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Rational Representations of $\ell_2[0, \infty)$ Systems

Siep Weiland  
Department of Electrical Engineering  
Eindhoven University of Technology  
P.O. Box 513  
5600 MB Eindhoven  
The Netherlands  
e-mail: s.weiland@ele.tue.nl

Anton Stoorvogel  
Department of Mathematics and Computing Science  
Eindhoven University of Technology  
P.O. Box 513  
5600 MB Eindhoven  
The Netherlands  
e-mail: wscoas@win.tue.nl

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1 Introduction

In this paper we aim to set-up a theory for well-posedness, stabilizability and stability of dynamical systems that departs from the usual input-output way of thinking. For the class of linear discrete time systems defined on right-sided time axes we study basic representation issues via kernel and image representations. Study system behaviors both in the time and the frequency domain. Using the behavioral theory, we introduce and characterize notions of stability and stabilizability of dynamical systems in which input and output variables have not been selected a priori.

This leads to a theory of linear left-shift invariant closed subspaces of the Hilbert space of square summable functions on $\mathbb{Z}^+$ in which classical concepts as well-posed feedback interconnections and stabilizability are naturally defined and studied.

The concepts that are introduced here are very closely related to the notion of a graph of an input-output operator. In [1] Georgiou and Smith studied representation issues and stability of feedback interconnections in a similar framework. We will clearly point out the intimate relationship between our approach and the one in [1]. Our technique has the main advantage that we are to give a representation of a closed loop system which, since it is often autonomous (i.e. no inputs) can not be derived in a straightforward manner using a graph of an input-output operator as a starting point.

We believe that our definitions might have important conceptual consequences for the analysis of stability of feedback systems. Stabilizability of dynamical systems is studied in our setting and we work out conditions under which a dynamical system can be stabilized by interconnecting it with various classes of systems.

2 Notation

Let $T \subset \mathbb{Z}$ be a set and $(W, \| \cdot \|)$ a normed space.

- $\mathbb{Z}_+ := \{ t \in \mathbb{Z} \mid t \geq 0 \}$
3 A CLASS OF $\ell_2$ SYSTEMS

- $\ell(T, W) = W^T$
- $\ell_2(T, W) := \{w \in \ell(T, W) \mid \sum_{t \in T} ||w(t)||^2 < \infty\}$
- $\ell_2^\downarrow := \ell_2(\mathbb{Z}_+, \mathbb{R}^q)$
- $\mathcal{H}_2^+, \mathcal{H}_2^\downarrow$: Hardy spaces of square integrable functions on the unit circle with analytic continuation outside and inside the unit circle, respectively.
- $\mathcal{H}_\infty^+, \mathcal{H}_\infty^\downarrow$: Hardy spaces of complex valued functions which are bounded on the unit circle with analytic continuation in $|z| < 1$ and $|z| > 1$, respectively.
- $\Pi_+: \mathcal{H}_2^\downarrow \oplus \mathcal{H}_2^\downarrow \rightarrow \mathcal{H}_2^\downarrow$ is the canonical projection $\Pi_+ w := w_+$ where $w$ is uniquely decomposed as $w = w_+ + w_-$ with $w_+ \in \mathcal{H}_2^\downarrow$ and $w_- \in \mathcal{H}_2^\downarrow$.

3 A class of $\ell_2$ systems

In this paper we concentrate on the class of discrete time systems which consist of signals defined on the time set $T = \mathbb{Z}_+$. Following the behavioral framework, a system is defined by a family of signals $w: T \rightarrow W$ where, for the purpose of this paper, the signal space $W$ is a $q$ dimensional real vector space $W = \mathbb{R}^q$ with $q$ some fixed non-negative integer.

**Definition 3.1** An $\ell_2$ system is a triple $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, \mathcal{B})$ whose behavior $\mathcal{B}$ is a closed subset of $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$.

For signals $w$ defined on the time set $\mathbb{Z}_+$ we distinguish two types of shift operators. The left-shift operator\(^1\) is a map $\sigma_L: \ell(\mathbb{Z}_+, \mathbb{R}^q) \rightarrow \ell(\mathbb{Z}_+, \mathbb{R}^q)$ defined, given $w \in \ell(\mathbb{Z}_+, \mathbb{R}^q)$, by

$$ (\sigma_Lw)(t) := w(t + 1) \quad (3.1) $$

The right-shift operator is a map $\sigma_R: \ell(\mathbb{Z}_+, \mathbb{R}^q) \rightarrow \ell(\mathbb{Z}_+, \mathbb{R}^q)$ defined by

$$ (\sigma_Rw)(t) = \begin{cases} 0 & \text{for } t = 0 \\ w(t - 1) & \text{for } t > 0. \end{cases} \quad (3.2) $$

Note that the composition $\sigma_L \sigma_R$ is the identity map on $\ell(\mathbb{Z}_+, W)$.

**Definition 3.2** A system $(\mathbb{Z}_+, \mathbb{R}^q, \mathcal{B})$ is said to be left-shift invariant if $\sigma_L \mathcal{B} \subseteq \mathcal{B}$ and right-shift invariant if $\sigma_R \mathcal{B} \subseteq \mathcal{B}$. It is said to be complete if a trajectory $w$ belongs to $\mathcal{B}$ whenever its restrictions $w|_{[t_0, t]}$ belong to $\mathcal{B}|_{[t_0, t]}$ for all (finite) intervals $[t_0, t] \subset \mathbb{Z}_+$.

**Remark 3.3** We emphasize that for the analysis of dynamical systems with time set $T = \mathbb{Z}_+$ left-shift invariance is a more appealing property than right-shift invariance. Since any trajectory in a right-shift invariant behavior can be preceded by an arbitrary number of zeros it is intuitively clear that $\sigma_R$-invariant subspaces of $\ell(\mathbb{Z}_+, W)$ correspond to behaviors with "zero initial conditions". If one investigates the graph of an input-output operator as for instance done by Georgiou-Smith [1] then this graph is $\sigma_R$-invariant and image representations of the graph are immediately obtained using the Beurling-Lax theorem (see below). However, in a theory for right-shift invariant subspaces of $\ell_2(\mathbb{Z}_+, \mathbb{R}^q)$ autonomous behaviors (typically obtained by feedback interconnections) are necessarily trivial (See theorem 6.4 below). This has the consequence that for an important class of feedback interconnections of right-shift invariant dynamical systems the closed-loop systems are not rich enough to further investigate performance or stability issues. On the other hand, $\sigma_L$-invariant system behaviors allow, by necessity, for initial conditions and therefore naturally model autonomous systems, transient phenomena in dynamical systems and feedback configurations.

\(^1\)The terminology used here refers to left shifting of the signal with respect to the time axis.
Based on definition 3.2 we introduce the following classes of dynamical systems.

Definition 3.4 For $q > 0$ the model set $\mathbb{B}_q$ consists of all behaviors $B$ of the class of linear and left-shift invariant $l_2$-systems $(\mathbb{Z}_+, \mathbb{R}^q, B)$. The model set $\mathbb{B}_{\text{comp}}$ consists of all behaviors $B$ of the class of linear, left-shift invariant and complete systems $(\mathbb{Z}_+, \mathbb{R}^q, B)$. Further, $\mathbb{B}_q^{\text{comp}} := \mathbb{B}_q \cap \mathbb{B}_{\text{comp}}$.

Any $B \in \mathbb{B}_{\text{comp}}$ induces a behavior $B_2 \in \mathbb{B}_2$ by the restriction

$$B_2 = B \cap l_2^+$$

However, this restriction is in general not injective. In other words, there are $B', B'' \in \mathbb{B}_{\text{comp}}$ with $B' \neq B''$ for which

$$B' \cap l_2^+ = B'' \cap l_2^+.$$

One of the main advantages to consider $l_2$-systems is that we can interchangeably consider their behavior in the "time domain" and in the "frequency domain". Specifically, define for all $B \in \mathbb{B}_2$:

$$\hat{B} = \{ \hat{w} \in H_2^+ \mid w \in B \}$$

where $\hat{w}$ is the image of $w \in l_2^+$ under the $z$-transform

$$\hat{w}(z) := \sum_{t=0}^{\infty} w(t)z^{-t}.$$

4 Left- and right-shift invariant systems

If $B$ is left-shift invariant then $\hat{B}$ is left-shift invariant in the sense that $\hat{\sigma}_L B \subset \hat{B}$ where $\hat{\sigma}_L : H_2^+ \rightarrow H_2^+$ is defined by $(\hat{\sigma}_L w)(z) = \Pi_+ zw(z)$ for all $w \in H_2^+$. Similarly, $B \subset l_2^+$ is right-shift invariant if and only if $\hat{B}$ is $\hat{\sigma}_R$-invariant in the sense that $\hat{\sigma}_R \hat{B} \subset \hat{B}$, where the right-shift operator $\hat{\sigma}_R : H_2^+ \rightarrow H_2^+$ is defined by $(\hat{\sigma}_R w)(z) := z^{-1}w(z)$ with $z \in \mathbb{C}$.

Theorem 4.1 (Shift-invariant subspaces) Let $\Sigma = (\mathbb{Z}_+, W, B)$ be a linear $l_2$ system. Then

1. $\Sigma$ is left-shift invariant and right-shift invariant if and only if $\Sigma$ is memoryless i.e. for all $w', w'' \in B$ and $t_0 > 0$ the concatenation

$$w := w' \wedge_{t_0} w'' = \begin{cases} w(t) = w'(t) & \text{for } t \leq t_0 \\ w(t) = w''(t) & \text{for } t > t_0 \end{cases}$$

belongs to $B$.

2. $\Sigma$ is autonomous and right-shift invariant if and only if $B = \{0\}$.

3. $\sigma_L B \subseteq B \iff \sigma_R B^\perp \subseteq B^\perp$

4. $\sigma_L B \subseteq B \iff \sigma_R B^\perp \subseteq B^\perp$

Proof. 1. (only if) Let $w', w'' \in B$ and let $n \in \mathbb{Z}_+$. Since $B$ is $\sigma_R$ and $\sigma_L$ invariant, both $\sigma_R^n \sigma_L w'$ and $\sigma_R^n \sigma_L w''$ belong to $B$. Linearity of $B$ implies that then also $\sigma_R^n \sigma_L w' + (1 - \sigma_R^n \sigma_L) w' \in B$ which is precisely the concatenation of $w'$ and $w''$ at time $n$.

(if) Trivial.

2. Let $B$ be autonomous and right-shift invariant and suppose that $w \in B$. Let $t_0 > 0$ and consider $w_{t_0} = \sigma_R^n w$. Then $w_{t_0} \in B$ and $w_{t_0}(t) = 0$ for $t \leq t_0$. Since $w$ is uniquely determined by
its restriction \( w(t_{0}, t') \) for some \( t' > 0 \), also \( w_{t_{0}} \) is uniquely determined by its restriction \( w_{t_{0}}|_{[0, t']} \). However, the latter vanishes for \( t_{0} > t' \) from which it follows that both \( w_{t_{0}}(t) = 0 \) for all \( t \in \mathbb{Z}_{+} \). The reverse implication is trivial.

3. Let \( \sigma_{v} \subseteq \mathcal{B} \) and \( w \in \mathcal{B}^{\bot} \). Then \( (w, v) = 0 \) for all \( v \in \mathcal{B} \). By shift invariance of \( \mathcal{B} \) also \( \langle \sigma_{v} w, v \rangle = 0 \) for all \( v \in \mathcal{B} \). Thus, \( \sigma_{v} w \in \mathcal{B}^{\bot} \) which implies that \( \sigma_{v} \mathcal{B}^{\bot} \subseteq \mathcal{B}^{\bot} \). The reverse implication follows from a similar argument.

4. The proof of 4 is a straightforward modification of the proof of statement 3.

\[ \square \]

## 5 Kernel representations

Suppose that \( \Theta \in \mathcal{H}_{\infty}^{-} \) is a matrix of dimension \( g \times q \). Associate with \( \Theta \) the mapping \( \Theta : \mathcal{H}_{\infty}^{+} \to \mathcal{L}_{2} \) defined by the multiplication \( (\Theta w)(z) := \Theta(z)w(z), \ z \in \mathbb{C} \). Associate with \( \Theta \) a subset of \( \mathcal{H}_{\infty}^{+} \) defined as

\[ \mathcal{B}_{\text{ker}}(\Theta) := \{ w \in \mathcal{H}_{\infty}^{+} \mid \Pi_{+}(\Theta w)(z) = 0 \text{ for all } z \in \mathbb{C} \} \]  \hspace{1cm} (5.1)

\[ = \text{Ker} \Pi_{+} \Theta. \]  \hspace{1cm} (5.2)

Clearly, \( \mathcal{B}_{\text{ker}}(\Theta) \) is a linear and closed subset of \( \mathcal{H}_{\infty}^{+} \).

**Definition 5.1** A subset \( \mathcal{B} \subseteq \mathcal{H}_{\infty}^{+} \) is said to have a **kernel representation** \( \Theta \) if there exist \( \Theta \in \mathcal{H}_{\infty}^{-} \) such that \( \mathcal{B} = \mathcal{B}_{\text{ker}}(\Theta) \).

Using the isometry between \( \mathcal{H}_{\infty}^{+} \) and \( \ell_{2}^{+} \), the set \( \mathcal{B}_{\text{ker}}(\Theta) \) has an equivalent interpretation in the time domain. Let \( \Theta_{k} \in \mathbb{R}^{q \times q} \), \( k \in \mathbb{Z} \), be the constant real matrices which uniquely define the Laurent series expansion

\[ \Theta(z) = \sum_{k = -\infty}^{\infty} \Theta_{k}z^{-k} \]  \hspace{1cm} (5.3)

where \( z \in \mathbb{C} \). Note that, since \( \Theta \in \mathcal{H}_{\infty}^{-} \), the coefficients \( \Theta_{k} \) are non-zero for \( k \leq 0 \) so that only non-negative powers of \( z \) appear in (5.3). Introduce the map \( \Theta(\sigma_{v}) : \ell_{2}(\mathbb{Z}_{+}, \mathbb{R}^{q}) \to \ell_{2}(\mathbb{Z}_{+}, \mathbb{R}^{q}) \) defined by \( \Theta(\sigma_{v}) : w \mapsto v \) where \( v \in \ell_{2}^{+} \) is the convolution of the sequences \( \{ \Theta_{k} \}_{k \in \mathbb{Z}} \) and \( \{ w(k) \}_{k \in \mathbb{Z}_{+}} \), i.e.,

\[ v(t) := \sum_{k \in \mathbb{Z}_{+}} \Theta_{t-k}w(k), \ t \in \mathbb{Z}_{+}. \]  

Clearly, for all \( \Theta \in \mathcal{H}_{\infty}^{-} \),

\[ \mathcal{B}_{\text{ker}}(\Theta) := \{ w \in \ell_{2}^{+} \mid \Theta_{\sigma_{v}}w = 0 \} \]  \hspace{1cm} (5.4)  

\[ = \text{Ker} \Theta(\sigma_{v}) \]  \hspace{1cm} (5.5)

belongs to \( \mathcal{B}_{2} \).

**Definition 5.2** A subset \( \mathcal{B} \subseteq \ell_{2}^{+} \) is said to have a **kernel representation** if there exists \( \Theta \in \mathcal{H}_{\infty}^{-} \) such that \( \mathcal{B} = \mathcal{B}_{\text{ker}}(\Theta) \).

The notation \( \mathcal{B}_{\text{ker}}(\Theta) \) in (5.1) and \( \mathcal{B}_{\text{ker}}(\Theta) \) in (5.4) are indeed consistent as is shown in the following theorem.

**Theorem 5.3** For all \( \Theta \in \mathcal{H}_{\infty}^{-} \) there holds \( \mathcal{B}_{\text{ker}}(\Theta) = \mathcal{Z}(\mathcal{B}_{\text{ker}}(\Theta)) \) where \( \mathcal{Z} : \ell_{2}^{+} \to \mathcal{H}_{\infty}^{+} \) denotes the \( z \)-transform \( \mathcal{Z}w := \hat{w} \).
5 KERNEL REPRESENTATIONS

Proof. Let \( \hat{w} \in \mathcal{H}_2^+ \) and \( \Theta \in \mathcal{H}_0^- \). Then there holds

\[
\Theta(z)\hat{w}(z) = \sum_{k=-\infty}^{\infty} \Theta_z z^{-k} \sum_{j=0}^{\infty} w(j) z^{-j} = \sum_{t=-\infty}^{\infty} v(t) z^{-t}
\]

where for all \( t \in \mathbb{Z} \), we have \( v(t) = \sum_{k=0}^{\infty} \Theta_{1-t} w(k) \). Therefore \( \hat{w} \in \mathcal{B}_{\text{fin}}(\Theta) \) if and only if \( v(t) = 0 \) for all \( t < 0 \). On the other hand

\[
v(t) = \sum_{k=0}^{\infty} \Theta_{1-t} w(k) = \sum_{j=0}^{\infty} \Theta_{1+t} w(t + j) = (\Theta(\sigma_1) w)(t)
\]

since \( \Theta_{1-t} = 0 \) for \( t < k \) because \( \Theta \in \mathcal{H}_0^- \). Hence \( \hat{w} \in \mathcal{B}_{\text{fin}}(\Theta) \) if and only if \( (\Theta(\sigma_1) w)(t) = 0 \) for all \( t < 0 \).

We conclude this section with a characterization of subset inclusions and non-uniqueness of kernel representations.

Theorem 5.4 For \( i = 1,2 \), let \( \Theta_i \in \mathcal{R}\mathcal{H}_0^- \) be a full rank kernel representation of \( \mathcal{B}_i = \mathcal{B}_{\text{fin}}(\Theta_i) \). Then

1. \( B_1 \subseteq B_2 \) if and only if there exists \( U \in \mathcal{R}\mathcal{H}_0^- \) such that \( \Theta_2 = U \Theta_1 \).

2. \( B_1 = B_2 \) if and only if there exist a unit\(^2 \) \( U \in \mathcal{R}\mathcal{H}_0^- \) such that \( \Theta_2 = U \Theta_1 \).

Proof. 1. (if) Suppose that \( \Theta_2 = U \Theta_1 \) for some \( U \in \mathcal{H}_0^- \). Let \( w \in B_1 \) and observe that \( \hat{w} := \Theta_1 \hat{w} \in \mathcal{H}_2^- \). Then also \( U \hat{w} = U \Theta_1 \hat{w} = \Theta_2 \hat{w} \in \mathcal{H}_2^- \) which implies that \( \Pi_\perp \Theta_2 \hat{w} = 0 \). Hence \( w \in B_2 \).\( \square \)

(only if) We need to show that \( B_1 \subseteq B_2 \Rightarrow \Theta_2 = U \Theta_1 \) for some \( U \in \mathcal{H}_0^- \). To see this, first observe that

\[
\mathcal{B}_{\text{fin}}(\Theta_1) = \{ \hat{w} \in \mathcal{H}_2^+ \mid (\Theta_1 \hat{w}, \hat{v}) = 0 \text{ for all } \hat{v} \in \mathcal{H}_2^+ \} = \\
\{ \hat{w} \in \mathcal{H}_2^+ \mid (\hat{w}, \Theta_1^* \hat{v}) = 0 \text{ for all } \hat{v} \in \mathcal{H}_2^+ \} = \\
(\text{Im } \Theta_1^*)^\perp
\]

(5.6)

where \( \Theta_1^* : \mathcal{H}_2^+ \to \mathcal{H}_2^+ \) is the dual operator \( \Theta_1^*(z) = \Theta_1^* z^{-1} \in \mathcal{H}_2^+ \). Thus, \( B_1 \subseteq B_2 \) implies that \( B_2^\perp \subseteq B_1^\perp \) which in turn implies that \( \text{Im } \Theta_2^* \subseteq \text{Im } \Theta_1^* \).

We can extend \( \Theta_1^* \) from a multiplicative operator from \( \mathcal{H}_2^+ \) to \( \mathcal{H}_2^+ \) to a multiplicative operator from \( \mathbb{R}(z) \) to \( \mathbb{R}(z)^3 \). We claim that also for this extended operator there holds \( \text{Im } \Theta_1^* \subseteq \text{Im } \Theta_2^* \). Indeed, suppose contrary to what we want to prove that there exists \( v \in \mathbb{R}(z) \) such that \( \Theta_2^* v \notin \text{Im } \Theta_1^* \).

Let \( u \in \mathbb{R}(z) \) be invertible and such that \( vu \in \mathcal{H}_2^+ \). By assumption, there exists \( h \in \mathcal{H}_2^+ \) such that \( \Theta_2^* vu = \Theta_2^* h \). But then \( \Theta_2^* v = \Theta_2^* hu^{-1} \) with \( vu^{-1} \in \mathbb{R}(z) \) which yields a contradiction.

\( \Theta_1^* \) has full column rank as a rational matrix. Hence, there exists a rational matrix \( \Theta_2^* \) such that \( \Theta_1^* \Theta_2^* = I \). Then it is easy to check that the inclusion of the images over the field \( \mathbb{R}(z) \) implies that \( \Theta_2^* = \Theta_1^* U^* \) where \( U^* := \Theta_2^* \Theta_2^* \). It remains to show that \( U \in \mathcal{H}_0^- \).

Since \( \text{Im } \Theta_2^* \subseteq \text{Im } \Theta_1^* \) with \( \Theta_1^* \) viewed as an operator from \( \mathcal{H}_2^+ \) to \( \mathcal{H}_2^+ \), for all \( \hat{v}_2 \in \mathcal{H}_2^+ \) there exists \( \hat{v}_1 \in \mathcal{H}_2^+ \) such that \( \Theta_2 \hat{v}_2 = \Theta_1 \hat{v}_1 \). Note that \( \Theta_2\left(U^* \hat{v}_2 - \hat{v}_1\right) = 0 \). Since \( \Theta_1^* \) has full column rank as a rational matrix, this implies \( U^* \hat{v}_2 = \hat{v}_1 \). Since \( \hat{v}_2 \) is an arbitrary element of \( \mathcal{H}_2^+ \), we find that \( U^* \) maps \( \mathcal{H}_2^+ \) to \( \mathcal{H}_2^+ \) and hence \( U \in \mathcal{H}_0^- \).

\(^2\)A unit \( U \in \mathcal{R}\mathcal{H}_0^- \) is a square matrix with entries in \( \mathcal{H}_0^- \) whose inverse \( U^{-1} \) exists and also belongs to \( \mathcal{H}_0^- \).

\(^3\)\( \mathbb{R}(z) \) is the field of all rational functions in the indeterminate \( z \)
6 Image representations

In this section we will introduce image representations for the model classes defined in section 3.

The Beurling-Lax theorem is known to provide the existency of an image representation for shift invariant subspaces $B$ of a Hilbert space $H$.

**Theorem 6.1 (Beurling-Lax)** Let $\sigma$ be an isometry on a Hilbert space $H$ such that
\[ ||\sigma^* w|| \to 0 \text{ for all } w \in H \text{ as } k \to \infty \] (6.1)

Then for any closed, $\sigma$-invariant linear subspace $B$ of $H$ there exists a bounded isometric linear operator $\Psi$ from some Hilbert space $W$ to $H$ such that $B = \text{Im} \Psi$.

**Remark 6.2** The left-shift $\sigma_l$ does not satisfy the conditions of the Beurling-Lax theorem and hence Beurling-Lax cannot be used to prove the existence of image representations of behaviors $B \in B_2$. On the other hand, if $B \subseteq H^+_2$ is $\sigma_R$ invariant then theorem 6.1 can be applied with $\sigma = \sigma_R$ to obtain the existence of $\Psi \in H^+_\infty$ such that
\[ B = B_{im}(\Psi) := \text{Im} \Psi. \] (6.2)

Here, $\Psi : H^+_2 \to H^+_2$ is the multiplicative operator $(\Psi v)(z) := \Psi(z)v(z)$. Moreover, $\Psi$ can be chosen to be an inner (or norm-preserving) map from $H^+_2$ to $B$.

In order to represent the model classes of left-shift invariant behaviors of section 3 we introduce a different type of image representation. Let $\Psi_a, \Psi_c \in H^+_\infty$ be elements of $H^+_\infty$ and consider the set
\[ B_{im}(\Psi_a, \Psi_c) = \left\{ \Pi_+ (\Psi_a \Psi_c) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} | s_1 \in L_2 \text{ and } s_2 \in H^-_2 \right\} \] (6.3)

**Definition 6.3** A subset $B \subseteq H^+_2$ is said to have an image representation if there exist $\Psi_a, \Psi_c \in H^+_\infty$ such that $B = B_{im}(\Psi_a, \Psi_c)$.

The matrices $\Psi_a$ and $\Psi_c$ constitute a decomposition of $B$ in an autonomous and a controllable part. Precisely, $B = B_c + B_a$ where $B_c = \Pi_+ \Psi_c L_2$ is the controllable part of $B$ and $B_a = \Pi_+ \Psi_a H^-_2$ is a (non-unique) autonomous part of $B$. Non-uniqueness of this type of image representations is characterized as follows.

**Theorem 6.4** For $i = 1, 2$, let $\Psi_a^i, \Psi_c^i \in \mathcal{J} H^+_\infty$ be image representations of $B_i = B_{im}(\Psi_a^i, \Psi_c^i)$. Then the following statements are equivalent

1. $B_{im}(\Psi_a^1, \Psi_c^1) = B_{im}(\Psi_a^2, \Psi_c^2)$

---

* $\Psi$ is an inner function if $\Psi \in H^\infty$ and $\Psi^* \Psi = I$ where $\Psi^*(z) = \Psi^T(z^{-1})$. 
2. There exist constant real matrices $U^1_a, U^2_a$, and elements $X^1, X^2, U^1_a$ and $U^2_a$ in $\mathcal{H}_\infty$ such that
\[
\begin{pmatrix}
\Psi^2_a & \Psi^2_e \\
\Psi^1_a & \Psi^1_e \\
\end{pmatrix}
\begin{pmatrix}
U^1_a & 0 \\
X^1 & U^2_a \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
\Psi^2_a & \Psi^2_e \\
\Psi^1_a & \Psi^1_e \\
\end{pmatrix}
\begin{pmatrix}
U^2_a & 0 \\
X^2 & U^2_a \\
\end{pmatrix}
\]

7 Main results

One of the main results of this paper is to prove that behaviors in $\mathbb{B}_2$ allow for both a kernel and an image representation.

**Theorem 7.1** The following statements are equivalent.

1. $B \in \mathbb{B}_2$

2. $B$ admits a kernel representation $B = B_{in}(\Theta)$.

3. $\tilde{B}$ admits an image representation $\tilde{B} = \tilde{B}_{im}(\Psi_a, \Psi_e)$.

**Proof.** (1 $\Leftrightarrow$ 2) We know from the Beurling-Lax theorem that any $\sigma^*_e$-invariant subspace (i.e. $\sigma_n$ invariant subspace) has an image representation. It is easy to check that $B$ is $\sigma_e$-invariant implies that $B^\perp$ is $\sigma^*_e$-invariant. Therefore, by the Beurling-Lax theorem, there exists $\Psi \in \mathcal{H}_\infty^+$ such that $B^\perp = \text{Im}\Psi$. The proof of the theorem is then completed by the following sequences of equalities

\[
B = [\text{Im}\Psi]^\perp = \left\{ \begin{array}{l}
\nu \in \mathcal{H}_\infty^+ \mid \langle \nu, \tilde{\nu} \rangle = 0 \text{ for all } \tilde{\nu} \in \text{Im}\Psi \\
\nu \in \mathcal{H}_\infty^+ \mid \langle \Psi \tilde{\nu}, \nu \rangle = 0 \text{ for all } \tilde{\nu} \in \mathcal{H}_\infty^+ \\
\nu \in \mathcal{H}_\infty^+ \mid \langle \tilde{\nu}, \Psi^* \nu \rangle = 0 \text{ for all } \tilde{\nu} \in \mathcal{H}_\infty^+ \\
\nu \in \mathcal{H}_\infty^+ \mid \langle \tilde{\nu}, \Pi_+ \Psi^* \nu \rangle = 0 \text{ for all } \tilde{\nu} \in \mathcal{H}_\infty^+ \\
\nu \in \mathcal{H}_\infty^+ \mid \Pi_+ \Psi^* \nu = 0 \\
\nu \in \mathcal{H}_\infty^+ \mid \text{Ker}(\Pi_+ \Psi^*)
\end{array} \right.
\]

Setting $\Theta = \Psi^*$ yields the result.

(2 $\Leftrightarrow$ 3) From the previous we know that $B = B_{in}(\Theta)$ for some $\Theta \in \mathcal{H}_\infty^+$. Moreover, by theorem 5.4 we can assume without loss of generality that $\Theta$ is co-inner. Construct $\Theta_e$ such that:

\[
\begin{pmatrix}
\Theta \\
\Theta_e
\end{pmatrix}
\]

is a square and inner matrix in $\mathcal{H}_\infty^+$. (if $\Theta$ is square then $\Theta_e$ is a trivial 0-dimensional matrix). We first prove that:

\[
\left\{ \Pi_+ \begin{pmatrix}
\Theta_e^{*} & \Theta^* \\
\end{pmatrix}
\begin{pmatrix}
s1 \\
s2
\end{pmatrix}
\mid s_1 \in \mathcal{L}_2^m, s_2 \in (\mathcal{H}_2^+)^{(s-m)} \right\} \subset \tilde{B}
\]

Let $w$ be an element of the set on the left hand side, i.e. there exist $s_1 \in \mathcal{L}_2^m$ and $s_2 \in (\mathcal{H}_2^+)^{(s-m)}$ such that

\[
w = \Pi_+ \Theta_e^{*} s_1 + \Pi_+ \Theta^* s_2
\]

Since $B$ is closed and $B^\perp = \text{Im}\Theta^*$ we find that $w \in B$ if and only if $\langle w, \Theta^* v \rangle = 0$ for all $v \in \mathcal{H}_2^+$. This follows easily from the following:

\[
\langle w, \Theta^* v \rangle = \langle \Theta_e^{*} s_1 + \Theta^* s_2, \Theta^* v \rangle = \langle \Theta^* s_2, \Theta^* v \rangle = \langle s_2, v \rangle = 0
\]
where we used that $\Theta_{\iota}^{*} = 0$ and $\Theta^{*} = I$. To prove the reverse inclusion it is sufficient to show that:

$$
\left\{ \Pi_{+} \begin{pmatrix} \Theta_{\iota}^{*} & \Theta^{*} \end{pmatrix} \begin{pmatrix} s_{1} \\ s_{2} \end{pmatrix} \bigg| s_{1} \in L_{2}^{m}, \ s_{2} \in (H_{2}^{+})^{(m-m)} \right\} + B^{\perp} = \mathcal{H}_{2}^{+}
$$

The latter is straightforward to check using our image representation of $B^{\perp}$ and the fact that $\Pi_{+} (\Theta_{\iota}^{*} \Theta^{*})$ is surjective as a map from $L_{2}$ to $\mathcal{H}_{2}^{+}$.

The following result characterizes the model set $\mathcal{B}_{2}^{*\text{comp}}$ as those behaviors that allow rational kernel and image representations.

**Theorem 7.2** The following statements are equivalent.

1. $\mathcal{B} \in \mathcal{B}_{2}^{*\text{comp}}$

2. $\mathcal{B}$ admits a kernel representation $\mathcal{B} = \mathcal{B}_{\text{int}}(\Theta)$ where $\Theta \in \mathcal{H}_{\infty}^{*}$ is rational

3. $\mathcal{B}$ admits an image representation $\mathcal{B} = \mathcal{B}_{\text{int}}(\Psi_{a}, \Psi_{c})$ where $[\Psi_{a}, \Psi_{c}] \in \mathcal{H}_{\infty}^{+}$ is rational.

8 System interconnections and well-posedness

8.1 Well-posed feedback interconnections

Consider two dynamical systems $\Sigma_{i} = (\mathbb{Z}_{+}, W_{i} \times W_{\text{int}}, B_{i})$, $i = 1, 2$, which have a common non-empty subset $W_{\text{int}}$, the interconnection space, in their respective signal spaces.

**Definition 8.1** The interconnection of $\Sigma_{1} = (\mathbb{Z}_{+}, W_{1} \times W_{\text{int}}, B_{1})$ and $\Sigma_{2} = (\mathbb{Z}_{+}, W_{2} \times W_{\text{int}}, B_{2})$ is the system

$$
\Sigma_{1} \sqcap \Sigma_{2} := (\mathbb{Z}_{+}, W_{1} \times W_{2} \times W_{\text{int}}, B_{1} \sqcap B_{2})
$$

where

$$
B_{1} \sqcap B_{2} := \{(w_{1}, w_{2}, w_{\text{int}}) \mid (w_{i}, w_{\text{int}}) \in B_{i}, \ i = 1, 2\}.
$$

$w_{\text{int}}$ are the interconnection variables. If both $W_{1}$ and $W_{2}$ are void then $\Sigma_{1} \sqcap \Sigma_{2}$ is called a full interconnection of $\Sigma_{1}$ and $\Sigma_{2}$.

Note that in a full interconnection $B_{1} \sqcap B_{2} = B_{1} \cap B_{2}$.

A feedback interconnection will be defined as an interconnection in which additional axioms are satisfied. We introduce a few more properties of dynamical systems.

**Definition 8.2** Consider a time-invariant system $\Sigma = (\mathbb{Z}_{+}, W, B)$ defined on a product set $W = W_{1} \times W_{2}$. The variable $w_{2}$ processes $w_{1}$ in $\Sigma$ if there exists $t_{0} \geq 0$ such that

$$
\{(w_{1}, w'_{2}), (w_{1}, w_{2}) \in B, \ w'_{2}(t) = w_{2}(t) \text{ for } t \leq t_{0} \} \implies \{w'_{2} = w_{2}\}
$$

**Definition 8.3** Let $\Sigma_{i} = (\mathbb{Z}_{+}, W_{i} \times W_{\text{int}}, B_{i})$, $i = 1, 2$, be two left shift-invariant dynamical systems. $\Sigma_{1}$ and $\Sigma_{2}$ are said to be instantaneous interconnectable if for all $t > 0$, $(w_{1}, w'_{\text{int}}) \in B_{1}[0,t]$ and $(w_{2}, w''_{\text{int}}) \in B_{2}[0,t]$ there exists $w_{1e}, w_{2e}, w'_{\text{int},e}$ and $w''_{\text{int},e}$ with

$$
w_{1e}[0,t] = w_{1}, \ w_{2e}[0,t] = w_{2}, \ w'_{\text{int},e}[0,t] = w'_{\text{int}} \quad \text{and} \quad w''_{\text{int},e}[0,t] = w''_{\text{int}}
$$

such that $(w_{1e}, w'_{\text{int},e}) \in B_{1}$ and $(w_{2e}, w''_{\text{int},e}) \in B_{2}$ with

$$
w'_{\text{int},e}[t,\infty) = w''_{\text{int},e}[t,\infty).
$$
This condition is quite natural since it states that any past for $B_1$ together with any past for $B_2$ can, after interconnection at time $t$, yield a common future. Well-posed feedback interconnections are now defined as follows.

**Definition 8.4** Let $\Sigma_i = (\mathbb{Z}_+, W_i \times W_{i\text{ext}}, B_i)$, $i = 1, 2$, be given. The interconnection $\Sigma = \Sigma_1 \cap \Sigma_2$ is said to be well-posed if $w_{\text{ext}}$ processes $(w_1, w_2)$ in $\Sigma$. The interconnection $\Sigma$ is said to be a regular feedback interconnection if it is well posed and if $\Sigma_1$ and $\Sigma_2$ are instantaneous interconnectable.

We proceed this section with the definition of stabilizability. Intuitively, in a stabilizable system any trajectory can at any time be concatenated with a future system trajectory that is asymptotically converging to zero. We formalize this as follows.

**Definition 8.5** A dynamical system $\Sigma = (\mathbb{Z}_+, W, B)$ is said to be stabilizable if for all $w' \in B$ and $t_0 > 0$ there exists $w'' \in B$ with $\lim_{t \to \infty} w''(t) = 0$ such that $w'\|_{[0,t_0]} = w''\|_{[0,t_0]}$.

Just like the notion of controllability, stabilizability is therefore defined as a property of the external behavior of a dynamical system. The notion of stabilizability of dynamical systems has an elegant characterization.

**Theorem 8.6** Let $\Sigma = (\mathbb{Z}_+, W, B)$ be a system with behavior $B \in \mathbb{B}^{\text{ext}}$. Then this system is stabilizable if and only if

$$
B = \overline{B \cap \ell_2^+}
$$

where the closure is taken in the topology of pointwise convergence.

**Proof.** We first proof that the equality (8.4) implies that the system is stabilizable. Choose $w \in B$. Equality (8.4) implies that for all $t_0$ we have

$$
B\|_{[0,t_0]} = \overline{(B \cap \ell_2^+)\|_{[0,t_0]}}
$$

On the other hand these are finite dimensional spaces and hence always closed. Since $w\|_{[0,t_0]} \in B\|_{[0,t_0]}$ we find $w\|_{[0,t_0]} \in (B \cap \ell_2^+)\|_{[0,t_0]}$. Hence there exists $v \in B \cap \ell_2^+$ such that $w\|_{[0,t_0]} = v\|_{[0,t_0]}$. Since this is possible for all $w \in B$ this implies by definition that $B$ is stabilizable.

Again choose an arbitrary element $w \in B$. We have to show that $B$ stabilizable guarantees that $w \in B \cap \ell_2^+$. $B$ stabilizable implies by definition that (8.5) is satisfied for all $t_0$. Hence for all $t$ there exists $v_t$ such that $w\|_{[0,t]} = v\|_{[0,t]}$ and $v_t \in B \cap \ell_2^+$. But then it is straightforward to check that $v_t \to w$ as $t \to \infty$ in the topology of pointwise convergence. Therefore $w \in B \cap \ell_2^+$.

**Remark 8.7** For the class of $\ell_2$ systems with behaviors $B_i \in \mathbb{B}_2$, $i = 1, 2$, we emphasize that the concept of instantaneous interconnectability guarantees at any time $t_0 > 0$ the existence of a common future signal $w_{\text{ext}}(t)$ with $t \geq t_0$ which is compatible with arbitrary past trajectories $(w_1, w_{\text{ext}}) \in B_i$. In particular, this implies that instantaneous interconnectable stabilizable systems result in a stabilizable interconnection. More precisely, if $B_1$ and $B_2$ are stabilizable, i.e.,

$$
B_1 = B_1 \cap \ell_2^+, \quad B_2 = B_2 \cap \ell_2^+
$$

and $B_1$ and $B_2$ are instantaneous interconnectable then we have:

$$
B_1 \cap B_2 = B_1 \cap B_2 \cap \ell_2^+.
$$

Clearly the latter implies that the full interconnection of $B_1 \cap B_2$ is stabilizable.
9 Stability of $\ell_2$-systems

In [1], Georgiou and Smith study the stability of closed loop systems in terms of the graphs of system and controller. Their definitions involve closed loop systems which are right-shift invariant and autonomous. Since the graph associated with an input-output operator defined on $\ell_2^+$ is necessarily defined for a fixed set of (zero) initial conditions these graphs correspond to right-shift invariant subspaces. This has the disadvantage that their representation is not rich enough to characterize closed loop autonomous behaviors. In this section we relate our definitions of stability to the conditions presented by Georgiou and Smith.

The graph of a linear system is a right-shift invariant subspace. Therefore by theorem 6.1 there exists $W \in \mathcal{H}_\infty^+$ such that the graph is represented as the image of $\Psi$ as a map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$. Given a system with transfer matrix $G$ we can construct the graph associated to the system from a right-coprime factorization of $G$. If $G = NM^{-1}$ then the graph of $G$ is equal to $\mathcal{B}_m(\Psi) = \text{Im} \, \Psi$ as defined in (6.2) where

$$\Psi = \begin{pmatrix} N \\ M \end{pmatrix}$$

Note that since $N$ and $M$ are coprime the rational matrix $\Psi$ does not have any zeros. We emphasize that $\mathcal{B}_m(\Psi)$ is not a left-shift invariant subspace of $\mathcal{H}_2^+$. The interconnection of a system $\Sigma_s$ with transfer matrix $G_s$ and a controller $\Sigma_c$ with transfer matrix $G_c$ is stable if the following transfer matrix

$$\begin{pmatrix} (I - G_sG_c)^{-1} & (I - G_sG_c)^{-1}G_s \\ G_s(I - G_sG_c)^{-1} & G_s(I - G_sG_c)^{-1}G_s \end{pmatrix}$$

belongs to $\mathcal{H}_\infty^+$. We have the following theorem from [1] relating the above stability condition to the graphs associated with the system and the controller.

**Theorem 9.1 (Georgiou-Smith)** Let a system $\Sigma_s$ with graph $\mathcal{B}_s = \text{Im} \, \Psi_s$ and a controller $\Sigma_c$ with graph $\mathcal{B}_c = \text{Im} \, \Psi_c$ be given. Assume $\Psi_c$ and $\Psi_s$ have no zeros. Then the following statements are equivalent

1. the full interconnection $\Sigma_s \cap \Sigma_c$ is stable.
2. the matrix $(\Psi_c, \Psi_s)$ is stable and has a stable inverse.
3. $\mathcal{B}_s + \mathcal{B}_c = \mathcal{H}_2^+$ and $\mathcal{B}_s \cap \mathcal{B}_c = \{0\}$.

We emphasize that here the graph of a system $\mathcal{B}_s$ is a right-shift invariant subspace of $\mathcal{H}_2^+$ and corresponds to the set of all input-output pairs in $\mathcal{H}_2^+ \times \mathcal{H}_2^+$ which are compatible with the system where zero initial conditions are assumed.

Let $\mathcal{B}_s$ denote the smallest left-shift invariant subspace containing $\mathcal{B}_s$. Similarly, denote by $\mathcal{B}_c$ the smallest left-shift invariant subspace containing $\mathcal{B}_c$. It is easy to check that:

$$\mathcal{B}_s = \{ \Pi_+ \Psi_s \mid w \in L_2 \}.$$

$$\mathcal{B}_c = \{ \Pi_+ \Psi_c \mid w \in L_2 \}.$$

In particular this yields that these sets correspond to the behaviors of controllable systems. We have the following theorem relating the stability condition in terms of the right-shift invariant subspace $\mathcal{B}_s$ and $\mathcal{B}_c$ given in theorem 9.1 to a condition on the left-invariant subspaces $\mathcal{B}_s$ and $\mathcal{B}_c$.

**Theorem 9.2** Let $\mathcal{B}_s, \mathcal{B}_c$ and $\mathcal{B}_s, \mathcal{B}_c$ be as defined above.

1. $\mathcal{B}_s \cap \mathcal{B}_c = \{0\}$ if and only if $\mathcal{B}_s \cap \mathcal{B}_c$ is autonomous.
2. If one of the equivalent statements of 1 holds, then $\mathcal{B}_s + \mathcal{B}_e = \mathcal{H}_2^+$ if and only if $\mathcal{B}_s$ and $\mathcal{B}_e$ are instantaneous interconnectable.

**Proof.** We have:

$$\mathcal{B}_s = \{ \Pi_+ \Psi_s w \mid w \in \mathcal{H}_2^+ \} + \{ \Pi_+ \Psi_e w \mid w \in \mathcal{H}_2^+ \}$$

Note that the first component is equal to $\mathcal{B}_s$ and, since $\Psi_s$ is rational, the second component is finite-dimensional. Therefore $\mathcal{B}_s \cap \mathcal{B}_e$ is finite dimensional if and only if $\mathcal{B}_s \cap \mathcal{B}_e$ is finite-dimensional. Since $\mathcal{B}_s \cap \mathcal{B}_e$ is finite dimensional if and only if $\mathcal{B}_s \cap \mathcal{B}_e$ is autonomous, we infer from the right-shift invariance of $\mathcal{B}_s \cap \mathcal{B}_e$ and theorem 4.1 that $\mathcal{B}_s \cap \mathcal{B}_e = \{0\}$. Conclude from this that this is equivalent of saying that $\mathcal{B}_s \cap \mathcal{B}_e$ is autonomous. This completes the proof of the first half of the theorem.

For the second step we note that it is easy to check that the system is instantaneously interconnectable if and only if for all $w_c \in \mathcal{H}_2^-$ and $w_s \in \mathcal{H}_2^-$ there exists $v_c \in \mathcal{H}_2^+$ and $v_s \in \mathcal{H}_2^+$ such that

$$\Pi_+ \Psi_c v_c + \Pi_+ \Psi_s w_s = \Pi_+ \Psi_c v_c + \Pi_+ \Psi_s w_s$$

This is equivalent to the existence of $v_c$ and $v_s$ such that

$$\begin{pmatrix} \Psi_c & \Psi_s \end{pmatrix} \begin{pmatrix} v_c \\ v_s \end{pmatrix} = \Pi_+ \begin{pmatrix} \Psi_c & \Psi_s \end{pmatrix} \begin{pmatrix} w_c \\ w_s \end{pmatrix}$$

Since $\mathcal{B}_s + \mathcal{B}_e = \mathcal{H}_2^+$ is equivalent to the requirement that $(\Psi_c \quad \Psi_s)$ is surjective as a map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$. From the above it is obvious that $\mathcal{B}_s + \mathcal{B}_e = \mathcal{H}_2^+$ implies that the interconnection is instantaneously interconnectable.

Remains to prove that instantaneously interconnectable implies $\mathcal{B}_s + \mathcal{B}_e = \mathcal{H}_2^+$. We know instantaneously interconnectable implies

$$\Pi_+ (\Psi_c \quad \Psi_s) w \in \text{Im} (\Psi_c \quad \Psi_s) \quad (9.1)$$

for all $w \in \mathcal{H}_2^-$. Clearly we also have (9.1) for all $w \in \mathcal{H}_2^+$ since in this case the projection $\Pi_+$ can be removed. This implies (9.1) is satisfied for all $w \in \mathcal{L}_2$. On the other hand we know the interconnection is autonomous which implies that $(\Psi_c \quad \Psi_s)$ is square and has full normal rank. This implies that $(\Psi_c \quad \Psi_s)$ is surjective as a mapping from $\mathcal{L}_2$ to $\mathcal{L}_2$. Therefore the fact that (9.1) holds for all $w \in \mathcal{L}_2$ implies that $(\Psi_c \quad \Psi_s)$ is surjective as a map from $\mathcal{H}_2^+$ to $\mathcal{H}_2^+$. $\square$

**References**