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Dirac bases:
A measure theoretical concept of basis
based on Carleman operators

by

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Abstract

The notion of Dirac basis is introduced in the setting of a Sobolev-like triple $\mathbb{R}(X) \subset X \subset \mathbb{R}^{-1}(X)$. It is a continuum substitute of the notion of orthonormal basis for Hilbert spaces. As a side result a generalization of the Sobolev embedding theorem is given in terms of Carleman operators and geometric measure theory. Finally, the concept of canonical Dirac basis is introduced. It is the basic concept for solving the generalized eigenvalue problem and corresponding generalized expansion problems.

0. A conceptual introduction

Our papers [EG 1] contain a mathematical interpretation of several aspects of Dirac's formalism. Herein we needed a continuum substitute of the ordinary notion of orthonormal basis. Therefore, we introduced the notion of Dirac basis in the setting of the Gelfand triple

$$S_{X, A} \subset X \subset T_{X, A}.$$  

Here $X$ denotes a separable Hilbert space, and $A$ a positive self-adjoint unbounded operator in $X$. The space $S_{X, A}$ is the inductive limit

$$U = \bigcup_{t > 0} e^{-tA}(X).$$

The space $T_{X, A}$ is a projective limit; it consists of mappings $F: (0, \infty) \to X$ with the property $F(t + \tau) = e^{-\tau A} F(t), \ t, \tau > 0$.

In the present paper we introduce the notion of Dirac basis in the bare setting of a triple of Hilbert spaces

$$R(X) \subset X \subset R^{-1}(X)$$

where $R$ denotes a bounded positive operator. It turns out that the concepts of Dirac basis and of Carleman operators are indissolubly connected.

This introduction contains the preliminary concepts.

The first one is the concept of Sobolev triple. Let $\overline{R}$ denote a positive bounded operator on the separable Hilbert space $X$ with possibly unbounded inverse $R^{-1}$. The dense subspace $R(X)$ of $X$ is the maximal domain of $R^{-1}$.

In $R(X)$ we introduce the non-degenerate sequilinear form
\[ (w, v)_1 = \langle R^{-1}w, R^{-1}v \rangle, \quad w, v \in \mathcal{R}(X) \]

with \((\cdot, \cdot)_1\) the inner product of \(X\). Then \(\mathcal{R}(X)\) is a Hilbert space with \((\cdot, \cdot)_1\) as its inner product.

Let \(\mathcal{R}^{-1}(X)\) denote the completion of \(X\) with respect to the norm \(f \mapsto \| Rf \|, f \in X\). Then \(R\) extends to an isometry from \(\mathcal{R}^{-1}(X)\) into \(X\).

The non-degenerate form \((\cdot, \cdot)_{-1}\) on \(\mathcal{R}^{-1}(X)\) is defined by

\[ (F, G)_{-1} = \langle RF, RG \rangle, \quad F, G \in \mathcal{R}^{-1}(X). \]

Thus \(\mathcal{R}^{-1}(X)\) becomes a Hilbert space.

It yields the triple of Hilbert spaces

\[ \mathcal{R}(X) \subseteq X \subseteq \mathcal{R}^{-1}(X) \]

The spaces \(\mathcal{R}(X)\) and \(\mathcal{R}^{-1}(X)\) establish a dual pair. Their pairing \(\langle \cdot, \cdot \rangle\)

is given by

\[ \langle w, G \rangle = \langle R^{-1}w, RG \rangle, \quad w \in \mathcal{R}(X), \quad G \in \mathcal{R}^{-1}(X). \]

We note that \(\mathcal{R}(X) = X = \mathcal{R}^{-1}(X)\) if and only if \(\mathcal{R}^{-1}\) is bounded.

The second concept is the notion of Carleman operator.

(0.1) Definition

Let \(X\) denote a separable Hilbert space, and \(\mathcal{M}\) a measure space with \(\sigma\)-finite measure \(\mu\). An operator \(T : X \to L_2(\mathcal{M}, \mu)\) is called a Carleman operator, if there exists a measurable function \(k : \mathcal{M} \to X\) with the following property:
For each $f \in D(T)$, the function

$$\alpha \mapsto (f, k(\alpha))$$

is a representant of the class $Tf \in L_2(M, \mu)$. (Cf. [Wei].)

A Carleman operator $T : X \to L_2(M, \mu)$ is Hilbert-Schmidt iff the function

$$\alpha \mapsto \|k(\alpha)\|^2$$

is $\mu$-integrable. Then the Hilbert-Schmidt norm equals

$$\left( \int_{M} \|k(\alpha)\|^2 \, d\mu(\alpha) \right)^{\frac{1}{2}}.$$

The following result is straightforward.

(0.2) **Lemma**

Let $T : X \to L_2(M, \mu)$ denote a Carleman operator induced by the measurable function $k : M \to X$. Let $(v_n)_{n \in \mathbb{N}}$ be an orthonormal basis in $D(T)$. Fix representants $(Tv_n)^{\wedge}$ for each $n \in \mathbb{N}$. Then there exists a null set $N \subset M$ such that for all $\alpha \in M \setminus N$

$$\sum_{n=1}^{\infty} |(Tv_n)^{\wedge}(\alpha)|^2 = \|k(\alpha)\|^2.$$

Put differently $k(\alpha) = \sum_{n=1}^{\infty} (Tv_n)^{\wedge}(\alpha)v_n$, $\alpha \in M \setminus N$

The third fundamental concept is a generalization by Federer of Vitali's differentiation theorem for the Lebesque measure on $\mathbb{R}^n$. Federer considers a topological measure space $M$ metrized by the metric $d$, and a regular Borel measure $\mu$ on $M$, such that bounded subsets of $M$ have finite $\mu$-measure. In the monograph [Fe], conditions are introduced on the metric space $(M, d)$ which lead to the following result.
Theorem

Let the function $f : M \to \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set $N_f$ such that for all $r > 0$ and all $a \in M \setminus N_f$ the closed ball $B(\alpha, r)$ with radius $r$ and centre $\alpha$ has positive measure. Further, the limit

$$\tilde{f}(\alpha) = \lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} f \, d\mu$$

exists for all $\alpha \in M \setminus N_f$. The function $\alpha \mapsto \tilde{f}(\alpha)$ defines a $\mu$-measurable function with $f = \tilde{f}$ almost everywhere.

Proof. Cf. [Fe], Theorem 2.8.18. \qed

Examples of such metric spaces are the following

- Finite dimensional vector spaces $M$ with $d(x,y) = \|x - y\|$ where $\| \cdot \|$ is any norm on $M$.
- A Riemannian manifold (of class $\geq 2$) with its usual metric.
- The disjoint union of metric spaces $(M_j, d_j)$, $j \in \mathbb{N}$, which satisfy Federer's conditions. Here $d$ is defined by

$$
\begin{cases}
    d(x_l, y_j) = 1 & , l \neq j \\
    d(x_l, y_j) = d_j(x_l, y_j)
\end{cases}
$$
1. The concept of Dirac basis

Let $R > 0$ be a bounded operator on $X$. We consider the triple $R(X) \subseteq X \subseteq R^{-1}(X)$. Further, let $M$ denote a measure space with $\mathcal{S}$-finite measure $\mu$. A function $\phi : M \to R^{-1}(X)$ is called (weakly) $\mu$-measurable if for each $w \in R(X)$ the function $\alpha \to \langle w, \phi(\alpha) \rangle$ is $\mu$-measurable.

In the space of measurable functions from $M$ to $R^{-1}(X)$ we introduce the natural equivalence relation

$$\phi \sim \psi : \iff \phi(\alpha) = \psi(\alpha)$$

almost everywhere.

In order to arrive at a proper introduction of the concept of Dirac basis, the following auxiliary result is needed.

1.1. Lemma

There exists an orthonormal basis $(u_n)_{n \in \mathbb{N}}$ in $X$ which is a Schauder basis in $R(X)$, i.e. for all $w \in R(X)$ the series $\sum_{n=1}^{\infty} \langle w, u_n \rangle u_n$ converges in $R(X)$.

Proof.

We may as well assume that $0 < R \leq 1$. If $R$ has pure point spectrum than an orthonormal basis of eigenvectors satisfies the requirements.

Now suppose that $R$ has no pure point spectrum. We construct an operator $\tilde{R}$ as follows. Let $(E_{\lambda})_{\lambda \in \mathbb{R}}$ denote the spectral resolution of the identity associated to $R$. Put $\tilde{R} = \sum_{n=1}^{\infty} \frac{1}{n+1} P_n$ with

$$P_n = \int_{\mathbb{R}} \chi_{\left[ \frac{1}{n+1}, \frac{1}{n} \right]} \frac{1}{dE_{\lambda}} \ .$$
We have \( \|R^{-1}R\| = 1 \) and \( \|R^{-1}R\| = 2 \).

Hence \( \hat{R}(X) = R(X) \) as Hilbert spaces. Finally, observe that \( \hat{R} \) has a pure point spectrum. \( \square \)

1.2. Definition

Let \((G,M,\mu,R,X)\) denote an equivalence class of \(\mu\)-measurable functions \(G : M \to R^{-1}(X)\). Then \((G,M,\mu,R,X)\) is called a Dirac basis if for a certain orthonormal basis \((u_k)_{k \in \mathbb{N}}\) in \(X\), which is a Schauder basis in \(R(X)\), the following relations are valid.

\[
(1.2') \quad \int_M <u_k, \hat{G} > <u_k, \hat{G} > du = \delta_{k\ell}, \quad k, \ell \in \mathbb{N}.
\]

Now let \((G,M,\mu,R,X)\) be a Dirac basis. Let \(\hat{G} \in (G,M,\mu,R,X)\) and \((u_k)_{k \in \mathbb{N}}\) be an orthonormal basis, which is a Schauder basis in \(R(X)\) such that \((1.2')\) is satisfied. Then for each \(k \in \mathbb{N}\), the function \(\hat{\phi}_k : \alpha \mapsto <u_k, \hat{G}_\alpha>\) is square integrable. Let \(\hat{\phi}_k \in L_2(M,\mu)\) denote the equivalence class of \(\hat{\phi}_k\), \(k \in \mathbb{N}\). We introduce the operator \(\mathcal{V}\) on the linear span \(<\{u_k | k \in \mathbb{N}\}>\) by

\[
\mathcal{V}u_k = \hat{\phi}_k, \quad k \in \mathbb{N}.
\]

From \((1.1')\) it follows that \((\hat{\phi}_k, \hat{\phi}_k) = \delta_{k\ell}\). So \(\mathcal{V}\) can be extended to an isometry from \(X\) into \(L_2(M,\mu)\). Let \(w \in R(X)\), and put \((\mathcal{V}w)^\wedge = \sum_{k=1}^{\infty} (w,u_k) \hat{\phi}_k\).

It is clear that \((\mathcal{V}w)^\wedge \in \mathcal{V}w\). Moreover, \(<w, \hat{G}_\alpha> = \sum_{k=1}^{\infty} (w,u_k) \hat{\phi}_k(\alpha)\).

Hence \((\mathcal{V}w)^\wedge = \sum_{k=1}^{\infty} (w,u_k) \hat{\phi}_k\) where the convergence is pointwise. For \(w_1, w_2 \in R(X)\),
(1.3.) Theorem

Let $(G,M,\mu,R,X)$ be a Dirac basis.

(i) For each representant $G \in (G,M,\mu,R,X)$, and all $w_1, w_2 \in R(X)$

$$
(w_1, w_2) = \int_{M} (\langle w_1 \rangle \hat{G} \tilde{G} \hat{G} \tilde{G}) d\mu = \int_{M} \langle w_1, \hat{G} \rangle \langle w_2, \hat{G} \rangle d\mu.
$$

(Plancherel)

There exists an isometry $V : X \rightarrow L_2(M,\mu)$ with the properties:

(ii) For each representant $G \in (G,M,\mu,R,X)$ and each $w \in R(X)$ the function

$$
(\alpha \mapsto \langle w_G \rangle) \in \mathcal{U}w.
$$

(So the definition of $V$ does not depend on the choice of $G$).

(iii) The operator $\mathcal{U}R$ is a bounded Carleman operator.

Proof

Part (i) follows from the observations above this theorem. To prove part (ii), let $G \in (G,M,\mu,R,X)$. As we have seen, there exists an isometry

$V : X \rightarrow L_2(M,\mu)$ such that for all $w \in R(X)$ the function $\alpha \mapsto \langle w_G \rangle$ is a representant of $\mathcal{U}w$. For each $G \in (G,M,\mu,R,X)$, there exists a null set $N$ such that $\hat{G}_\alpha = \hat{G}_\alpha$ for all $\alpha \in M \setminus N$. It follows that

$$
(\alpha \mapsto \langle w_G \rangle) \in \mathcal{U}w, \quad w \in R(X).
$$

Let $G \in (G,M,\mu,R,X)$, and let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis in $X$.

Then for all $\alpha \in M$
\[ \|V^R G\|_{L_2}^2 = \|R^\wedge\|_{L_2}^2 = \sum_{k=1}^{\infty} |\langle e_k^\wedge, G \rangle|^2 = \sum_{k=1}^{\infty} |(V^Re_k)^\wedge(\alpha)|^2. \]

Put \( k : \alpha \mapsto \sum_{k=1}^{\infty} (V^Re_k)^\wedge(\alpha)e_k \). Then \( k \) induces \( VR \) as a Carleman operator. \( \square \)

The reverse of the previous theorem is also valid. So there exists a one-to-one correspondence between Dirac bases \((G, M, \mu, R, X)\) and isometries \( U : X \to L^2(M, \mu) \) such that \( VR \) is a Carleman operator.

(1.4) Theorem

Let \( U \) denote an isometry from \( X \) into \( L^2(M, \mu) \). Suppose the operator \( VR \) is Carleman. Then there exists a Dirac basis \((G, M, \mu, R, X)\) such that for each \( \hat{G} \in (G, M, \mu, R, X) \) and all \( w \in R(X) \)

\[ (\alpha \mapsto \langle w, \hat{G} \rangle) \in Uw. \]

Proof

Let \( (e_k)_{k \in \mathbb{N}} \) denote an orthonormal basis in \( X \), and let \( (V^Re_k)^\wedge \) denote a representant of the class \( V^Re_k \in L^2(M, \mu), k \in \mathbb{N} \). By Lemma (0.2) there exists a null set \( \hat{N} \) such that for all \( \alpha \in M \setminus \hat{N} \)

\[ \sum_{k=1}^{\infty} |(V^Re_k)^\wedge(\alpha)|^2 < \infty. \]

Now define \( \hat{G} : M \to R^{-1}(X) \) by
\[
\begin{align*}
G_\alpha &= 0 & \text{if } \alpha &\in \mathbb{N} \\
G_\alpha &= \sum_{k=1}^{\infty} (\mathcal{V}\mathcal{R} e_k)^\wedge(\alpha) \mathcal{R}^{-1} e_k
\end{align*}
\]

Let \( w \in \mathcal{R}(X) \). Then

\[
<w, G_\alpha> = \sum_{k=1}^{\infty} (\mathcal{R}^{-1}w, e_k)(\mathcal{V}\mathcal{R} e_k)^\wedge(\alpha).
\]

Hence \( \alpha \mapsto <w, G_\alpha> \) is measurable and \( (\alpha \mapsto <w, G_\alpha>) \in \mathcal{W} \). It follows that \( \hat{G} \) is weakly measurable. Let \( w_1, w_2 \in \mathcal{R}(X) \). Then we have

\[
(w_1, w_2) = (\mathcal{V}w_1, \mathcal{V}w_2)_{L_2} = \int <w_1, \hat{G}_\alpha><w_2, \hat{G}_\alpha> d\mu_\alpha.
\]

If \((G, \mathcal{M}, \mu, \mathcal{R}, X)\) denotes the equivalence class of \( \hat{G} \), then \((G, \mathcal{M}, \mu, \mathcal{R}, X)\) is the wanted Dirac basis.

**Remark:**

If the support of the measure \( \mu \) consists of atoms only, then any Dirac basis \((G, \mathcal{M}, \mu, \mathcal{R}, X)\) is an orthogonal basis.

Let \((G, \mathcal{M}, \mu, \mathcal{R}, X)\) be a Dirac basis, and let \( \hat{G} \in (G, \mathcal{M}, \mu, \mathcal{R}, X) \). From the Plancherel-type result stated in Theorem (1.3) we obtain the weak expansion
\[ w = \int \frac{<w, G > G}{\mu} \, d\mu, \quad w \in R(X) \]

in the sense that for all \( v \in R(X) \)

\[ (w, v) = \int \frac{<w, G > <v, G >}{\mu} \, d\mu. \]

A sharper result is valid, if \( R \) is a Hilbert-Schmidt operator on \( X \).

(1.5) Theorem

Let \( R > 0 \) be a Hilbert-Schmidt operator. Let \( ^{\wedge} G \) be a representant of the Dirac basis \((G, M, \mu, R, X)\). Then for each \( w \in R(X) \) the function

\[ \alpha \mapsto \left\| \frac{\wedge}{\wedge} w, \frac{\wedge}{\wedge} G \right\| \]

is \( \mu \)-integrable. So we get the strong expansion

\[ w = \int \frac{\wedge}{\wedge} \frac{\wedge}{\wedge} w, \frac{\wedge}{\wedge} G \, d\mu \]

i.e. in strong \( X \)-sense

\[ R w = \int \frac{\wedge}{\wedge} \frac{\wedge}{\wedge} w, \frac{\wedge}{\wedge} G \, d\mu. \]

Proof

Let \( V \) denote the corresponding isometry from \( X \) into \( L_2(M, \mu) \), and let
$(e_k)_{k \in \mathbb{N}}$ denote an orthonormal basis in $X$. Then

$$\left\| R G^\wedge_\alpha \right\|^2 = \sum_{k=1}^{\infty} \left| (VR e_k)^\wedge_\alpha \right|^2$$

with $(VR e_k)^\wedge_\alpha = (e_k, R G^\wedge_\alpha)$, $\alpha \in \mathcal{M}$, $k \in \mathbb{N}$.

Thus we obtain

$$\int_{\mathcal{M}} \left\| R G^\wedge_\alpha \right\|^2 d\mu_\alpha = \sum_{k=1}^{\infty} \int_{\mathcal{M}} \left| (VR e_k)^\wedge_\alpha \right|^2 d\mu_\alpha = \sum_{k=1}^{\infty} \left\| R e_k \right\|^2 < \infty .$$

Hence

$$\int_{\mathcal{M}} \left\| \langle w, G^\wedge_\alpha \rangle R G^\wedge_\alpha \right\| d\mu_\alpha \leq \left( \int_{\mathcal{M}} \left| \langle w, G^\wedge_\alpha \rangle \right|^2 d\mu_\alpha \right)^{\frac{1}{2}} \cdot \left( \int_{\mathcal{M}} \left\| R G^\wedge_\alpha \right\|^2 d\mu_\alpha \right)^{\frac{1}{2}} .$$

Another problem concerns the existence of Dirac bases for each $(\mathcal{M}, \mu)$.

This problem is solved in the following lemma.

(1.6) **Lemma**

Let $R$ be a positive bounded operator in $X$, and let $\mathcal{M}$ be a measure space with $\sigma$-finite measure $\mu$.

Let the densely defined operator $R^{-1}$ be unbounded. Then there exists an isometry $V$ from $X$ into $L^2(\mathcal{M}, \mu)$ such that $VR$ is a Carleman operator.

Let also $R^{-1}$ be bounded. Then there exists an isometry $V$ from $X$ into $L^2(\mathcal{M}, \mu)$ such that $VR$ is Carleman iff the support of $\mu$ consists of atoms only.
Proof

The proof can be obtained from [Wei], Theorem 7.1 and 7.2.

Consequently, we obtain

(1.7) Corollary

Let $\mathcal{R}$ be a positive bounded operator in $X$, and $\mathcal{M}$ a measure space with $\sigma$-finite measure $\mu$.

(i) Let $\mathcal{R}^{-1}$ be unbounded. Then there exists a Dirac basis $(G, \mathcal{M}, \mu, \mathcal{R}, X)$ with respect to the triple $\mathcal{R}(X) \subseteq X \subseteq \mathcal{R}^{-1}(X)$.

(ii) Let $\mathcal{R}^{-1}$ be bounded. Then the only Dirac bases are the orthogonal bases (Note that $\mathcal{R}(X) = X = \mathcal{R}^{-1}(X)$).

(iii) Let $\mathcal{R}$ be Hilbert-Schmidt. Then any isometry $\mathcal{U}: X \rightarrow L_2(\mathcal{M}, \mu)$ gives rise to a Dirac basis.

If we put restrictions on the measure space $(\mathcal{M}, \mu)$, a so called canonical choice can be made in each equivalence class $(G, \mathcal{M}, \mu, \mathcal{R}, X)$. In the next section, we clarify this statement. We prove a measure theoretical Sobolev lemma based on Carleman operators.

2. A measure theoretical Sobolev lemma

Let $\mathcal{R} > 0$ be a bounded operator, and let $\mathcal{M}$ be a metrizable topological measure space with regular Borel measure $\mu$. We assume that the pair $(\mathcal{M}, \mu)$
satisfies Federer's conditions, i.e. Theorem (0.3) is valid. Let 
$\mathcal{D} : X \to L_2(M, \mu)$ be a densely defined operator with $R(X)$ contained in 
its domain.

On the pair $\mathcal{D}, R$ we impose the following conditions

(2.1.i) The operator $\mathcal{D} \circ R : X \to L_2(M, \mu)$ is a bounded Carleman operator.

Let the function $k : M \to X$ induce $\mathcal{D} R$. (Cf. Definition (0.1).)

(2.1.ii) The function $\alpha \mapsto \|k(\alpha)\|^2$ is integrable on bounded Borel sets.

Remarks

- Condition (2.1.ii) is not redundant. To show this, consider the following
  example: Define $k : \mathbb{R} \to L_2(\mathbb{R}, dx)$ by

\[
k(t) = \begin{cases} \chi(0, |t|^{-1}) & t > 0 \\ 0 & t = 0 \end{cases}
\]

Then for $t \neq 0$, $\|k(t)\|^2 = |t|^{-1}$ and hence condition (2.1.ii) is not
satisfied.

- In our paper [EG 2] a measure theoretical Sobolev lemma has been
  proved based on Hilbert-Schmidt operators. So we started with a positive
  Hilbert-Schmidt operator $R$, and on $\mathcal{D}$ we imposed the condition that $\mathcal{D} \circ R$
is Hilbert-Schmidt. In that case the condition (2.1.ii) is always ful-
filled because $(\alpha \mapsto \|k(\alpha)\|^2) \in L_2(M, \mu)$.

Now let $(v_k)_{k \in \mathbb{N}}$ denote an orthonormal basis in $X$. Since $\mathcal{D} R v_k \in L_2(M, \mu)$
there exists a null set $N_1$ such that for all $k \in \mathbb{N}$
(2.2.i) \[ \phi_k(\alpha) = \lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} |\mathcal{D}v_k|^2 d\mu, \quad \alpha \in M \setminus N \]

exists. Each function \( \alpha \mapsto \phi_k(\alpha) \) extends to a representant of the class \( \mathcal{D}v_k^k, k \in \mathbb{N} \).

Since \( |\mathcal{D}v_k|^{2} \in L_{1}(\mathbb{M}, \mu) \) for all \( k \in \mathbb{N} \), Theorem (0.3) yields a null set \( N_2 \) such that for all \( k \in \mathbb{N} \) and \( \alpha \in M \setminus N_2 \)

\[
(2.2.ii) \quad |\phi_k(\alpha)|^2 = \lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} |\mathcal{D}v_k|^2 d\mu .
\]

Finally, since the measurable function \( \alpha \mapsto \|\mathcal{K}(\alpha)\|^2 \) is integrable on bounded Borel sets, Lemma (0.2) and Theorem (0.3) yield a null set \( N_3 \) such that for all \( \alpha \in M \setminus N_3 \)

\[
(2.2.iii) \quad \sum_{k=1}^{\infty} |\phi_k(\alpha)|^2 = \lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (\sum_{k=1}^{\infty} |\mathcal{D}v_k|^2) d\mu .
\]

Throughout this section \( N \) denotes the null set \( N_1 \cup N_2 \cup N_3 \).

We put \( \phi_k(\alpha) = 0 \) for all \( \alpha \in N, k \in \mathbb{N} \).

(2.3) Lemma

(i) Let \( \alpha \in M \). Define \( e_\alpha = \sum_{k=1}^{\infty} \frac{\phi_k(\alpha)}{Rv_k} \). Then \( e_\alpha \in \mathcal{R}(X) \).

(ii) Let \( \alpha \in M \setminus N \). Then for all \( r > 0 \)
e_{\alpha}(r) = \mu(B(\alpha, r))^{-1} \sum_{k=1}^{\infty} \left( \int_{B(\alpha, r)} (DRv_k) d\mu \right) Rv_k

belongs to \( R(X) \). Furthermore

\[ \lim_{r \to 0} \| e_{\alpha} - e_{\alpha}(r) \|_1 = 0. \]

**Proof**

(i) Let \( \alpha \in \mathbb{N} \). Then \( e_{\alpha} = 0 \).

Let \( \alpha \in \mathbb{M} \setminus \mathbb{N} \). Then \( \sum_{k=1}^{\infty} |\phi_k(\alpha)|^2 < \infty \), whence \( e_{\alpha} \in R(X) \).

(ii) Let \( \alpha \in \mathbb{M} \setminus \mathbb{N} \). The Hölder inequality yields

\[ \sum_{k=1}^{\infty} \left| \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (DRv_k) d\mu \right|^2 \leq \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} \left( \sum_{k=1}^{\infty} |DRv_k|^2 \right) d\mu. \]

Because of condition (2.1.ii) the latter expression is finite, whence \( e_{\alpha}(r) \in R(X) \).

Let \( \varepsilon > 0 \). Take a fixed \( n_0 \in \mathbb{N} \) so large that

\[ \sum_{k=n_0+1}^{\infty} |\phi_k(\alpha)|^2 < \frac{\varepsilon^2}{4}. \]

Next take \( r_0 > 0 \) so small that for all \( r, 0 < r < r_0 \)
\[ (**) \quad \left| \phi_k(\alpha) - \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (\phi R_k) \, du \right| < \frac{\varepsilon}{\sqrt{n}} \]

for all \( n = 1, 2, \ldots, n_0 \), and also

\[ (***) \quad \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} \left( \sum_{k=n_0+1}^{\infty} |\phi R_k|^2 \right) \, du < \varepsilon^2. \]

(Cf. 2.2.(i)-(iii) and (*).)

We estimate as follows

\[ \| e_\alpha - e_\alpha(r) \|_1^2 = \left( \sum_{k=1}^{n_0} + \sum_{k=n_0+1}^{\infty} \right) \left| \phi_k(\alpha) - \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (\phi R_k) \, du \right|^2. \]

By (**)

\[ \sum_{k=1}^{n_0} \left| \phi_k(\alpha) - \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (\phi R_k) \, du \right|^2 < \varepsilon^2 \]

and by (*) and (***)

\[ \sum_{k=n_0+1}^{\infty} \left| \phi_k(\alpha) - \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} (\phi R_k) \, du \right|^2 \leq \]

\[ \leq 2 \sum_{k=n_0+1}^{\infty} |\phi_k(\alpha)|^2 + 2\mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} \left( \sum_{k=n_0+1}^{\infty} |\phi R_k|^2 \right) \, du < 3\varepsilon^2 \]
Thus we have proved that for all $r$, $0 < r < r_0$

$$\|e_\alpha - e_\alpha(r)\|_1 < 2\varepsilon.$$  

(2.4) **Theorem (Measure theoretical Sobolev lemma)**

For each $w \in R(X)$ a representant $\tilde{D}w$ can be chosen with the following properties

(i) $\tilde{D}w = \sum_{k=1}^{\infty} (R^{-1}w, v_k) \phi_k$ with pointwise convergence.

(ii) Let $\alpha \in M$. Then the linear functional $w \mapsto \tilde{D}w(\alpha)$ is continuous on $R(X)$; its Riesz representant in $R(X)$ equals $e_\alpha$.

(iii) Let $\alpha \in M \setminus N$. Then

$$\tilde{D}w(\alpha) = \lim_{r \to 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (Dw) d\mu.$$  

(iv) Suppose $\alpha \mapsto \|R(\alpha)\|^2$ is essentially bounded on $M$. Then there exists a null set $N_0$ such that the convergence in (i) is uniform on $M \setminus N_0$.

Further,

$$\exists K > 0 \forall \alpha \in M \setminus N_0 : \|\tilde{D}w(\alpha)\| \leq K\|w\|_1.$$  

**Proof**

Let $w \in R(X)$, and put $\tilde{D}w = \sum_{k=1}^{\infty} (R^{-1}w, v_k) \phi_k$. Then $\tilde{D}w \in Dw$, because

$$Dw = \sum_{k=1}^{\infty} (R^{-1}w, v_k) DRv_k.$$
(i) Since \((w,e_\alpha)_1 = \sum_{k=1}^{\infty} (R^{-1} w, v_k) \phi_k\), and since \(e_\alpha \in R(X)\) the series converges pointwise on \(M\).

(ii) Trivial, because \(Dw(\alpha) = (w,e_\alpha)_1\).

(iii) Let \(\alpha \in M \setminus N\). Then by Lemma (2.3)

\[
\tilde{D}_w(\alpha) = \lim_{r \to 0} (w,e_\alpha(r))_1 =
\]

\[
= \lim_{r \to 0} \mu(B(\alpha,r))^{-1} \sum_{k=1}^{\infty} (R^{-1} w, v_k)(\int_{B(\alpha,r)} DRv_k d\mu).
\]

We show that summation and integration can be interchanged.

\[
\left( \sum_{k=1}^{\infty} \int_{B(\alpha,r)} \mu(B(\alpha,r))^{-1} ((R^{-1} w, v_k)) |DRv_k| d\mu \right)^{1/2} \leq \left( \sum_{k=1}^{\infty} |R^{-1} w, v_k|^2 \right)^{1/2} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} (\sum_{k=1}^{\infty} |DRv_k|^2) d\mu \right)^{1/2}.
\]

This yields

\[
\tilde{D}_w(\alpha) = \lim_{r \to 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} \left( \sum_{k=1}^{\infty} (R^{-1} w, v_k) DRv_k \right) d\mu,
\]

which proves statement (2.4.iii).
There exists $K > 0$ and a null set $N_0 \subset M$ such that

$$\sum_{k=1}^{\infty} |\phi_k(\alpha)|^2 \leq K^2, \quad \alpha \in M \setminus N_0.$$ 

So for all $L_1 < L_2$, $L_1, L_2 \in \mathbb{N}$,

$$\left| \sum_{k=L_1}^{L_2} (R^{-1}w, v_k) \phi_k(\alpha) \right| \leq K \left( \sum_{k=L_1}^{L_2} ((R^{-1}w, v_k))^2 \right)^{1/2}.$$

Illustration: "The classical Sobolev embedding theorem on $\mathbb{R}^n$"

Let $dx = dx_1 \cdots dx_n$ denote the Lebesque measure on $\mathbb{R}^n$.

Let $\Delta$ denote the positive self-adjoint operator

$$\Delta = 1 - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

in $L^2(\mathbb{R}^n, dx)$. For each $m > 0$, put $R_m = \Delta^{-m/2}$. Then $R_m$ is positive and bounded. It is well-known that the Sobolev space $H^m(\mathbb{R}^n)$ of order $m$ equals $R_m(L^2(\mathbb{R}^n))$. We have the classical Sobolev triple $H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H^{-m}(\mathbb{R}^n)$.

(2.5) Corollary (Cf. [Yo], p.174.)

Let $m > n/2$ and let $0 \leq \ell < m - n/2$, $\ell \in \mathbb{N} \cup \{0\}$. Then there is a null set $N_n(\ell)$ such that for each $u \in H^m(\mathbb{R})$ there exists a representant $\tilde{u}$ with the following property: For all $s \in (\mathbb{N} \cup \{0\})^m$, $|s| \leq \ell$, there
exists $\gamma_s > 0$ such that
\[ \forall x \in \mathbb{R}^n \setminus \mathbb{N}_m : |(D^s u)(x)| \leq \gamma_s \|u\|_m. \]

Here $D^s = \left( \frac{\partial}{\partial x_1} \right)^{s_1} \ldots \left( \frac{\partial}{\partial x_n} \right)^{s_n}$, $s = s_1 + \ldots + s_n$ and $\|\cdot\|_m$ denotes the norm of $H^m(\mathbb{R}^n)$.

**Proof**

Let $\hat{F}_n$ denote the Fourier transform on $L_2(\mathbb{R}^n)$. The operator $D^s R_m$ is a bounded Carleman operator induced by the function $k_{s,m} : \mathbb{R}^n \rightarrow L_2(\mathbb{R}^n)$,

\[ k_{s,m}(x;y) = (\hat{F}_n \hat{g}_{s,m})(x - y), \quad x, y \in \mathbb{R}^n \]

with

\[ g_{s,m}(y) = (i)^s \frac{\prod_{j=1}^{s_1} y_1 \cdots \prod_{j=1}^{s_n} y_n}{\sqrt{1 + \sum_{j=1}^{s_1} y_1^2 + \ldots + \sum_{j=1}^{s_n} y_n^2}}. \]

So for all $x \in \mathbb{R}^n$,

\[ \|k_{s,m}(x;\cdot)\|_{L_2(\mathbb{R}^n)} = \|g_{s,m}\|_{L_2(\mathbb{R}^n)}. \]

---

3. **Canonical Dirac basis with applications to the generalized eigenvalue problem**

In Section 1 we have defined a Dirac basis $(G,\mathbb{M},\nu,\mathbb{R},X)$ as an equivalence class of $\nu$-measurable functions from $\mathbb{M}$ to $\mathbb{R}^{-1}(X)$. No restrictions have been put on the measure space $\mathbb{M}$ and on the $\sigma$-finite measure $\nu$. However,
if we restrict to topological measure spaces $M$ and regular Borel measures $\mu$, which satisfy Federer's conditions, then for certain $R > 0$, a canonical choice can be made in the equivalence class $(G, M, \mu, R, X)$. Such a choice is called a canonical Dirac basis.

(3.1) Definition

Let $(G, M, \mu, R, X)$ be a Dirac basis. A representant $G \in (G, M, \mu, R, X)$ is called a canonical Dirac basis if there exists a null set $N$ such that for all $w \in R(X)$ and all $a \in M \setminus N$

$$\lim_{r \to 0} \mu(B(a, r))^{-1} \int_{B(a, r)} <w, G_\beta> d\mu_\beta = <w, G_a>.$$ 

We use the notation $(G_a)_{a \in M}$.

Throughout this section we assume that $M$ is a metrizable topological measure space which satisfies Federer's conditions, and $\mu$ a regular Borel measure on $M$ such that bounded sets have finite $\mu$-measure. So Theorem (0.3) is valid.

(3.2) Lemma

Let $V : X \to L^2(M, \mu)$ be an isometry with the property that $VR$ is a Carleman operator. Assume that the function $k : M \to \mathbb{C}$ which induces $VR$, 


satisfies (2.1.i1). Then in the Dirac basis $(G,M,\mu,R,X)$ associated to
\( V \) (cf. Theorem (1.4)) there exists a canonical representant $\{\bar{\zeta}_\alpha\}_{\alpha \in \mathcal{M}}$.

**Proof**

Following Lemma (2.3) and Theorem (2.4) there are $\bar{g}_\alpha \in \mathcal{R}(X)$, $\alpha \in \mathcal{M}$, and there is a null set $\mathcal{N}$ such that for all $w \in \mathcal{R}(X)$

$$(\alpha \mapsto (w,\bar{g}_\alpha)) \in \mathcal{V}w$$

and

$$(w,\bar{g}_\alpha) = \lim_{r \to 0} \mu(B(\alpha,r))^{-1} \int_{B(\alpha,r)} <w,\bar{\zeta}_\beta> d\mu_\beta.$$

For each $\alpha \in \mathcal{M}$, we define $\bar{G}_\alpha \in \mathcal{R}^{-1}(X)$ by

$$(w,\bar{G}_\alpha) = <w,\bar{G}_\alpha>, \quad w \in \mathcal{R}(X).$$

Then $\{\bar{G}_\alpha\}_{\alpha \in \mathcal{M}} \in (G,M,\mu,R,X)$ (cf. Theorem (1.4)), and $\{\bar{g}_\alpha\}_{\alpha \in \mathcal{M}}$ is canonical Dirac basis.

**Remarks**

Let $\{u_k\}_{k \in \mathbb{N}}$ be an orthonormal basis in $X$. For each $\alpha \in \mathcal{M}$, we put

$$g_\alpha(r) = \mu(B(\alpha,r))^{-1} \sum_{k=1}^{\infty} \left( \int_{B(\alpha,r)} \overline{VRu_k} d\mu \right) u_k.$$
Then by Lemma (2.3) there exists a null set \( \tilde{N} \) such that

\[
\lim_{r \to 0} \| \tilde{g}_a - \tilde{g}_a(r) \|_1 = 0, \quad a \in M \setminus \tilde{N}.
\]

We observe that \( \tilde{g}_a(r) = \mu(B(a,r))^{-1} RV^* \chi_{B(a,r)} \) where \( \chi_{B(a,r)} \) denotes the characteristic function of \( B(a,r) \).

In the next theorem we present a sufficient condition which guarantees that a Dirac basis \((G,M,\mu,R,X)\) contains a canonical representant.

(3.4) Theorem

Let \((G,M,\mu,R,X)\) be a Dirac basis, and \( \hat{G} \) denote a representant. Assume that the measurable function \( a \mapsto \| R_{\tilde{a}} \|_2 \) is integrable on bounded Borel sets. Then \((G,M,\mu,R,X)\) contains a canonical Dirac basis \( \hat{G}_a \in M \).

Proof

Theorem (1.3) yields an isometry \( V \) from \( X \) into \( L_2(M,\mu) \) such that \( VR \) is Carleman, and

\[
(\alpha \mapsto <VRf,\hat{G}_a>) \in VRf, \quad f \in X.
\]

It is clear that the Carleman operator \( VR \) is induced by the function \( \hat{R}_\alpha : a \mapsto R_{\tilde{a}} \). By assumption \( \hat{R}_\alpha \) satisfies (2.1.ii). Hence from Lemma (3.2) the assertion follows.

In a natural way the notion of canonical Dirac basis is associated to
the generalized eigenvalue problem. We describe this connection here. Let \( \phi \) be a complex valued measurable function on \( M \), which is bounded on Borel sets. In \( L_2(M, \mu) \) we define the multiplication operator \( M_\phi \) by

\[
D(M_\phi) = \{ h \in L_2(M, \mu) \mid \int_M |\phi h|^2 d\mu < \infty \}
\]

and

\[ M_\phi h = \phi h , \quad h \in D(M_\phi) . \]

Because of the conditions on \( \phi \), \( \chi_{B(\alpha, r)} \in D(M_\phi) \) for all \( r > 0 \) and all \( \alpha \in M \). We note that \( M_\phi \) is a normal operator.

**Lemma (3.5)**

Let \((G, M, \mu, R, X)\) denote a Dirac basis. Let \( \hat{G} \) be a representant such that the function \( \alpha \mapsto \|\hat{R}_\alpha\|_2 \) is integrable on bounded Borel sets. Further let \( V : X \to L_2(M, \mu) \) denote the isometry associated to \((G, M, \mu, R, X)\). Then there exists a canonical representant \( (\hat{G}_\alpha)_{\alpha \in M} \) and a null set \( N \) such that for all \( \alpha \in M \setminus \tilde{N} \)

\[
\lim_{r \to 0} \left\| \hat{G}_\alpha - \mu(B(\alpha, r))^{-1} \{ V^* \chi_{B(\alpha, r)} \} \right\| = 0 .
\]

Let \( \phi : M \to \mathbb{C} \) be a measurable function which is bounded on bounded Borel sets. Then there exists a null set \( N_\phi \) such that for all \( \alpha \in M \setminus N_\phi \)

\[
\lim_{r \to 0} \left\| \phi(\alpha) \hat{G}_\alpha - \mu(B(\alpha, r))^{-1} \{ V^* M_\phi \chi_{B(\alpha, r)} \} \right\| = 0 .
\]
Proof

The first statement follows immediately from the previous theorem and remark (3.3.i).

With respect to the second assertion, we are ready if we can prove that there exists a null set $N_2$ such that for all $\alpha \in M \setminus N_2$

\[
(*) \quad \lim_{r \to 0} \mu(B(\alpha, r))^{-1} \| \varphi^*(\phi(a)I - M_\phi) \chi_{B(\alpha, r)} \|_{-1} = 0.
\]

Consider the estimation for all $\alpha \in M$ with $\mu(B(\alpha, r)) \neq 0$, $r > 0$.

\[
\mu(B(\alpha, r))^{-2} \| \varphi^*(\phi(a)I - M_\phi) \chi_{B(\alpha, r)} \|_{-1}^2 = \sum_{k=1}^{\infty} \{ \mu(B(\alpha, r))^{-1} \left( \int_{B(\alpha, r)} (\phi(a) - \phi(\lambda)) \varphi R u_k \, du_\lambda \right) \}^2 \leq \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} |\phi(a) - \phi(\lambda)|^2 \, du_\lambda \mu(B(\alpha, r))^{-1} \sum_{k=1}^{\infty} \left( \int_{B(\alpha, r)} \|\varphi R u_k\|^2 \, du \right).
\]

Since $\phi$ is bounded on bounded Borel sets, both $\phi$ and $|\phi|^2$ are $\mu$-integrable on bounded Borel sets. Hence there exists a null set $N_{21}$ such that for all $\alpha \in M \setminus N_{21}$

\[
\lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} |\phi(a) - \phi(\lambda)|^2 \, du_\lambda = 0.
\]
Further, for all \( r > 0 \) and all \( \alpha \)

\[
\int_{B(\alpha, r)} \left( \sum_{k=1}^{\infty} |VR_u_k|^2 \right) d\mu = \int_{B(\alpha, r)} \|R_G^\alpha\|^2 d\mu_{\lambda}.
\]

So there exists a null set \( N_{22} \) such that for all \( \alpha \in M \setminus N_{22} \)

\[
\lim_{r \to 0} \mu(B(\alpha, r))^{-1} \int_{B(\alpha, r)} \left( \sum_{k=1}^{\infty} |VR_u_k|^2 \right) d\mu = \|R_G^\alpha\|^2.
\]

Now put \( N_2 = N_{21} \cup N_{22} \), then (\*) follows.

\[\square\]

(3.6) Corollary

Suppose \( V \) is unitary and \( V^* M_\phi V \) extends to a closable operator in \( R^{-1}(X) \), i.e. \( RV^* M_\phi V^{-1} \) is closable in \( X \). Then we have

\[
V^* M_\phi V ^\alpha = \phi(\alpha) G^\alpha
\]

where \( V^* M_\phi V \) denotes the closure in \( R^{-1}(X) \) of \( V^* M_\phi V \).

Proof

We have

\[
\lim_{r \to 0} \mu(B(\alpha, r))^{-1} RV^* \chi_{B(\alpha, r)} = R_G^\alpha
\]

and

\[
\lim_{r \to 0} \mu(B(\alpha, r))^{-1} RV^* M_\phi \chi_{B(\alpha, r)} = \phi(\alpha) R_G^\alpha.
\]
Hence
\[ V^* M V \tilde{g}_\alpha = \phi(\alpha) \tilde{g}_\alpha. \]

An application of the previous results is the following.

Let $T$ be a self-adjoint operator in $X$ with a simple spectrum. Then there exists a finite Borel measure $\mu$ on $\mathbb{R}$ and a unitary operator $U : X \to L_2(\mathbb{R}, \mu)$ such that $U^* U$ equals the self-adjoint operator of multiplication by the identity function. It is clear that $(\mathbb{R}, \mu)$ satisfies Federer's conditions.

Now let $R$ be a bounded positive operator with the property that $UR : X \to L_2(\mathbb{R}, \mu)$ is a Carleman operator satisfying Condition (2.1.ii).

Then following Lemma (3.2) and (3.5), there exists a canonical Dirac basis $(E_\alpha)_{\alpha \in \mathbb{R}}$ and a null set $N_T$ such that for all $\alpha \in \mathbb{R} \setminus N_T$

\[ \lim_{r \to 0} \| E_\alpha - \mu(B(\alpha, r))^{-1} \{ U^* \chi_{B(\alpha, r)} \} \|_{-1} = 0 \]

and

\[ \lim_{r \to 0} \| a E_\alpha - \mu(B(\alpha, r))^{-1} \{ U^* \chi_{B(\alpha, r)} \} \|_{-1} = 0. \]

(Here $B(\alpha, r) = [\alpha - r, \alpha + r]$.) So for each $\alpha \in \mathbb{R} \setminus N_T$, $E_\alpha$ is a candidate (generalized) eigenvector.

If $RTR^{-1}$ is closable in $X$, then the closure $\overline{T}$ of $T$ in $R^{-1}(X)$ exists, and for all $\alpha \in \mathbb{R} \setminus N_T$

\[ \overline{T} \tilde{E}_\alpha = \alpha \tilde{E}_\alpha \]

So the rather mild condition that $RTR^{-1}$ is closable in $X$ yields 'genuine' eigenvectors.
(3.7) Remarks

(i) If $R$ is a positive Hilbert-Schmidt operator, then for any unitary operator $U$, the operator $UR$ is Hilbert-Schmidt. It follows that for each self-adjoint operator in $X$ with simple spectrum, there exists a canonical Dirac basis of candidate (generalized) eigenvectors. In our paper [EG 5] we have proved the same result for any self-adjoint operator.

(ii) Along similar lines as in [EG 5] the following can be proved:

Let $T$ be any self-adjoint operator in $X$. Let $U$ be its diagonalizing unitary operator in the sense of the multiplicity theorem (Cf. Theorem 1.2 in [EG 5]), and let $R > 0$ such that $UR$ is a Carleman operator which satisfies (1.2.ii). Then the unitary operator $U$ gives rise to a canonical Dirac basis $(G_\alpha)_{\alpha \in M}$. Here $M$ denotes the disjoint union

$$M = \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} IR_{m,j} \cup \left( \bigcup_{j=1}^{m} IR_{\infty,j} \right)$$

where each $IR_{m,j}$, $m = \infty, 1, 2, \ldots$, $1 \leq j < m + 1$, is a copy of $IR$.

To almost each point $\lambda$ in the spectrum $\sigma(T)$ of $T$ with multiplicity $m_\lambda$ there belong $m_\lambda$ candidate (generalized) eigenvectors which are elements of $\{G_\alpha \mid \alpha \in M\}$. 
References

Part c: Free field operators, preprint.


