The Hilbert transform in complex Envelope Displacement Analysis (CEDA)

Citation for published version (APA):
The Hilbert Transform in Complex Envelope Displacement Analysis (CEDA)

Niels Geerts
WFW-report 96.121
# Contents

Summary 3

1 Introduction 4
  1.1 CEDA 4
  1.2 Strategy 5

2 The Hilbert Transform, the Analytic Signal and the Complex Envelope 6
  2.1 The Hilbert Transform 6
  2.2 The analytic signal 7
  2.3 The Complex Envelope 8

3 Signal Transformations 9

4 Complex Envelope Displacement Analysis 14
  4.1 Theory 14
  4.2 Example 15
  4.3 Concluding remarks 17

5 Boundary conditions in CEDA 18
  5.1 Pseudo-inverse 19

6 Conclusions and recommendations 20
  6.1 Conclusions 20
  6.2 Recommendations 20
  6.3 Future research 21

Bibliography 23

A Fourier Theory 24
  A.1 Discrete and Fast Fourier Transform 24
  A.2 FPT in MATLAB 25

B MATLAB implementation of the Hilbert Transform 26
  B.1 Errors in the computations 26

C CEDA transformation for systems with hysteretic damping 28
Summary

Analysis of linear dynamic (acoustic) systems has become a routine job, with the development of the Finite Element Method. There is however a restriction in the maximum frequency that can be analyzed. Finite Element Method analyses for high frequencies demand a large number of degrees of freedom. Above that deterministic calculations become meaningless with this high frequencies. Therefore stochastic properties have to be taken into account.

A literature survey of possible solutions to this problem has been made by Raaymakers (1995a). A conclusion from Raaymakers (1995a) is that Complex Envelope Displacement Analysis (CEDA) is the most promising alternative to cover the mid and high frequency range. CEDA is developed by Carcaterra & Sestieri (1994). It is based on the Hilbert Transformation, a signal transformation that is sometimes used in communication theory. With the Hilbert transform, the analytic signal can be defined. The analytic signal is a complex signal which most important property is that it only contains positive frequency components. An important step in the CEDA theory is a frequency shift that is performed on the analytic signal. This results in the so-called 'complex envelope'.

This report investigates the theoretical and numerical aspects of Complex Envelope Displacement Analysis, or CEDA.

In chapter 1 the scope of the subject is discussed. In chapter 2 the definitions of the Hilbert Transform and the Analytic signal are given. After that the Complex Envelope is presented. In chapter 3 the signal transformations will be demonstrated. A simple application of the use of these signal transformations and CEDA are given in chapter 4. Chapter 5 mentions the problems that rise with the boundary conditions and chapter 6 gives some conclusions and recommendations.
Chapter 1

Introduction

Analysis of linear dynamic systems with deterministic loads, boundary conditions and material parameters has become a routine job, using the Finite Element Method. However, two problems can be distinguished.

First, the analyses are restricted in the maximum frequency that can be taken into account, because high frequencies demand an unacceptably large number of degrees of freedom. Solving the related numeric problem will become practically impossible. A solution may be found in techniques that reduce the number of degrees of freedom. Reduction techniques however often imply a loss of information.

Second, the responses of dynamic systems at high frequencies are strongly dependent on, for example, the position of the excitation and the material parameters, making calculation of a deterministic response meaningless. A solution may be found by taking into account the stochastic properties of the material parameters or excitation position.

Possible solutions to these problems are summarized in a literature survey by Raaymakers (1995a). For high frequencies, solutions are found in methods like Statistic Energy Analysis (SEA). In this method the systems' properties are averaged over a frequency band, where the assumption is made that the systems' parameters are statistical. This method implies a loss of information since averaging techniques are used and is therefore not very useful when responses are calculated. Other attempts have been made to deal with the high frequencies using "thermal analogies". These methods have a great lack of insight in the physical problem. There are also methods, like envelope methods, that cover the mid and high frequency range. A major advantage of these methods is that they are based on the same physical laws and properties that are used in deterministic calculations. To deal with the stochastic properties of the system a stochastic method has to be used in addition. It is however possible to deal with the stochastic properties by doing only deterministic calculations. This is done by combining the results of a series of deterministic calculations to obtain probability density functions of the response. Such a method may not be ideal, but it works. The Monte-Carlo method is such a method. For a set of parameters with statistic distribution the responses are calculated deterministically. From these responses the statistic distribution is derived.

1.1 CEDA

The most promising method to do high frequency dynamic analyses is CEDA (Complex Envelope Displacement Analysis) which is an envelope method developed by Carcaterra & Sestieri (1994). It is based on the Hilbert Transformation and the analytic signal which will be explained more thoroughly in the next chapters. With these transformations high frequency real signals can be transformed into low frequency complex signals. In practice however physical problems are often stated mathematically as differential equations with prescribed input signals. Since the CEDA transformations are linear, they can not only be applied to signals but also to linear differential
equations. The input signals and the solutions to these equations become low frequency complex signals. From these low frequency complex signals the solutions to the original problem can be reconstructed by interpolation and a backward transformation.

1.2 Strategy

At this point we want to understand more about CEDA. Therefore the method is investigated more thoroughly and the results are printed in this report. A good overview of the possibilities and impossibilities of CEDA is necessary to decide whether it is worth the effort of implementing this method.

After that the CEDA theory must be formulated in a way that it can be used in combination with a Finite Element Method formulation. This implies that the transformation of the equations and application of the boundary conditions (which is problematic according to Raaymakers (1995b)) must be dealt with automatically. Another goal is to find a stochastic method that can be added to the CEDA concept. This however will not be dealt with in this report.
Chapter 2

The Hilbert Transform, the Analytic Signal and the Complex Envelope

The Hilbert transform is used in communication theory for the modulation of signals. The Hilbert transform is also part of the definition of the analytic signal. Analytic signals are defined in Thomas (1969) as signals that have only positive frequency components. Since the definition of the analytic signal contains the Hilbert transform, the Hilbert transform will be discussed first. The analytic signal can be used for the definition of the Complex Envelope of a signal, that will be discussed in the end of this chapter. It is important to mention that the transformations that are described in this chapter do not lose any information on the original signal. Therefore the reconstruction of the original signal is very easy as will be demonstrated in the end of this chapter.

2.1 The Hilbert Transform

The Hilbert transform $\tilde{x}(t)$ of a function $x(t)$ is defined in Bracewell (1965) as the convolution of $x(t)$ with $\frac{1}{\pi t}$:

$$\tilde{x}(t) = \frac{1}{\pi t} * x(t) = \int_{-\infty}^{\infty} \frac{1}{\pi t} x(t - \tau) d\tau$$  (2.1)

With the Fourier transform the convolution in the time domain can be transformed to a multiplication in the frequency domain. (More on Fourier theory can be found in appendix A.) The Fourier transform of $\frac{1}{\pi t}$ is $j \text{sign}(s)$, which is equal to $+j$ for positive $s$ and $-j$ for negative $s$.

$$\tilde{X}(s) = j \text{sign}(s) X(s)$$  (2.2)

Some properties of the Hilbert transform are:

- The Hilbert transform of an even function is an odd function\(^1\).
- The Hilbert transform of an odd function is an even function.
- When the Hilbert transformation is performed twice on a signal the result is the negative original signal.

The Hilbert transform corresponds to a filtering in which the amplitudes are unchanged but the phases are increased by $\pi/2$ for positive frequencies and decreased by $\pi/2$ for negative frequencies.

\(^1\)A function $f(t)$ is an even function if $f(t) = f(-t)$. The function $f(t)$ is an odd function if $f(t) = -f(-t)$. The Fourier transform of an even function is real and the Fourier transform of an odd signal is purely imaginary.
2.2 The analytic signal

The signal $x(t)$ and the Hilbert transform $\tilde{x}(t)$ are used to derive\(^2\) the analytic signal $\hat{x}(t)$:

$$\hat{x}(t) = x(t) + j\tilde{x}(t).$$  \hspace{1cm}  (2.3)

This is a complex signal. The analytic signal of – for example – a cosine wave can now be calculated:

$$\begin{align*}
x(t) &= \cos(\omega t) \\
\tilde{x}(t) &= \sin(\omega t) \\
\hat{x}(t) &= e^{j\omega t}
\end{align*}$$  \hspace{1cm}  (2.4)

It can be seen that the amplitude of the analytic signal equals 1, and the phase has an angular frequency $\omega$. The analytic signal corresponding with a cosine wave with angular frequency $\omega$ is a helix, as can be seen in figure 2.1. The phase of the helix changes with angular speed $\omega$.

![Figure 2.1: The analytic signal corresponding to a cosine wave is a helix. The real and imaginary parts are the original signal and the Hilbert transform.](image)

A very important property of the analytic signal is that its frequency spectrum contains only positive components. For negative frequencies the frequency content is zero. For positive frequencies it is twice as high as the spectrum of the original signal as demonstrated in figure 2.2. This property is used by the MATLAB implementation of the Hilbert transform. It must be mentioned that the original signal $x(t)$ needs to be real, because only real signals have a symmetric frequency spectrum. In the transformation the negative part of the frequency spectrum is set equal to zero. For complex signals this would result in loss of information.

**Instantaneous amplitude and frequency**

According to Bracewell (1965) the analytic signal can be written as:

$$A(t)e^{j\omega(t)t}$$  \hspace{1cm}  (2.5)

\(^2\)In some literature the Hilbert transform is defined as a convolution with $-\frac{1}{\pi t}$ what also leads to a negative sign in equation (2.3).
The instantaneous amplitude or envelope of a signal is defined as the absolute value of the analytic signal $|A(t)|$, and the instantaneous frequency is $\omega(t)$. This representation makes the interpretation of the analytic signal simple. The 'carrier' of the signal is a helix that rotates with angular frequency $\omega(t)$ in the direction of positive time $t$, and $A(t)$ describes the time-dependent envelope or amplitude of the signal.

2.3 The Complex Envelope

The Complex Envelope $\mathcal{F}$ as used in CEDA is derived by multiplying the analytic signal $\hat{x}$ with $e^{-j\omega_0 t}$.

$$\mathcal{F}(t) = \hat{x}(t)e^{-j\omega_0 t}. \quad (2.6)$$

When $x(t)$ is a cosine wave with frequency $\omega$ then the Complex Envelope $\mathcal{F}$ will equal:

$$\mathcal{F}(t) = e^{j(\omega-\omega_0)t}. \quad (2.7)$$

This signal has an amplitude 1 and the phase has an angular frequency $(\omega - \omega_0)$, which is a smooth signal with low frequency when $\omega_0$ is chosen close to $\omega$. The property of the low frequency would be very useful in more complex cases where very high sample frequencies are needed. A transformation to the complex envelope would reduce the amount of samples significantly.

Reconstruction

From the Complex Envelope the original signal can be reconstructed by the inverse transformation:

$$x(t) = \text{Re}[\mathcal{F}(t)e^{j\omega_0 t}] \quad (2.8)$$

The multiplication with $e^{j\omega_0 t}$ increases the frequency of $x(t)$ and therefore $x(t)$ will have higher frequency contents than $\mathcal{F}(t)$. When working with discrete signals it is therefore necessary to resample $\mathcal{F}(t)$ with an interpolation routine (e.g. linear interpolation; cubic spline interpolations are used here) to guarantee that the reconstructed signal meets the Shannon sampling criteria.
Chapter 3

Signal Transformations

In this chapter the effects of some signal transformations will be analyzed to demonstrate the effects of the transformations in both the time and the frequency domain. First a simple harmonic signal is analyzed. After that the Hilbert transform, the analytic signal and the complex envelope based on the harmonic signal are analyzed. It is also shown that the original signal can be reconstructed easily from the transformed signal.

Harmonic signal

A simple sine wave with a frequency of 10 Hz is given in the time domain from 0 to 1 second (See figure 3.1). There will be ten periods of 0.1 seconds in this range. The signal can be sampled at a sample frequency $f_s$ of 100 Hz.

Figure 3.1: The sine wave is a purely real signal with purely imaginary frequency components.
This harmonic signal is purely real, the imaginary part is zero. Note that the sine wave is an odd signal. Therefore the frequency spectrum is purely imaginary. This signal can be analyzed with the Fast Fourier Transform. The absolute value of the FFT shows that the frequency of the signal is found at 10 Hz.

**Hilbert Transform**

The Hilbert transform of this signal can be calculated. In theory the Hilbert transform of a sine wave is a cosine wave. This is an even signal and has a real frequency spectrum as can be seen in figure 3.2.

When the original signal $x(t)$ would not have been exactly periodic on the interval $[0, 1]$ the Hilbert Transform would not exactly equal a cosine wave. This is due to the way the Hilbert Transform is calculated and is explained in appendix B.

![Graph](image)

**Figure 3.2:** The Hilbert transform of a sine wave is a negative cosine wave, which is still a purely real signal. Its FFT however is purely real.
Analytic signal

The analytic signal is a complex signal. Its real part is the original real signal and its imaginary part is the Hilbert transform of that signal. For the signal analyzed here the FFT is given in figure 3.3. The analytic signal contains only positive frequency components but the intensity of the positive frequencies is twice as high. Note that the frequency spectrum is imaginary.

![Image of analytic signal and FFT](image)

**Figure 3.3:** The analytic signal of the sine wave has a real part that is equal to that sine wave; its imaginary part is equal to the Hilbert transform of the sine wave and is therefore a negative cosine wave. The FFT of this signal shows only positive frequency components, which are purely imaginary.

Complex envelope

From the analytic signal the complex envelope can be determined by multiplication with $e^{-2\pi j f_0 t}$. When the frequency $f_0$ is chosen close to the frequency $f$ of the original real signal, the complex envelope will be a low frequency signal with frequency $f - f_0$. This is demonstrated by taking $f_0 = 9$ Hz, so the complex envelope will be a signal with a frequency of 1 Hz, as can be seen in figure 3.4. The main result of this operation is the shift of the frequency peak from 10 Hz to 1 Hz.

One might wonder why the multiplication with $e^{-2\pi j f_0 t}$ is not performed on the original signal, which is also a signal with frequency $f$. This can be easily seen when the sine wave is written as:

$$\sin(2\pi ft) = \frac{e^{2\pi j ft} - e^{-2\pi j ft}}{2j}$$

So multiplication with $e^{-2\pi j f_0 t}$ will lead to a low frequency term with frequency $f - f_0$, but also to a term with frequency $f + f_0$. 

11
Figure 3.4: The complex envelope and the FFT. The only difference between the complex envelope and the analytic signal is the scaling of the horizontal axis, which can be seen as a frequency shift.

**Reconstruction**

From the complex envelope the original signal can be reconstructed by multiplying the complex envelope with $e^{2\pi j \omega t}$ and taking the real part. From figure 3.5 can be seen that the reconstructed signal is equal to the original signal. The error signal $\hat{x} - x_r$, where $x_r$ is the reconstructed signal, is not plotted because the error is of the order of the floating point relative accuracy, which is about $10^{-16}$.

In this example the sine wave was perfectly periodic on the interval that was taken into account. For non-periodic signals the calculation of the Hilbert Transforms is suffering from signal leakage (The Hilbert Transform uses an FFT and inverse FFT operation). This leads to the introduction of a non-periodic Hilbert Transform, analytic signal and complex envelope. These errors however have a negligible effect on the reconstruction.
Figure 3.5: The reconstructed signal and the FFT. The signal that is reconstructed from the complex envelope equals the original sine wave.
Chapter 4

Complex Envelope Displacement Analysis

In chapter three, the complex envelope of a signal was computed. When the analytic signal is modulated with the right frequency, the complex envelope is a smooth signal with low frequency. Therefore it might be represented by a smaller amount of samples, than would be necessary for the original high frequency signal.

In practice physical problems can often be stated mathematically in the form of differential equations. The solution to these differential equations may contain high frequency signals. The CEDA theory now presents a method to transform these differential equations in such a way that the solution is the complex envelope corresponding to the (high frequency) solution of the original differential equations. This solution can be derived with larger integration steps (or in Finite Element Method with a more course mesh). From the calculated complex envelope the correct solution to the original problem can be derived by interpolation and a backward transformation.

4.1 Theory

In general a second order differential equation can be written as:

\[ a\ddot{x}(t) + b\dot{x}(t) + cx = f(t) \quad (4.1) \]

Where \( \dot{x}(t) \) and \( \ddot{x}(t) \) are the first and second derivative of \( x \) to time \( t \), and \( a, b \) and \( c \) are real constants. Suppose the input signal \( f(t) \) is a known high frequency signal, then \( x(t) \) will also be a high frequency signal. In this case the solution can be derived analytically, but the CEDA theory will be formulated in such a way that it is also applicable for problems that need to be solved numerically.

Since the Hilbert Transformation is a linear transformation, equation (4.1) is also valid for the Hilbert transform of \( x(t) \),

\[ a\ddot{x}(t) + b\dot{x}(t) + cx = \tilde{f}(t) \quad (4.2) \]

and also for the analytic signal:

\[ a\ddot{x}(t) + b\dot{x}(t) + cx = \check{f}(t) \quad (4.3) \]

With the relation for the complex envelope and the analytic signal (equation (2.6)), written inversely:

\[ \dot{z}(t) = \bar{x}(t)e^{j\omega_0 t} \quad (4.4) \]

we can write for the derivatives:

\[ \ddot{z}(t) = [\ddot{\bar{x}}(t) + j\omega_0 \bar{x}(t)]e^{j\omega_0 t} \quad (4.5) \]

\[ \dot{z}(t) = [\ddot{x}(t) + 2j\omega_0 \dot{x}(t) - \omega_0^2 \bar{x}(t)]e^{j\omega_0 t} \quad (4.6) \]
The differential equation can now be written as (the term $e^{j \omega_0 t}$ is factored out):

$$a \ddot{x}(t) + (b + 2aj \omega_0) \dot{x}(t) + (c + bj \omega_0 - aw_0^2)x = f(t)$$

(4.7)

This differential equation is forced by a complex input signal $\bar{f}$ and therefore the solution $\bar{x}$ will be complex as well. Numeric software packages like MATLAB can deal with this, but for programs that can not deal with complex signals the equations can be rewritten in real form. Therefore the complex variables $\bar{x}$ and $\bar{f}$ are written as (the dependence of time $(t)$ is omitted):

$$\bar{x} = x_r + jx_i$$

(4.8)

$$\bar{f} = f_r +jf_i$$

(4.9)

The equations can now be written as:

$$a \ddot{x}_r + b \dot{x}_r - 2a \omega_0 \ddot{x}_i + (c - a \omega_0^2)x_r - b \omega_0 x_i = f_r$$

(4.10)

$$a \ddot{x}_i + b \dot{x}_i + 2a \omega_0 \ddot{x}_r + (c - a \omega_0^2)x_i + b \omega_0 x_r = f_i$$

(4.11)

Or written in matrix notation:

$$\begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}\begin{bmatrix}
\ddot{x}_r \\
\ddot{x}_i
\end{bmatrix} + \begin{bmatrix}
b & -2a \omega_0 \\
b & b \omega_0
\end{bmatrix}\begin{bmatrix}
\dot{x}_r \\
\dot{x}_i
\end{bmatrix} + \begin{bmatrix}
c - a \omega_0^2 & -b \omega_0 \\
b \omega_0 & c - a \omega_0^2
\end{bmatrix}\begin{bmatrix}
x_r \\
x_i
\end{bmatrix} = \begin{bmatrix}
f_r \\
f_i
\end{bmatrix}$$

(4.12)

Note that when $\omega_0$ equals zero, the equations result in two decoupled differential equations, equal to equation (4.1).\footnote{Raaymakers (1995b) (page 22) used a matrix representation, with complex elements, where only $\bar{x}$ was split up in a real and imaginary part. In that case the column of inputs had the complex input $\bar{f}$ and zero in it. The set of differential equations in the equation (4.12) is a better representation because the equations are completely real (real variables and real coefficients).}

The Hilbert transformation can only be performed on real signals. When the differential equation (4.1) is used to describe systems with hysteretic damping, the coefficient $c$ will be complex. In that case the solution of the differential equation will be a complex signal. Therefore an adjustment has to be made to transform the equations to the CEDA notation. This will be discussed in appendix C.

4.2 Example

The aim of the CEDA transformation is to solve differential equations in the high frequency domain with fewer integration steps. Suppose we want to determine the response on the interval from 0 to 10 seconds of a system described by equation (4.1) with parameters $a = 1$, $b = 2$ and $c = 3$ for an harmonic (sine wave) excitation with an excitation frequency of 1 Hz. All initial conditions are set to zero. The 'exact' solution can be determined by taking a timescale with intervals of 0.05 seconds. The response of this simple linear system is determined with three different timescales with timesteps of 0.1, 0.2 and 0.4 seconds. The results are given in figure 4.1. In the righthand figures the solutions are compared with the 'exact' solution and one can see that with timesteps of 0.2 seconds there is an error of several percents. The solution where a timestep of 0.4 seconds is applied is unacceptable.

The differential equations are rewritten as in equation (4.12) and the excitation is transformed to a complex signal. Its real and imaginary part are used as input signals. The timescale has intervals of 0.1, 0.2 and 0.4 seconds like in the original calculation. The solution that is found is a complex signal, from which the absolute value is printed in the righthand figures of figure 4.2. The complex envelope describes the absolute values of the signal very good, even with the larger timesteps.
Figure 4.1: Response on an excitation of 1 Hz with time intervals of 0.1, 0.2 and 0.4 seconds.

Figure 4.2: The complex envelope response of a system on an excitation of 1 Hz calculated with time intervals of 0.1, 0.2 and 0.4 seconds. x=Real part, o=Imaginary part

With the backward transformation as given in chapter 3 the solution to the original differential
equations is found. A graphical demonstration of the reconstruction is given in figure 4.3.

![Reconstruction of the signal](image)

Figure 4.3: Reconstruction of the signal. The solution to the CEDA differential equations is represented with the o-symbols. A cubic spline interpolation is also plotted with the dotted line. This signal is multiplied with $e^{j\omega t}$ which results in the dash-dotted line. This is the analytic signal corresponding with the solution. The real part of this curve is the solution of the original differential equations. It is compared with the 'exact' solution (dotted line) in the projection on the plane where the imaginary part equals 1.

4.3 Concluding remarks

From the demonstration in the foregoing section it can be concluded that the CEDA transformation can be used to solve differential equations. Signal leakage influences the solutions of the CEDA equations since the complex envelope of a non-periodic signal is distorted as demonstrated in appendix B. The distortion of the CEDA solution is canceled partly in the reconstruction. Therefore it can be concluded that the effect of signal leakage is small.

The example was an initial value problem, but Raaymakers did also apply CEDA on 2 point boundary value problems. This introduces some problems concerning the boundary conditions in the CEDA representation. This will be investigated in the next chapter.
Chapter 5

Boundary conditions in CEDA

As mentioned in the previous chapter the use of boundary conditions in CEDA will cause certain problems. This will be illustrated with the following differential equation.

\[ a \ddot{z} + b \dot{z} + c z = f \]  \hspace{1cm} (5.1)

The boundary conditions (in this case initial conditions) can be formulated as:

\[ z(0) = z_0 \]
\[ \dot{z}(0) = v_0 \]  \hspace{1cm} (5.2)

When the differential equation is formulated in CEDA notation, the equations can be split in a real and an imaginary part:

\[ a \ddot{x}_r + b \dot{x}_r - 2 \omega_0 \dot{x}_i + (c - a \omega_0^2) x_r - b \omega_0 x_i = \bar{f}_r \]  \hspace{1cm} (5.3)
\[ a \ddot{x}_i + b \dot{x}_i + 2 \omega_0 \dot{x}_r + (c - a \omega_0^2) x_i + b \omega_0 x_r = \bar{f}_i \]  \hspace{1cm} (5.4)

with four initial conditions:

\[ \ddot{x}_r(0) = x_{r0} \]
\[ \dot{x}_r(0) = v_{r0} \]
\[ \ddot{x}_i(0) = x_{i0} \]
\[ \dot{x}_i(0) = v_{i0} \]  \hspace{1cm} (5.5)

The translation to the CEDA equations is explained in the previous chapter. The transformation of the boundary conditions is ambiguous. Two relations between the original and the CEDA boundary conditions can be formulated:

\[ \text{Re}[\dot{x}e^{i\omega_0 t}]_{t=0} = x_0 \]
\[ \text{Re}[\ddot{x}e^{i\omega_0 t}]_{t=0} = v_0 \]  \hspace{1cm} (5.7)

This implies that:

\[ \ddot{x}_r(0) = x_{r0} = x_0 \]
\[ \dot{x}_r(0) = v_{r0} = v_0 \]  \hspace{1cm} (5.8)

The imaginary parts of the initial conditions of the complex envelope displacement can be taken arbitrarily. The choice however will have influence on the solution procedure.
5.1 Pseudo-inverse

A possible solution to this problem can be found by applying the pseudo-inverse relation. Equation (5.7) can be rewritten for a 2 point boundary condition problem as:

\[
\begin{align*}
\text{Re}[\overline{x}(\sin(\omega_0 t) + j \cos(\omega_0 t))]_{t=a} &= x_a \\
\text{Re}[\overline{x}(\sin(\omega_0 t) + j \cos(\omega_0 t))]_{t=b} &= x_b
\end{align*}
\]

or in matrix notation as:

\[
\begin{bmatrix}
\sin(\omega_0 a) & -\cos(\omega_0 a) & 0 & 0 \\
0 & 0 & \sin(\omega_0 b) & -\cos(\omega_0 b)
\end{bmatrix}
\begin{bmatrix}
\overline{x}_r(a) \\
\overline{x}_r(b) \\
\overline{x}_i(a) \\
\overline{x}_i(b)
\end{bmatrix} =
\begin{bmatrix}
x(a) \\
x(b)
\end{bmatrix}
\]  

(5.10)

The solution to the inverse equation can be found with the pseudo-inverse of the matrix. Since the rows in the matrix are independent the pseudo-inverse equals its transposed. Further investigations on the use of the pseudo-inverse will be made.
Chapter 6
Conclusions and recommendations

6.1 Conclusions

From the foregoing chapters the following can be concluded:

- The Hilbert Transform and the analytic signal are useful tools to do signal transformations. With the described transformations it is possible to transform a high frequency signal into a low frequency (complex) signal and the corresponding differential equations into complex differential equations. This complex differential equations can be dealt with easily by separation of the complex variables into real and imaginary parts.

- The Ceda transformations can be applied to linear differential equations. With these transformed equations the solution can be found with a lower sampling frequency. It is not investigated however if this reduction weighs up to the extra steps that are necessary for the transformation.

6.2 Recommendations

Based on the conclusions the following recommendations can be formulated:

- It must be investigated whether the CEDA transformation can be applied to more dimensional systems. In the examples described in this report only one dimensional systems are taken into account. Extension to more dimensions is preferred to be able to analyze more dimensional constructions.

- It must be investigated how the CEDA transformations can be used when the excitation consists of more frequencies or frequency bands. A possible solution may be found by analyzing a number of frequency intervals separately.

- The efficiency of the transformations must be investigated. The solution to the differential equations can be found by applying less sample intervals or a more course mesh. The transformations to the CEDA formulation and the reconstruction must take considerable less computation time than the achieved reduction to make this method efficient.
6.3 Future research

In figures 6.1 to 6.6 is pointed out what may be achievable with the CEDA method. The response of a plate (or membrane) that is forced by an high frequency excitation in the centre can be computed with a Finite Element Method software package. To calculate the modes of vibration sufficiently accurate a very fine mesh may be necessary. With the CEDA method the differential equations and the excitation can be transformed. The solution can be calculated with a more course mesh, which takes less calculation time. The solutions has to be interpolated before a backward transformation can be performed. This leads to the reconstructed solution of the original problem.

The normal method to do response calculations of a linear dynamic system is to make a fine mesh and solve the differential equations, as demonstrated in figures 6.1 and 6.2. This however will take an enormous amount of calculation time. The steps to be taken in the CEDA method are given in figures 6.3 to 6.6.
Normal approach

Figure 6.1: Fine mesh of a plate or membrane

Figure 6.2: Solution calculated with the fine mesh

CEDA approach

Figure 6.3: Course mesh of a plate or membrane

Figure 6.4: Solution calculated with the course mesh

Figure 6.5: Interpolation of the solution

Figure 6.6: Solution after reconstruction
Bibliography


Appendix A

Fourier Theory

In this appendix a brief review of the concept of the Fourier Transform is given. For some applications signals can better be analyzed in the frequency domain than in the time domain. The Fourier theory gives some tools to evaluate signals in the frequency domain.

A.1 Discrete and Fast Fourier Transform

Whether signals are obtained experimentally or in numeric simulation they are almost always represented digitally, i.e. the signal is sampled. The value of the signal is only known for a discrete number of time-points. The frequency spectrum or Fourier Transform of discrete signals can be determined by the Discrete Fourier Transform (DFT) (See Enden & Verhoeckx (1989), Kraker (1992) and Papoulis (1977)).

For an arbitrary function \( z(t) \) the Fourier Transform integral is defined as:

\[
X(f) = \int_{-\infty}^{\infty} z(t)e^{-2\pi j ft} dt \quad \text{or} \quad X(\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t} dt
\]

And the inverse relation:

\[
z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f)e^{2\pi j ft} df \quad \text{or} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega
\]

For the DFT algorithm the signal \( z(t) \) is sampled on the interval \([0, T]\), with sample frequency \( f_s = N/T \) to obtain \( N \) equidistant parts \( \Delta T = T/N \). The discrete values \( x_k \) are given by \( x_k = x(k\Delta T) \) for \( k = 0, 1, 2, \ldots, N - 1 \).

Signal \( x(t) \) is only sampled on the interval \([0, T]\). Information of the signal outside this interval is not available. Therefore we assume the signal to be periodic with period \( T \). To represent this signal it can be proved (see Enden & Verhoeckx (1989) or Kraker (1992)) that only the discrete frequencies \( f_n = nf_0 \) \((n = 0, 1, \ldots, N)\) are needed. The frequency spectrum \( X[f_n] \) is periodic with period \( f_N = Nf_0 \).

Now the Discrete Fourier Transform can be written as:

\[
X[n] = \sum_{k=0}^{N-1} x[k]e^{-2\pi j nk/N} \Delta T \quad \text{for:} \quad n = 0, 1, 2, \ldots, N - 1
\]

In numeric software packages like MATLAB the FFT-algorithm may work a little different. This will be dealt with further on in this appendix.

The fast Fourier Transform is a special case of the Discrete Fourier Transform. In recent years many different methods are developed to calculate \( N \)-point DFT’s with considerably fewer operations (See Kraker (1992) and Enden & Verhoeckx (1989)). All these methods, that can be denoted by the collective name ‘Fast Fourier Transform’, always follow a procedure to calculate
a number of DFT’s of shorter length and then combine the results appropriately. When \( N \) is chosen to be a power of two, the discrete series \( x[k] \) can be divided in two iteratively until each part includes only one element. The DFT’s of these \( N \) elements can be calculated and the results can be combined with use of the expression:

\[
W_k^n = e^{-\frac{2\pi j k}{N}}
\]

When \( N \) is chosen to be a power of 2, \((N = 2^a, a \in \mathbb{N})\), extra calculation time can be saved by changing the order of summations and multiplications. The term \( e^{-2\pi j k^p / N} \) equals 1 when \( nk = pN \) \((p \in \mathbb{N})\), but also for other combinations there is a reduction as can be seen in the equations below. If:

\[
n_kk_p = n_a k_a + pN, \quad p \in \mathbb{N}
\]

then:

\[
e^{-2\pi j (n_a k_a + pN) / N} = e^{-2\pi j k_a \frac{n}{N}} e^{-2\pi j \frac{pN}{N}} = e^{-2\pi j k_a \frac{n}{N}} 1 = e^{-2\pi j k_a \frac{n}{N}}
\]

This proves that \( e^{-2\pi jk^p / N} \) is periodic. When this property is used a lot of calculation time can be saved. Instead of adding terms \( a \cdot b, a \cdot c, a \cdot d \) etc., which needs a lot of multiplications, a summation and one multiplication can be applied: \( a \cdot (b + c + d + \ldots) \).

### A.2 FFT in MATLAB

Signals can be described by their frequency contents with the aid of the FFT. The signal has to be sampled to obtain a discrete time signal. The (discrete) FFT of the discrete time signal is a series of complex numbers, in contrast with the real sampled signal.

With the numeric software package MATLAB the FFT of a discrete time signal \( x[k] \) on the interval \([0, T]\) with \( N \) samples is calculated as:

\[
X[n] = \sum_{k=0}^{N-1} x[k] e^{-2\pi j kn / N}
\]

The corresponding frequency axis is \([0, f_N]\), where \( f_N = N/T \). The frequency spectrum can be repeated periodically, so we can also represent the spectrum on the frequency interval \([−\frac{1}{2}f_N, \frac{1}{2}f_N]\).

For the frequency spectrum plots in this report the MATLAB output is adapted by shifting the right half of the calculated frequency spectrum \((\frac{1}{2}f_N, f_N]\) to the negative axis \((-\frac{1}{2}f_N, 0]\).

The amplitude of the frequency spectrum is a measure for the energy present in the signal. The energy theorem (or Parseval’s formula) states:

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df
\]

This energy theorem can also be applied on sampled signals, by changing the integration into a summation. The amplitudes of the FFT as formulated in equation (A.5) is dependent on the number of samples \( N \). Therefore the energy theorem can not be applied, because the energy will be different at different sampling frequencies. To make the outcome of the FFT independent on \( N \), the results have to be multiplied with \( \Delta T \) (\( \Delta T = T/N \)).

The results of FFT’s that are plotted in this report are constructed with a modified FFT-routine, that multiplies the amplitudes of the MATLAB-FFT with the time-interval \( \Delta T \), and plots the frequency spectrum on the interval \([-\frac{1}{2}f_N, \frac{1}{2}f_N]\).
Appendix B

MATLAB implementation of the
Hilbert Transform

To compute the Hilbert Transform of a signal in the time domain a convolution has to be performed. In the frequency domain it can be computed by a changing the phase of the signal. Neither of these methods are used in the implementation of the Hilbert Transform in MATLAB.

In chapter 2 an important property of the analytic signal was given. The frequency spectrum for positive frequencies was twice as high as that of the original signal and for negative frequencies it equals zero. The Hilbert Transform can be seen as the imaginary part of the analytic signal, so if the analytic signal is computed the Hilbert Transform is also known.

To compute the analytic signal the MATLAB function hilbert can be used. It performs a Fast Fourier Transform on signal $x$ and multiplies the frequency spectrum with a block-shaped signal that has height 2 when it corresponds with the positive frequencies of signal $x$ and height 0 when it corresponds with negative frequencies of signal $x$. At frequency zero the filter has a value of 1. On this modified frequency spectrum an inverse FFT is performed. The analytic signal now is computed by using its most important property. The Hilbert Transform of $x$ can easily be found by taking the imaginary part of the analytic signal.

B.1 Errors in the computations

As mentioned in chapter 1 signal leakage occurs as FFT's are computed of signals that are not exactly periodic in the interval that is used. This will also happen in the FFT that is part of the Hilbert Transform. When such a Hilbert Transform is calculated that is not periodic in the interval it is defined on, the Hilbert Transform will not be exact. This is made clear by figure B.1 where a Hilbert Transform of a sine wave is calculated for an interval of two, and of two and a half periods. The exact solution should be a cosine wave.
Figure B.1: Hilbert Transform of sine wave of 2 respectively $2\frac{1}{2}$ periods
Appendix C

CEDA transformation for systems with hysteresis damping

The ordinary second order differential equation (4.1) with real coefficients can be used to describe systems with structural or viscous damping. For the description of hysteresis damping coefficient $c$ will be complex. The equations now read (structural damping is also represented):

$$a \ddot{x} + b \dot{x} + cx + dx = f$$  \hspace{1cm} (C.1)

The solution $x$ of this differential equation will be a complex signal. The CEDA transformations are not applicable to these equations, because the Hilbert transformation can only be performed on real signals. Therefore we rewrite the equations, using:

$$x = x_r + jx_i$$  \hspace{1cm} (C.2)

The differential equation can now be written in matrix notation, as if there were two real signals, $x_r$ and $x_i$.

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \dddot{x}_r \\ \dddot{x}_i \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \ddot{x}_r \\ \ddot{x}_i \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} + \begin{bmatrix} f_r \\ f_i \end{bmatrix}$$  \hspace{1cm} (C.3)

On the two real signals $x_r$ and $x_i$ the Hilbert transformation can be performed, so these differential equations are also valid for the Hilbert transforms $\dddot{x}_r$ and $\dddot{x}_i$ of $x_r$ and $x_i$. The same equations also result for the analytic signal $\dddot{x} = x + j\dddot{x}$, who's real and imaginary parts result from $^1$:

$$\begin{align*}
\dddot{x} &= x_r + jx_i + j(\dddot{x}_r + j\dddot{x}_i) \\
&= x_r - \dddot{x}_i + j(\dddot{x}_r + x_i) \\
\dddot{x}_r &= \text{Re}\{\dddot{x}\} = x_r - \dddot{x}_i \\
\dddot{x}_i &= \text{Im}\{\dddot{x}\} = \dddot{x}_r + x_i
\end{align*}$$  \hspace{1cm} (C.4)

Now the analytic signal can be transformed to the complex envelope which results in the following equations:

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \dddot{x}_r \\ \dddot{x}_i \end{bmatrix} + \begin{bmatrix} b & -2a\omega_0 \\ 2a\omega_0 & b \end{bmatrix} \begin{bmatrix} \ddot{x}_r \\ \ddot{x}_i \end{bmatrix} + \begin{bmatrix} c - a\omega_0^2 & -d - b\omega_0 \\ d + b\omega_0 & c - a\omega_0^2 \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} + \begin{bmatrix} f_r \\ f_i \end{bmatrix}$$  \hspace{1cm} (C.6)

These equations are equal to equation (4.12), apart from the $-d$ and $d$ in the stiffness matrix.

---

$^1\dddot{x}_r \neq x_r + j\dddot{x}_r$