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FREQUENCY - DEPENDENT TE - TM MODE CONVERSION IN ISOTROPIC, INHOMOGENEOUS WAVEGUIDES

by

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I. INTRODUCTION.

Based on previous work by Marcuvitz [1] and De Hoop [2], the present author has reported earlier on the theory of the isotropic, inhomogeneously filled uniform waveguide [3].

V.V. Nikolskii has described a similar method, called by him "the method of eigenfunctions", in a series of articles [4]. We shall adopt this name in this and forthcoming reports, as the method is indeed based on a description of the electromagnetic field in terms of eigenmodes of the homogeneous perfect guide, the inhomogeneous $\mathcal{E} = \mathcal{E}(x,y)$ and $\mathcal{H} = \mathcal{H}(x,y)$ causing the coupling between these modes, subject to Maxwell's equations ($x$ and $y$ indicates the transverse coordinates).

One of the main advantages of the method of eigenfunctions concerns the boundary conditions (on the perfectly conducting wall) being automatically satisfied by the field, as each term in the linear expansion satisfies the boundary condition.

An extension of the method to the case of anisotropic media has been accomplished by Jeuken [5].

We shall now proceed with a short recapitulation of the relevant results and formulas of the earlier report [3].

It was found there, that the propagation-characteristics of the field are almost entirely governed by the so-called $A$-matrix, $|A| = |Z||Y|$, where

$$
Z_{ji} = j\omega P_{ji} + \frac{1}{j\omega} Q_{ji} \\
Y_{ji} = j\omega S_{ji} + \frac{1}{j\omega} R_{ji}
$$

(1)

the indices $j$ and $i$ designating two different modes out of an infinite number of discreet modes; while $P$, $Q$, $R$, and $S$ are area-integrals over the cross-section of the guide.
In the empty guide \((\mathcal{E} = \mathcal{E}_0, \mu = \mu_0)\) we find, that the cross-coupling terms on account of orthogonality, vanish: \(P_{ji} = Q_{ji} = R_{ji} = S_{ji} = 0\).

In this homogeneous case the so-called self-coupling terms will have the following values:

\[
P_{jj} = P_{ii} = \frac{k^2_{c1}}{\mu_0}, \quad R_{jj} = R_{ii} = \frac{k^2_{c2}}{\mathcal{E}_0},
\]
\[
Q_{jj} = \frac{k^2_{c1}}{\mathcal{E}_0}, \quad Q_{ii} = \frac{k^2_{c1}}{\mathcal{E}_0}, \quad S_{jj} = \frac{k^2_{c3}}{\mu_0}, \quad S_{ii} = \frac{k^2_{c3}}{\mu_0}.
\]

The \(k^2_{c}\) are of course the well-known eigenvalues, dependent on cross-section and index-number; furthermore \(Q = 0\) when a TE - mode is involved, and \(S = 0\) for a TM - mode.

The diagonal elements of the \(A\)-matrix are then the only remaining elements, and they assume the easily recognizable form:

\[-\omega^2 \varepsilon_0 \varepsilon_0 + k^2_{c1} ; \quad -\omega^2 \varepsilon_0 \mu_0 + k^2_{c2} ; \quad -\omega^2 \varepsilon_0 \mu_0 + k^2_{c3} \quad \text{etc. etc.}\]

In the case of the inhomogeneously filled guide with \(\mathcal{E} = \mathcal{E}(x,y)\) and \(\mu = \mu(x,y)\), the permittivity and permeability thus in general being functions of the transverse coordinates, coupling between the modes will appear as the cross-coupling terms are now non-zero.

The self-coupling terms, and as a consequence the diagonal elements of the \(A\)-matrix, will also be modified by the inhomogeneity.

The complete expressions for the integrals are as follows ([3], appendix A):

\[
P_{ji} = \iint_D \frac{1}{\mu} \left( \nabla_t \mathcal{E}_j \right) \left( \nabla_t \mathcal{E}_i \right) \ dx \ dy.
\]
\[
Q_{ji} = \iint_D \frac{1}{\mathcal{E}} \left( \nabla_t \mathcal{E}_j \right) \left( \nabla_t \mathcal{E}_i \right) \ dx \ dy.
\]
\[
R_{ji} = \iint_D \mathcal{E} \left( \nabla_t \mathcal{E}_j \right) \left( \nabla_t \mathcal{E}_i \right) \ dx \ dy.
\]
\[
S_{ji} = \iint_D \frac{1}{\mu} \left( \nabla_t \mathcal{H}_j \right) \left( \nabla_t \mathcal{H}_i \right) \ dx \ dy.
\]
\[ \nabla_t \] is the transverse nabla-operator; 
\[ \vec{e}_j, \vec{e}_i; \vec{h}_j, \vec{h}_i; \] are the orthonormal electric mode-vectors, respectively magnetic mode-vectors for the \( j \)-th mode resp. \( i \)-th mode, as defined by Marcuvitz [1].

The propagation-constants \( \Gamma \) of the modified modes (sometimes called quasi-modes when weak coupling is assumed) can then be found by adding \( -\Gamma^2 \) to each diagonal element of the \( A \) - matrix (in the loss-free case \( \Gamma^2 = j^2 \beta^2 = -\beta^* \)), and solving the determinantal equation.

To each root \( (\Gamma^2) \) corresponds an eigenvector, the form of the latter indicating the composition and apportioning as regards the linear combination of eigenmodes, which is then none other than the corresponding normal mode of the inhomogeneous guide.

These normal modes are complete too, so a "pure" eigenmode field-configuration of the empty guide (for example the dominant mode), which we excite at the sending end of the inhomogeneous guide, may be decomposed into the new-found normal modes, each with appropriate "amplitude".

As these latter modes are completely decoupled and for each of them the propagation constant is known, the problem of the isotropic, inhomogeneously filled uniform waveguide may be considered solved in principle.
II. DEGENERATE MODES

When two different modes of the homogeneous guide have the same eigenvalue $k_c$ and therefore have the same phase-constant $\beta$ for any frequency, we call them degenerate modes.

In the $A$-matrix this means that the diagonal elements associated with these modes are identical.

For example, the $TE_{nm}$-mode and the $TM_{nm}$-mode of the rectangular $nm$ waveguide are degenerate; so are the $TE_{01}$-mode and the $TM_{11}$-mode in the circular waveguide.

If the latter guide is bent, the diagonal elements of the corresponding $A$-matrix remain identical, but the bend causes coupling between the $TE_{01}$- and the $TM_{11}$-mode. That is, the off-diagonal elements in the $A$-matrix will now have a value different from zero.

It is well-known from coupled-mode theory and it can be easily proven, that degeneracy when accompanied by coupling, has a catastrophic effect on the propagation of a "pure" mode.

Even very weak coupling (i.e. very small values of the off-diagonal elements in the $A$-matrix) cannot prevent the total conversion and periodic re-conversion of one mode to the other, provided of course, that the guide-length is adequately long.

We may conclude then, that in the case of degeneracy, the "equal-ness" of the diagonal elements of the $A$-matrix are all-important, whereas the off-diagonal elements only affect the so-called critical guide-length.

There are two ways to avoid conversion, or render it insignificant. The first is to remove the degeneracy; in the sense of our formalism this means making the diagonal elements different from each other. We may achieve this by putting a thin dielectric layer on the inside of the guide-wall, as the $TM_{11}$-mode "reacts" to this device in a different manner as compared with the $TE_{01}$-mode (in the bent round waveguide).
The second method consists of filling up the circular guide inhomogeneously in such a way, that the off-diagonal elements, i.e. the cross-coupling integrals, become identical zero [6].

Degeneracy has theoretically no effect now (in the two-mode approximation) but as we cannot in general avoid indirect coupling via other modes, the first method [7] is to be preferred.

What we intend to do now, is to investigate the "dual" problem. We seek an answer to the following question:

if we have two non-degenerate modes of the homogeneous waveguide, thus having different values of their appropriate diagonal elements in the A-matrix (for any frequency); can we or can't we find an inhomogeneous configuration in such a way, that the diagonal elements associated with these (now coupled) modes become identical?

It will be shown, that the answer is in general in the affirmative, subject to the restriction that degeneracy will only occur at one specific frequency.

The underlying reason for this investigation is our expectation, that it may help explain some experimentally found "anomalies", e.g. the peculiar behaviour of the attenuation curve of the coaxial-cable with spiralized dielectric as centering device, at some frequencies [8].

On the other hand, similar "anomalies" are to be expected for special inhomogeneous fillings and at one specific frequency or frequency-range (backward waves?), and these may now be theoretically predicted.
III. THE INHOMOGENEOUS RECTANGULAR WAVEGUIDE

Consider the above waveguide \( b = \frac{1}{2} a \) with \( \mu = \mu_0 \) and \( \varepsilon = \varepsilon(x,y) \).

Assuming weak coupling and restricting ourselves to a two-mode approximation; assign index 1 to the dominant TE\(_{10}\)-mode and index 2 to the lowest possible TM-mode, TM\(_{11}\); the diagonal elements of the \((2 \times 2)\) \( A \)-matrix become then:

\[
\begin{align*}
-\omega^2 \mu_0 R_{11} & \quad + \frac{\pi^2}{2a^2} \\
-\omega^2 \mu_0 R_{22} & \quad + R_{22} Q_{22}
\end{align*}
\]

(3)

In the empty guide: \( R_{11} = R_{22} = E_o \),

\[ Q_{22} = \frac{1}{E_o} k^2_c = \frac{1}{E_o} \frac{5 \pi^2}{a^2} \]

In the inhomogeneous guide:

With \( R_{22} Q_{22} \) apparently much larger than \( \frac{\pi^2}{2a^2} \), if \( R_{22} < R_{11} \), no real frequency exists, for which the two expressions (3) have the same negative value.

If on the other hand, \( R_{22} \gg R_{11} \), then however small the difference may be, degeneracy is assured at some specific frequency (which in the following we will call \( f_{xx} = f_d \)).

The larger the difference between \( R_{22} \) and \( R_{11} \), the lower will be \( f_d \), and vice versa.
The question now arises: if we have at our disposal a thin or small homogeneous dielectric slab or rod, then where do we have to put this slab or rod (alongside the axis of the guide), to ascertain, if possible without trial and error, that the condition \( R_{22} > R_{11} \) will be met?

We recognize that this looks like some sort of synthesis-problem, but it so happens that a brief consideration of the \( R \)-integral will supply us the necessary clue. 

\[
R_{11} = \int_{\mathcal{D}} \varepsilon(x,y) \mathbf{E}_1 \cdot \mathbf{E}_1 \, dx \, dy \\
R_{22} = \int_{\mathcal{D}} \varepsilon(x,y) \mathbf{E}_2 \cdot \mathbf{E}_2 \, dx \, dy.
\]

Now, if \( \varepsilon = \varepsilon_0 \) (constant), then \( R_{11} = R_{22} = \varepsilon_0 \) on account of the normalization rule: \( \int_{\mathcal{D}} \mathbf{E}_1 \cdot \mathbf{E}_1 \, dx \, dy = \int_{\mathcal{D}} \mathbf{E}_2 \cdot \mathbf{E}_2 \, dx \, dy = 1 \).

Thus, placing the dielectric rod in the middle of the guide will not (or almost not) contribute an increase in the \( R_{22} \)-value \( \Rightarrow R_{22} \approx \varepsilon_0 \).

The TM\(_{11}\) mode has a zero transverse field at the centre of the guide as \( \mathbf{H}_2 = 0 \) for \( (x=\frac{1}{2}a, y=\frac{1}{2}b) \).

The TE\(_{10}\) mode however, has a rather strong field at the centre, and as a consequence \( R_{11} > \varepsilon_0 \) for the above placing of the rod.

Of course, for our purpose \( R_{22} > R_{11} \) this is the wrong way, but the principle behind it is obvious:

if we want some \( R \)-integral to exceed another, then put the dielectric in that region of the cross-section where the transverse field of the mode associated with the former \( R \)-integral is stronger than that of the latter.

Now the TM\(_{11}\) mode has zero field at the centre and this means
that the transverse field is, so to speak, pushed to the sides.

For the TM_{11} mode: more to the upper and lower region than to the vertical sides (for the TE_{11} mode just the contrary).

As regards our aim to achieve degeneracy between the TE_{10} and TM_{11} mode, following the forementioned considerations, we then decide on the inhomogeneous configuration, as sketched below:

\[ b = \frac{1}{2} a \]
\[ \mu = \mu_0 \]
\[ \delta < 1 \]

\[ \mathcal{E} = \mathcal{E}_1 \text{ for } \begin{cases} 0 < x < a \quad & \mathcal{E} = \mathcal{E}_0 \text{ for } \begin{cases} 0 < y < \delta b \quad & \quad \frac{3}{10 \pi} (\mathcal{E}_1 - \mathcal{E}_0) \sin 2\pi \frac{y}{b} \end{cases} \end{cases} \]

This slab-form has the added advantage that the TE_{01} mode will not be coupled, as the TE_{0n} modes form an autonomous group for the above configuration ([3], p.24 for an analogous example).

Some simple integral-computations will get us the following results:

\[ R_{11} = \int_{\frac{\pi}{b}}^{\frac{\pi}{a}} \int_{\frac{\alpha}{b}}^{\frac{\pi}{a}} \mathcal{E}(y) \tilde{h}_1 \cdot \tilde{h}_1 \, dx \, dy = \mathcal{E}_o + (\mathcal{E}_1 - \mathcal{E}_o) \frac{\pi}{a} \frac{\pi}{b} \]

\[ R_{22} = \int_{\frac{\pi}{b}}^{\frac{\pi}{a}} \int_{\frac{\alpha}{b}}^{\frac{\pi}{a}} \mathcal{E}(y) \tilde{h}_2 \cdot \tilde{h}_2 \, dx \, dy = \mathcal{E}_o + (\mathcal{E}_1 - \mathcal{E}_o) \frac{\pi}{a} \frac{\pi}{b} + \frac{3}{10 \pi} (\mathcal{E}_1 - \mathcal{E}_o) \sin 2\pi \frac{y}{b} \]

(here \( \mathbf{h}_1 = \mathbf{h}_1^{P_{10}} = \frac{2}{a} \sin \frac{\pi x}{a} \cdot \mathbf{a}_x \))

\[ \mathbf{h}_2 = \mathbf{h}_2^{P_{11}} = \frac{2}{b} \sqrt{\frac{2}{b}} \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \cdot \mathbf{a}_x - \frac{2}{a} \sqrt{\frac{2}{b}} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \cdot \mathbf{a}_y \]

\( p \rightarrow \text{TE-mode} \quad q \rightarrow \text{TM-mode} \)
Comparing $R_{11}$ with $R_{22}$ assures us of the condition $R_{22} > R_{11}$ always being fulfilled, as soon as $\varepsilon_r > 1$ and $\delta > 0$.

Furthermore, it may be verified that the cross-coupling off-diagonal element $A_{12}$ of the $A$-matrix is non-zero (because $R_{12} \neq 0$), so at some definite distance from the guide-entrance near-total conversion from the TE$_{10}$- into the TM$_{11}$-mode is to be expected, at frequency $f_d$.

We remark that the selfcoupling $R$-integral of the TE$_{11}$-mode is always smaller than $R_{11}$, as it can be calculated to be:

$$\varepsilon_o + (\varepsilon_r - \varepsilon_o) \delta = \frac{3}{10 \pi} (\varepsilon_r - \varepsilon_o) \sin 2\pi \delta.$$

The TE$_{11}$-mode therefore will never reach degeneracy with the TE$_{10}$-mode.

To clarify some of the results, it seems best to proceed with a numerical example. Consider the configuration of the preceding page in order to limit the coupling somewhat (so as to prevent too much conversion to higher modes than the TM$_{11}$-mode), take the permittivity of the slab to be $\varepsilon_r = 2 \varepsilon_o$.

From the expression for the $R_{22}$-integral it is clearly seen that a $\delta$-value $\delta = \frac{1}{4}$ will produce the largest possible difference between $R_{22}$ and $R_{11}$, implying a frequency of degeneracy $f_d$ as low as possible.

For the above circumstances we will then acquire the following results:

$$\left\{ \begin{align*}
R_{11} &= \frac{1}{2} \varepsilon_r + \frac{1}{2} \varepsilon_o = 1 + \frac{1}{2} \varepsilon_o, \\
R_{22} &= (\frac{1}{2} \varepsilon_r + \frac{1}{2} \varepsilon_o) + \frac{3}{10 \pi} (\varepsilon_r - \varepsilon_o) = (1 + \frac{3}{10 \pi}) \varepsilon_o, \\
Q_{22} &= \sqrt{\int_0^1 \int_0^b \frac{1}{\varepsilon(y)} (\nabla \cdot \varepsilon_{11})^2 \, dx \, dy} \\
&= \frac{5\pi}{\sqrt{a^2}} \left[ \frac{2}{4\varepsilon_o} + \frac{1}{4\varepsilon_r} + \frac{1}{2\pi} \left( \frac{1}{\varepsilon_r} - \frac{1}{\varepsilon_o} \right) \right] = \frac{1}{\varepsilon_o} \cdot \frac{5\pi}{\sqrt{a^2}} \cdot \left( \frac{7}{8} + \frac{1}{4\pi} \right) \varepsilon_o.
\end{align*} \right.$$  

Equating the two diagonal elements of the $A$-matrix, i.e.:

$$- \frac{\omega}{\mu_o} R_{11} + \frac{\mu_o}{\varepsilon_o} = - \frac{\omega}{\mu_o} R_{22} + Q_{22} R_{22}$$
we find:  \[ \omega^2 \varepsilon_0 / \mu_0 \cdot \frac{3}{10 \pi^2} = \frac{\pi^2}{a^2} \left\{ 5 \left( \frac{3}{4} + \frac{2}{10 \pi^2} \right) \left( \frac{7}{8} + \frac{1}{4 \pi^2} \right) - 1 \right\} \]

As \( \omega^2 \varepsilon_0 / \mu_0 = \frac{4 \pi^2}{\lambda_o^2} \), with \( \lambda_o \) the free-space wavelength, we obtain for this configuration, the following wavelength of degeneracy:

\[ \lambda_o = 0.265 \ a \approx \frac{3}{8} a \quad \rightarrow \ f_d = \frac{3.10^{6}}{4} \ \text{c/s}. \]

(with \( a \) the largest transverse guide-dimension; Giorgi system of units)

A table of the values of the diagonal elements \( A_{11} \) and \( A_{22} \) as a function of \( \lambda_o \) is given below:

<table>
<thead>
<tr>
<th>( \lambda_o = 0.5 \ a )</th>
<th>( \lambda_o = 0.4 \ a )</th>
<th>( \lambda_o = 0.3 \ a )</th>
<th>( \lambda_o = 0.25 \ a )</th>
<th>( \lambda_o = 0.2 \ a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( 3a )</td>
<td>( 5a )</td>
<td>( 7a )</td>
<td>( 12a )</td>
</tr>
<tr>
<td>( \frac{\pi^2}{a^2} )</td>
<td>( \frac{9 \pi^2}{a^2} )</td>
<td>( \frac{25 \pi^2}{a^2} )</td>
<td>( \frac{49 \pi^2}{a^2} )</td>
<td>( \frac{144 \pi^2}{a^2} )</td>
</tr>
</tbody>
</table>

Having found \( f_d \), we next consider the question of the conversion-reconversion effects and of the guide-lengths involved, for the above example.

The value of the off-diagonal element of the \( A \) - matrix may be shown to be almost entirely decided by the cross-coupling integral

\[ R_{12} = \int_0^a \int_0^b \varepsilon(y) \ \vec{h}_1 \cdot \vec{h}_2 \ \ dx \ dy. \]

We remark once more, that inserting the dielectric in a special way (synthesis !) may result in \( R_{12} \) becoming zero, whereas the self-coupling \( R_{11} \) or \( R_{jj} \) in general, can never be smaller than \( \varepsilon_o \).

With \( \lambda_o \approx \frac{3}{8} a \), we find

\[ \omega^2 / \mu_0 R_{12} = 10^{6} \frac{\pi^2}{a^2}. \]

In the case of degeneracy it is well-known from coupled-mode theory and it can be easily proven by the method of eigenfunctions, that in the two-mode approximation the new normal-modes may be thought of as combinations of equal parts of the two modes involved; in our case (symbolically) \#TEx#xEx: "\( \frac{1}{2} \) TE_{10} + \frac{1}{2} TM_{11}" and "\( \frac{1}{2} \) TE_{10} - \frac{1}{2} TM_{11}".
The difference in sign in the symbolic expressions, indicates the fact, that at the entrance of the guide ( \( z = 0 \) ), the \( \text{TM}_{11} \) parts are in anti-phase, as for \( z = 0 \) no \( \text{TM}_{11} \)-mode field is as yet present, just the incoming \( \text{TE}_{10} \)-mode.

These two approximate normal-modes will propagate with different phase-constants \( \beta_1 \) and \( \beta_2 \), which can be calculated as follows:

\[
\beta_1^2 \approx 78 \frac{\pi^2}{a^2} + 18 \frac{\pi^2}{a^2} = 97 \frac{\pi^2}{a^2} \rightarrow \beta_1 \approx 10 \frac{\pi}{a}.
\]
\[
\beta_2^2 \approx 78 \frac{\pi^2}{a^2} - 18 \frac{\pi^2}{a^2} = 60 \frac{\pi^2}{a^2} \rightarrow \beta_2 \approx 8 \frac{\pi}{a}.
\]

Near-total conversion from \( \text{TE}_{10} \)- into the \( \text{TM}_{11} \)-mode will then take place for the first time at a distance \( z \) from the entrance, corresponding to

\[
(\beta_1 - \beta_2) \, z = \pi \rightarrow \left(10 \frac{\pi}{a} - 8 \frac{\pi}{a}\right) z = \pi \rightarrow z = \frac{1}{2} a.
\]

Thus, at the points \( z = \frac{1}{2} a \), \( z = 1\frac{1}{2} a \), \( z = 2\frac{1}{2} a \), etc.; we may expect regions of predominantly \( \text{TM}_{11} \) mode-configuration (in phase!).

For instance: if we take \( a = 8 \text{ cm.} \), \( b = 4 \text{ cm.} \); the operating frequency must have a free-space wavelength \( \lambda_0 = \frac{\pi}{a} = 2 \text{ cm.} \).

The periodically spaced regions where the incoming \( \text{TE}_{10} \)-mode will be almost totally converted into the \( \text{TM}_{11} \)-mode may then be found in the vicinity of the points \( z = 4 \text{ cm.}, \ z = 12 \text{ cm.}, \ z = 20 \text{ cm.} \), etc. etc.

As the \( \text{TM}_{11} \)-mode is a higher mode compared with the dominant \( \text{TE}_{10} \)-mode, from the viewpoint of eigenmodes of the empty guide, there apparently exist alternating regions of higher and lower phase-constants, (at \( f_d \)).

This means, that as we proceed from an operating frequency slightly lower than \( f_d \), e.g. \( (f_d - \Delta f) \), to \( f_d \), the fieldconfiguration in some fixed regions will change from predominantly \( \text{TE}_{10} \)-mode to predominantly \( \text{TM}_{11} \)-mode.

For these regions of the inhomogeneous guide then, a decrease in phase-constant has been effected in conjunction with an increase in frequency, as a consequence, within a narrow frequency-range, backward waves seem indeed possible.
We realize now, that for the occurrence of periodically spaced "backward wave" regions, we do not have to restrict ourselves to TE-TM coupling. Any lower mode - higher mode conversion, with lower mode as input wave and the higher mode more to benefit from the special way of inserting the dielectric, will eventually reach degeneracy at some frequency \( f_d \). However, TE - TM coupling is thought to be more feasible for experimental verification.

One final remark on the rectangular guide: if we want to avoid \( TE_{10} - TM_{11} \) coupling for the above example, this can indeed be done by simply slicing the slab horizontally in two equal parts and move one part to the upper wall. For this symmetrical configuration, although the condition \( R_{22} > R_{11} \) still holds, now the cross-coupling \( R_{12} \) integral will be identically zero!

**IV. THE INHOMOGENEOUS CIRCULAR WAVEGUIDE**

As the underlying principles of the theory are assumed to be reasonably understood by now, we give the results for certain inhomogeneous configurations of the round waveguide without much comment.

Only the frequencies of degeneracy will be calculated here. Further details about the periodic distances of alternating conversion-regions will be published in subsequent reports.

The first configuration to be examined is the circular symmetric case of the round dielectric rod placed along the axis of the guide.

\[
\begin{align*}
\mu &= \mu_0, \\
p &< 1, \\
\varepsilon &= \varepsilon(r).
\end{align*}
\]

\[
\left\{ \begin{array}{ll}
\varepsilon &= \varepsilon_1 \text{ for } 0 < r < p_a, \\
\varepsilon &= \varepsilon_0 \text{ for } p_a < r < a.
\end{array} \right.
\]

The same considerations as in the foregoing example applies here.
We want the **dominant** TE\(_{11}\) mode to couple with the lowest possible TM-mode and the two quasi-modes to reach degeneracy at a specific frequency.

For this configuration the TM\(_{0n}\) modes (and the TE\(_{0n}\) modes for that matter) form an autonomous group, so the TM\(_{01}\) mode will not be excited, the TM\(_{11}\) mode then being the lowest TM mode coupled to the dominant TE\(_{11}\) mode.

Assignate index 1 to the latter, and index 2 to the former (TM\(_{11}\)) mode. We can be reasonably sure of the condition \(R_{22} > R_{11}\) being satisfied, as we may easily verify that at the centre the TM\(_{11}\) field is slightly **stronger** than the TE\(_{11}\) field.

We obtain the following results after some elementary computations:

\[
R_{11} = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0) \left[ \frac{p^2}{d_{11}^2} \left( \frac{p^2 d_{11}^2}{d_{11}^2 - 1} \right) + \frac{(p^2 d_{11}^2 - 2)}{d_{11}^2 - 1} \right] = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0) K.
\]

\[
R_{22} = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0) \left[ p^2 \left( \frac{p^2}{d_{22}^2} \left( \frac{p^2 d_{22}^2}{d_{22}^2 - 1} \right) + \frac{(p^2 d_{22}^2 - 2)}{d_{22}^2 - 1} \right) \right] = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0) L.
\]

Unlike the former example of the inhomogeneous rectangular guide, here the condition \(R_{22} > R_{11}\) is not fulfilled for all \(p > 0\) (\(pa\) being the radius of the dielectric rod, \(a\) of the guide itself).

Certainty can be had however, by sketching \(K\) and \(L\) as functions of \(p\). For those values of \(p\) where \(L > K\) holds, so will \(R_{22} > R_{11}\).

Even here we may predict the outcome.

The TM\(_{11}\) mode has zero transverse field at some points corresponding to \(p \approx 0.6\); therefore the advantage of the stronger field at the centre will surely die out for a value of \(p\) below \(p = 0.6\).

We may check first, that for the above \(R_{11}\) - and \(R_{22}\) - expressions

- when \(p = 0 \rightarrow K = L = 0 \rightarrow R_{11} = R_{22} = \varepsilon_0\);
- when \(p = 1 \rightarrow K = L = 1 \rightarrow R_{11} = R_{22} = \varepsilon_1\),

as it should be. \((R_{12} \neq 0\), for \(0 < p < 1\))
From the above diagram we conclude, that dielectric rods with radii larger than 0.4 \(a\) will not satisfy the condition \(R_{22} > R_{11}\); i.e. no degeneracy between \(TE_{11}\) - and \(TM_{11}\) - mode is to be expected, at whatever frequency.

Clarricoats arrives at the opposite conclusion in his paper ([9], Fig. 3), but nevertheless his diagram and ours bear a striking resemblance as to the critical point \(p = 0.4\).

As the difference between \(K\) and \(L\) (for \(p < 0.4\)) is greatest for \(p = 0.2\), we take as numerical example a dielectric rod of radius \(0.2 \ a\) and a permittivity \(\varepsilon_r = 2 \varepsilon_o\), as before.

A table of the values of the diagonal elements \(A_{11}\) and \(A_{22}\) as a function of the free-space wavelength \(\lambda_o\) is given below:

<table>
<thead>
<tr>
<th>(p)</th>
<th>(K)</th>
<th>(L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>0.2</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>0.3</td>
<td>0.17</td>
<td>0.21</td>
</tr>
<tr>
<td>0.4</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>0.5</td>
<td>0.42</td>
<td>0.35</td>
</tr>
<tr>
<td>0.6</td>
<td>0.55</td>
<td>0.41</td>
</tr>
<tr>
<td>0.7</td>
<td>0.71</td>
<td>0.51</td>
</tr>
<tr>
<td>0.8</td>
<td>0.80</td>
<td>0.63</td>
</tr>
<tr>
<td>0.9</td>
<td>0.90</td>
<td>0.80</td>
</tr>
</tbody>
</table>

The specific frequency \(f_d\) can be computed to be: \(f_d = \frac{3.10^8}{\varepsilon_o \sqrt{a^2}}\) c/s.

( for this case: \(R_{11} = 1.08 \varepsilon_o\); \(R_{22} = 1.15 \varepsilon_o\) and \(Q_{22} = \frac{1}{\varepsilon_o} (a^2 b^2) \frac{d_{11}^2}{a^2}\)

with \(d_{11}\) the first root of the Bessel-equation \(J_1(k_c r) = 0\) for \(r = a\).
As regards the dual configuration sketched below:

\[ \mu = \mu_0, \]
\[ p < 1, \]
\[ \varepsilon = \varepsilon(r). \]
\[ \begin{cases} \varepsilon = \varepsilon_0 & \text{for } 0 < r < p a, \\ \varepsilon = \varepsilon_1 & \text{for } p a < r < a. \end{cases} \]

A circular waveguide with concentric dielectric layer adjacent to the wall, we remark that the \( \text{TM}_{11} \) field close to the wall is indeed slightly stronger than the \( \text{TE}_{11} \) field; i.e., favourable for conversion.

For the self-coupling \( R \) -integrals in this case, we obtain:

\[ R_{11} = \int_0^a \int_0^{2\pi} \varepsilon(r) \, \bar{h}_1^p \cdot h_1^p \, r \, dr \, d\varphi = \varepsilon_1 - (\varepsilon_1 - \varepsilon_0) \, K, \]
\[ R_{22} = \int_0^a \int_0^{2\pi} \varepsilon(r) \, \bar{h}_1^q \cdot h_1^q \, r \, dr \, d\varphi = \varepsilon_1 - (\varepsilon_1 - \varepsilon_0) \, L. \]

\( K \) and \( L \) are the same as that of the preceding example, so we may use the diagram of page 14.

The condition \( R_{22} > R_{11} \) now implies \( K > L \), as we conclude from the above expressions for \( R_{11} \) and \( R_{22} \).

This dual configuration then, leads to an analogous restriction as in the former case, namely that now it is the thickness of the layer, which must not exceed the value \( 0.6 \, a \), as for the interval \( p < 0.4 \) the value for \( K \) will be smaller than that for \( L \).

We may check for the extreme case (homogeneous):

when \( p = 0 \) \( \rightarrow K = L = 0 \) \( \rightarrow R_{11} = R_{22} = \varepsilon_1 \),

when \( p = 1 \) \( \rightarrow K = L = 1 \) \( \rightarrow R_{11} = R_{22} = \varepsilon_0 \),

as it should be. \( (R_{12} \neq 0, \text{for } 0 < p < 1) \)

The largest difference between \( R_{11} \) and \( R_{22} \) (i.e., between \( K \) and \( L \),
in the interval $0.4 < p < 1$ occurs for $p = 0.7$.

Thus, to assure a frequency of $\text{TE}_{11} - \text{TM}_{11}$ degeneracy as low as possible (for a given permittivity), the dielectric layer adjacent to the wall must have a thickness corresponding to $p = 0.7$, that is a thickness of $0.3\,a$.

For the numerical example we choose then: $\varepsilon_r = 2\varepsilon_o$ and $p = 0.7$

We find for these circumstances:

$$
R_{11} = 1.29\varepsilon_o \quad Q_{22} = \frac{1}{\varepsilon_o} \quad 0.98 \quad \frac{d_{11}^2}{a^2}
$$

$$
R_{22} = 1.49\varepsilon_o
$$

A table of the values of the diagonal elements $A_{11}$ and $A_{22}$ as a function of the free-space wavelength is given below:

<table>
<thead>
<tr>
<th>$\lambda_o = 0.8,a$</th>
<th>$\lambda_o = 0.7,a$</th>
<th>$\lambda_o = 0.6,a$</th>
<th>$\lambda_o = 0.5,a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-A_{11} \rightarrow$</td>
<td>$76.2/\lambda_o^2$</td>
<td>$99.8/\lambda_o^2$</td>
<td>$140.1/\lambda_o^2$</td>
</tr>
<tr>
<td>$-A_{22} \rightarrow$</td>
<td>$71.6/\lambda_o^2$</td>
<td>$97.7/\lambda_o^2$</td>
<td>$144.2/\lambda_o^2$</td>
</tr>
</tbody>
</table>

The specific frequency $f_d$ can be computed to be:

$$
f_d = \frac{3.1 \times 10^8}{0.66\,a} \text{ c/s}
$$

From the above examples it will be clear by now, that a dielectric insert may cause degeneracy between a lower and a higher quasi-mode at some specific frequency $f_d$.

Now, the preferred solution in the bent circular waveguide (for telecommunication purposes) is to remove the $\text{TE}_{01} - \text{TM}_{11}$ degeneracy by dielectrically layering the wall.

The $\text{TE}_{01}$ field adjacent to the wall is zero while that for the $\text{TM}_{11}$-mode is non-zero, therefore, removal of the degeneracy is assured from the beginning (cut-off) and will be permanent, as the $\omega$-$\beta$ characteristics for the two modes are divergent.

The operating frequency for the wave-cable is known to be ultra-high,
wavelengths as small as a few mm. are proposed, so even a very thin layer is sufficient.

In the light of the above theory, bearing in mind that the phase-constants of all modes (except the $\text{TE}_{\text{On}}$-group) benefit more or less, not only will such an extreme thinness be sufficient, it is absolutely necessary as well to prevent the next higher coupled mode ($\text{TE}_{12}$) reaching degeneracy with the $\text{TE}_{01}$-mode at some $f_d$ within the interval of operating frequencies, thereby spoiling the favourable attenuation-curve of the latter.

As a final example, let's have a look at the following sector-slab adjacent to the wall of the circular guide, a configuration which we believe has far-reaching implications in view of its similarities as regards the spiral coax.

For this inhomogeneous filling no circular symmetry exists, so the $\text{TM}_{\text{On}}$-modes no longer form an autonomous group, and the lowest possible coupled TM-mode will then be the $\text{TM}_{01}$-mode.

The electric field of the latter is radially directed, hence, as regards the associated phase-constant, will certainly benefit from the dielectric insert.

To assure the occurrence of degeneracy-effects, we choose as polarization plane of the incoming dominant $\text{TE}_{11}$-mode that one which corresponds to horizontal electric fieldlines. The reason is obvious:
this $\text{TE}_{11}$-mode gets almost no benefit from the insert, whereas the other independent (orthogonal) $\text{TE}_{11}$-mode does profit.

Assignate index 1 to the dominant mode, index 2 to the $\text{TM}_{01}$-mode, we find after some computation, that our expectation concerning $R_{22} > R_{11}$ indeed comes true:

$$R_{11} = \int_0^\pi \int_0^{2\pi} \mathcal{E}(\varphi) \rho_{11}^p \rho_{11}^p \ r \ dr \ d\varphi = \left[ \varepsilon_o + \frac{\partial}{2\pi} (\varepsilon_i - \varepsilon_o) \right] - \left[ 0.0324 (\varepsilon_i - \varepsilon_o) \sin 2 \beta \right].$$

$$R_{22} = \int_0^\pi \int_0^{2\pi} \mathcal{E}(\varphi) \rho_{01}^q \rho_{01}^q \ r \ dr \ d\varphi = \left[ \varepsilon_o + \frac{\partial}{2\pi} (\varepsilon_i - \varepsilon_o) \right].$$

The calculations involved, are a.o. as follows:

$$\rho_{11}^p = -\nabla_t \psi_{11}^p; \text{ with } \psi_{11}^p = \frac{\delta_1 \left( d_{11} \gamma \right)}{\gamma_{11} \left( d_{11} \right)} \cos \varphi \sqrt{\frac{2}{\pi (d_{11}^2 - 1)}}.$$

$$R_{11} = \left[ \varepsilon_o + \frac{\partial}{2\pi} (\varepsilon_i - \varepsilon_o) \right] + \frac{2}{\pi (d_{11}^2 - 1) \gamma_{11}^2 (d_{11}^2 - \frac{\gamma_{11}^2}{d_{11}^2})} \int_0^\pi \int_0^{2\pi} \mathcal{E}(\varphi) \cos \gamma \rho_{11}^p \ r \ dr \ d\varphi$$

$$+ 0.81 \text{ (positive)} - 0.08 \text{ (negative)} \text{ positive for } 0 < \beta < \frac{\pi}{2}.$$

In this expression:

$$\int_0^\pi \mathcal{E}(\varphi) \cos \gamma \rho_{11}^p \ d\varphi = \int_0^\pi \varepsilon_i \cos \gamma \rho_{11}^p \ d\varphi + \int_0^\pi \varepsilon_o \cos \gamma \rho_{11}^p \ d\varphi =$$

$$= \frac{1}{2} (\varepsilon_i - \varepsilon_o) \sin 2 \beta.$$

$$\tilde{h}_{01} = \frac{V_{\pi}}{d_0 \gamma_0 (d_0)} \cdot \frac{d_{11}}{d_0} \cdot \frac{\delta_1 \left( d_{11} \gamma \right)}{\gamma_{11} \left( d_{11} \right)} \cdot \n_0 \sqrt{2 \varphi} \cos \varphi.$$

$$R_{22} = \int_0^\pi \int_0^{2\pi} \mathcal{E}(\varphi) \ d\varphi = \int_0^\pi \mathcal{E}(\varphi) \ d\varphi$$

$$= \frac{1}{2 \pi} \int_0^{2\pi} \mathcal{E}(\varphi) \ d\varphi = \frac{1}{2 \pi} \left[ \varepsilon_i \Delta + (2\pi - \beta) \varepsilon_o \right] = \varepsilon_o + \frac{\partial}{2\pi} (\varepsilon_i - \varepsilon_o).$$
Degeneracy will be possible, as soon as $\lambda > 0$ and $\varepsilon_\nu > 1$.

We may check, as before: when $\lambda = 0 \Rightarrow R_{11} = R_{22} = \varepsilon_o$;
when $\lambda = \theta 2\pi \Rightarrow R_{11} = R_{22} = \varepsilon_i$.

$R_{12} \neq 0$ can be verified.

For the largest difference between $R_{11}$ and $R_{22}$ we must clearly have the value of $\lambda$ corresponding to $2\lambda = \pi/2$; $\lambda = \pi/4$ rad.

In the numerical example we choose, conform the preceding cases: $E_i = 2E_o$.

We obtain then:

$$R_{11} = \left( \frac{1}{8} \varepsilon_i + \frac{7}{8} \varepsilon_o \right) - 0.0344\varepsilon_i - 1.0926 \varepsilon_o.$$

$$R_{22} = \left( \frac{1}{8} \varepsilon_i + \frac{7}{8} \varepsilon_o \right) = 1.1250 \varepsilon_o.$$

Furthermore:

$$Q_{22} = \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{\varepsilon(\varphi)} \right) (\mathbf{e}, \mathbf{e}_o)^2 r dr d\varphi =$$

$$= \int_0^{2\pi} \mathbf{d}_{o1}^2 \int_0^{2\pi} \frac{r}{\varepsilon(\varphi)} d\varphi =$$

$$= \frac{1}{2\pi} \mathbf{d}_{o1}^2 \frac{7\pi}{4} + \frac{1}{2\pi} \frac{d_{o1}^2}{4} \varepsilon_o \frac{7\pi}{4} = \frac{d_{o1}^2}{4} \varepsilon_o \left( \frac{\pi}{4} \right).$$

Hence,

$$R_{22} Q_{22} = \frac{d_{o1}^2}{\lambda^2} (1.1250 \times \frac{15}{16}) = (1.06) \frac{d_{o1}^2}{\lambda^2}.$$

A table of the values of the diagonal elements $A_{11}$ and $A_{22}$ as a function of the free-space wavelength $\lambda_o$ is given below:

<table>
<thead>
<tr>
<th>$\lambda_o = \lambda$</th>
<th>$\lambda_o = 3/4 \lambda$</th>
<th>$\lambda_i = 3/4 \lambda$</th>
<th>$\lambda_o = 1/2 \lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{11}$</td>
<td>$39.8 \frac{1}{\lambda^2}$</td>
<td>$59.3 \frac{1}{\lambda^2}$</td>
<td>$92.6 \frac{1}{\lambda^2}$</td>
</tr>
<tr>
<td>$R_{22}$</td>
<td>$169.7 \frac{1}{\lambda^2}$</td>
<td>$37.3 \frac{1}{\lambda^2}$</td>
<td>$58.3 \frac{1}{\lambda^2}$</td>
</tr>
</tbody>
</table>

The simplest solution to avoid or render insignificant this $TE_{11}^{01}$ total conversion at the frequency of degeneracy $f_d$ is, of course, to use the other (vertical) polarisation of the $TE_{11}$ mode.
However, should we slowly spiralize the dielectric sector-slab within the guide, then this solution will not do, as the incoming $\text{TE}_{11}$-mode of whatever polarisation will periodically encounters lengths of guide favourable for conversion into the $\text{TM}_{01}$-mode at some $f_d$.

Only one method avails: it so happens that the $\text{TM}_{01}$-mode has zero transverse field at the origin, hence what we may do is to provide the slab with some extra dielectric located at the sharp edge, i.e. at the centre of the guide. The $\text{TE}_{11}$-mode will strongly benefit, the $\text{TM}_{01}$-mode almost not at all.

One possible reason for avoiding $\text{TE}_{11}$-$\text{TM}_{01}$ degeneracy when lengthy guides are involved, as the attenuation curve of the $\text{TM}_{01}$-mode (due to wall currents) is considerably worse compared with the $\text{TE}_{11}$ ([10], p. 5/9).

Now this is precisely the link with the spiralized dielectric of the coaxial cable mentioned earlier. As communicated privately to the author by ir.C.Kooy the experimentally found anomaly consists of a rather sudden and unexpected jump of the attenuation at some frequency.

This spiral-cable can be conceived as an inhomogeneous waveguide, with eigenfunctions and eigenmodes associated to the empty coaxial cable. In subsequent reports then, we will investigate the possibility of the above anomaly arising out of frequency-dependent mode-conversion (degeneracy), the $\text{TE}_{11}$-mode excited by the dominant $\text{TEM}$-mode, then totally converting its energy into the $\text{TM}_{01}$-mode at some frequency $f_d$, thereby causing a jump in the attenuation (stronger wall-currents).

Anticipating somewhat, we may propose the following solution to suppress the anomaly.
This eventual solution is based on the same kind of reasoning as used in the preceding example.

The transverse electric field configuration of the $\text{TM}_{01}$-mode of the coaxial cable, which is a radial one just like in the circular guide, has a concentrical zero-region about midway between outer and inner conductor.

The precise location can be calculated for any given dimension of the coax, making use of the formulas (combined Bessel-Neumann functions), as published in the Waveguide Handbook (Marcuvitz), page 76.

Hence, attaching concentric dielectric fins (see figure) just at the regions of the spiral, corresponding to the zero transverse field of the $\text{TM}_{01}$-mode, will not enhance the phase-constant of the latter, whereas the $\text{TE}_{11}$-mode benefit from this dielectric additions, on account of the non-zero $\text{TE}_{11}$ transverse field there. (The $\omega-\beta$ diagram of the $\text{TE}_{11}$ gains a little steepness, that for the $\text{TM}_{01}$ not at all)

As a consequence, the quasi $\text{TE}_{11}$ - quasi $\text{TM}_{01}$ mode-degeneracy (for a specific frequency) caused by the spiral, will now either be suppressed altogether or "delayed" to far higher frequency.

At the end of this preliminary report, the author wishes to express his gratitude to drs.M.Jeuken, ir.C.Kooy and ir.W. van Veenendaal, for the many clarifying discussions on the above subject.
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