_{Ri} \cdot r_{i} + h_{R} \cdot (x_{E} - b - \sum_{i} (x_{Si} + r_{i})) + \sum_{i} L_{i} (x_{Si} + r_{i})}
\]

\[
+ \alpha \cdot \int_{0}^{\infty} \ldots \int_{0}^{\infty} f(x_{E} - b + S, x_{S1} + r_{1} - D_{1}, \ldots, x_{Sn} + r_{n} - D_{n}) \cdot
\]

\[
\varphi_{S}(S) \cdot \varphi_{D1}(D_{1}) \ldots \varphi_{Dn}(D_{n}) \cdot dS \cdot dD_{1} \ldots dD_{n}
\]

According to the results of the finite-period case it is evident that also the value function \( \tilde{f}(x_{E}, x_{S1}, \ldots, x_{Sn}) \) is separable and convex.

Thus the optimal decision rules which minimize \( \tilde{f}(x_{E}, x_{S1}, \ldots, x_{Sn}) \) are characterized by a \((nM, U)\)-policy with stationary parameters \( M_{i} \) and \( U \).
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