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PROBLEMS

published in

"WISKUNDIGE OPGAVEN MET DE OPLOSSINGEN"

by N.G. de Bruijn

The references at the end of each problem indicate the publication of the solution in Wiskundige Opgaven; volume, issue, year, problem number. An additional D indicates that the problem was published in Dutch language.

Technological University,
Eindhoven, Netherlands.
September 1964.
(Suppens 1975)
Assume \( \varepsilon > 0, a_n > 0, b_n > 0 \) \((n=1,2,3,\ldots, \lim_{n \to \infty} b_n a_n^{-1} = \varepsilon)\). Rearranging the sequence \( a_1, a_2, \ldots \) we obtain the non-increasing sequence \( a_1 \geq a_2 \geq a_3 \geq \ldots \). Similarly we obtain from \( b_1, b_2, \ldots \) the sequence \( b_1 \geq b_2 \geq b_3 \geq \ldots \). Show that \( \lim_{n \to \infty} b_n a_n^{-1} = \varepsilon \).

If \( \varepsilon > 0 \), show that

\[
\sum_{\ell=1}^{N} \frac{1}{\ell^{2}} \prod_{p \mid \ell} \frac{p-1}{p+1} = \frac{1}{3} N^{3} \prod_{p} \left(1 - \frac{2}{p(p+1)}\right) + \sigma(N^{2} + \varepsilon)
\]

(p runs through the primes, and \( p/\ell \) means that \( p \) runs through all primes dividing \( \ell \)).

Let \( B(N) \) denote the number of triples \( (\ell, m, n) \) with \( \ell, m, n \) are positive integers \( \leq N \), mutually co-prime (that is \((\ell, m) = (m, n) = (n, \ell) = 1 \)). Show that

\[
B(N) = -2 + 3 \sum_{\ell=1}^{N} \sum_{k \in \varphi(\ell)} \mu(k) \left[ \sum_{d/\ell} \mu(d) \left[ \frac{\ell}{kd} \right]^{2} \right]
\]
(where \( k \in \varphi(\ell) \) means \( 1 \leq k \leq \ell \), \((k, \ell) = 1 \), and \( \mu \) denotes the Möbius' function), and that, for every \( \varepsilon > 0 \),

\[
B(N) = N^{3} \prod_{p} \left(1 - \frac{2p-2}{p^{3}}\right) + \sigma(N^{2} + \varepsilon).
\]

[17 (5) 1942, Nr. 173, D]
If $p \geq q \geq 0$, $p + q = n$, and
\[
\sum_{\lambda = 1}^{p} |a_{\mu \lambda}| \leq 1, \sum_{\lambda = p+1}^{n} |a_{\mu \lambda}| \leq 1 \quad (\mu = 1, \ldots, n)
\]
then show that
\[
|\det (a_{\mu \lambda})| \leq 2^q.
\]
and that there is a matrix $a_{\mu \lambda}$ for which the sign $=$ holds.

[17 (5) 1942, Nr. 177 jointly with D. van Dantzig, D]

Show that for every integer $n > 0$ and for every integer $x$ we have
\[
\sum_{d \mid n} \mu(d) x^{n/d} \equiv 0 \pmod{n},
\]
when $\mu(d)$ indicates Möbius' function.

[18 (1) 1943 Nr. 15, D]

Assume $0 < \alpha < 1$, $\beta$ complex. If $z > 0$ we define
\[
F(z, \beta, \alpha) = \int_{n=1}^{\infty} e^{-z^{\alpha} n^{\beta-1}} \frac{P(\beta)}{z^\beta} \quad (\beta \neq 0, -1, -2, \ldots),
\]
\[
\phi(z, \beta) = (-1)^{\beta+1} \frac{\log z}{(-\beta)\Gamma(\beta)} \quad (\beta = 0, -1, -2, \ldots).
\]

Show that $F(z, \beta, \alpha) - \frac{1}{\alpha} \phi(z, \beta)$ can be continued analytically throughout the complex $z$-plane.

[18 (1) 1943, Nr. 16, D]
Let \( f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots \) be holomorphic for \(|z| < 1\), and assume that for no value of \( z \) inside that circle \( f(z) \) takes a real value \(< - \frac{1}{4} \). Show that \( |a_n| \leq n \) \((n=1, 2, 3, \ldots)\).

[18 (2B) 1946, Nr. 64, D]

Assume \( d > 0, \; d \equiv 3 \pmod{4}, \) and let \( \mathbb{Q}(\sqrt{d}) \) denote the quadratic field obtained by adjoining \( \sqrt{d} \) to the rationals. If \( \xi \) is an integer of the field, and if \( \xi \) divides \( 4(d+1) \), then show that the norm of \( \xi \) is positive.

[18 (2B) 1946, Nr. 65, D]

Let \( G \) be a finite abelian group and let \( R \) be a subset having \( k \) elements. Assume that for every character \( \chi \) of \( G \) the sum \( \sum_{a \in R} \chi(a) \) is either 0 or \( k \). Show that \( R \) is a subgroup.

[18(3) 1946 Nr. 107, D]

Show that for every positive integer \( N \) we have

\[
\sum_{q=1}^{N} \sum_{N \leq b \leq N+q, (b,q) \neq 1} \frac{1}{b_q} = \frac{1}{2}
\]

[18(3) 1946, Nr. 108, D]
Let $s$ be fixed, $s > 1$. Let $F(z,s)$ be defined by

$$F(z,s) = \sum_{n=1}^{\infty} \frac{e^{-z \log n}^2}{n^s} - \frac{1}{\sqrt{2}} e^{\frac{(s-1)^2}{4z}} \int_{-s}^{s-1} e^{-u^2} du$$

if $z > 0$. Show that $F(z,s)$ can be continued analytically throughout the complex $z$-plane.

[18(3) 1946, Nr. 109, D]

On the interval $-1 \leq x \leq 1$ we are given $n$ point charges ($n \geq 2$) situated in $x_1, \ldots, x_n$, respectively, repulsing each other mutually with forces $|x_i - x_j|^{-1}$. They are in equilibrium. Show that $x_1, \ldots, x_n$ are the zeros of $(1-x^2) P_{n-1}(x)$, where $P_{n-1}$ is the $(n-1)^{st}$ Legendre polynomial.

[18(3) 1946 Nr. 110, D]

Let the complex number $f(n)$ be defined for all integers $n,m$ and satisfy

$$|f(n+m) + f(n-m) - 2f(n) - 2f(m)| < 1.$$ 

Show that there exists a complex constant $\omega$ such that

$$|f(n) - \omega n^2| < \frac{1}{2} \text{ for all } n.$$ 

[19(1) 1950, Nr. 22, D]

Let the sequences $\{a_n\}$ and $\{c_n\}$ satisfy the conditions that

$-\rho < a_n \leq 1$ (where $\rho$ is a constant, $-1 < -\rho \leq 1$) for all $n$, and that

$$\sum_{n=1}^{\infty} (c_{n+1} - a_n)$$

converges. Show that $\lim_{n \to \infty} c_n$ exists.

[19(1) 1950, Nr. 23, D]
If \( 0 < q < 1 \), show that
\[
\int_0^\infty \left( \sum_{n=1}^{\infty} q^n t \right)^2 \, dt = \frac{\pi}{2} \left( 1 + q^2 + q^4 + q^6 + \ldots \right)^2.
\]
[19 (1) 1950, Nr. 24, D]

Two players play the following game, moving alternately the game is lost by the player who is unable to move.

On an infinite row of squares, labeled 1, 2, 3, ..., we have a finite number of counters, at most one on each square. A move consists of shifting a counter to a square with a lower number, which is allowed only if the new square and all squares between the old square and the new one, are empty.

Show how to play this game in all possible situations.

[19(1) 1950, Nr. 25, D]

Let, for \( x \geq 1 \), the function \( f \) be decreasing and positive, and assume that \( \Sigma f(n) \) converges. Let \( a_1, \ldots, a_k \) be positive, and
\[
a_1^{-1} + \ldots + a_k^{-1} = 1.
\]
Show that
\[
\Sigma_{n=1}^{\infty} f(na_1) + \ldots + \Sigma_{n=1}^{\infty} f(na_k) \leq \Sigma f(n).
\]
[19(2) 1951, Nr. 51, D]
Assume $0 < x < 1$, $x^{-1} + (1-x)^{-1} = 1$, and let $\lfloor x \rfloor$ denote the integral part of $x$. Show that

$$
\sum_{n=1}^{\infty} \lfloor n \alpha \rfloor^{-2} + \sum_{n=1}^{\infty} \lfloor n \beta \rfloor^{-2} \leq \sum_{n=1}^{\infty} n^{-2},
$$

and that the equality sign holds if and only if $x$ is irrational.

[19(2) 1951, Nr. 52, D]

Show that (i) if $p \geq 1$, $\sum_{n=0}^{\infty} |a_n|^{p} < \infty$ then we have

$$
\int_{0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{a_n e^{-x^n}}{n!} \right|^p dx \leq \sum_{n=0}^{\infty} |a_n|^p.
$$

(ii) if $p > 1$, the equality sign holds only if $a_0 = a_1 = \ldots = 0$;

(iii) for no value of $p > 1$ we can replace the right hand side by $C \sum_{n=0}^{\infty} |a_n|^p$ with $C$ not depending on $a_0, a_1, \ldots$, and $0 \leq C < 1$.

[19(2) 1951, Nr. 53, D]

Consider partitions of the positive integer $n$ into positive integral parts, and identify two partitions if and only if the one can be transformed into the other by cyclic permutation of the summands (Thus $3+2+1+1+1$ and $1+1+3+2+1$ are the same partitions, but $1+1+2+3+1$ is different). Show that there are

$$
\frac{1}{n} \sum_{d|n} (2^{n/d} - 1) \varphi(d)
$$
different partitions, where $\varphi$ is Euler's function.

[19(2) 1951, Nr. 54, D]
Two players play the following game, moving alternately; the game is lost by the player who is unable to move. We have a number of heaps of matches. A move consists of removing either one match from one heap, or removing simultaneously one match from a number of heaps (one from each heap). The latter operation is only allowed if all heaps from which a match is taken have the same size (although there may be heaps of the same size, from which no match is taken).

Show how to play this game.

[19(2) 1951, Nr. 55, D]

Show that the conic passing through the centres of the circumscribed circle and all four inscribed circles of the triangle ABC, touches Euler's line.

[19(3) 1952, Nr. 91, D]

Show that

\[ \int_{0}^{\infty} \left\{ t(3t^3 + 1) - 3t^3 (1+t^3)^{\frac{3}{2}} \right\} dt = \frac{1}{20\pi} \left( r \left( \frac{1}{3} \right) \right)^{\frac{3}{2}} \sqrt{3}. \]

[19(4) 1953, Nr. 131, D]

Let \( f_0, f_1, \ldots, f_n \) be continuously differentiable functions of the variables \( x_1, \ldots, x_n \), and assume that they are periodic functions of each variable, with period 1. Let

\[ \{ f_1, \ldots, f_n \} \quad \text{and} \quad \{ x_1, \ldots, x_n \} = J(f_1, \ldots, f_n) \]

denote the Jacobian of \( f_1, \ldots, f_n \).
Show that

\( (i) \int_0^1 \cdots \int_0^1 J(f_1, \ldots, f_n) \, dx_1, \ldots, dx_n = 0; \)

\( (ii) \) If \( F_i = \Sigma_{i=0}^n a_{ij} f_j \) (i=0, \ldots, n), and

\[ \det_{i,j=0,\ldots,n} a_{ij} = 1, \]

we have

\( \int_0^1 \cdots \int_0^1 f_0 J(F_1, \ldots, F_n) \, dx_1, \ldots, dx_n = \)

\( = \int_0^1 \cdots \int_0^1 f_0 J(F_1, \ldots, F_n) \, dx_1, \ldots, dx_n. \)

[19(4) 1952, Nr. 132, D]

Consider the confocal system of hypersurfaces

\[ \frac{x_1^2}{a_1 + \lambda} + \cdots + \frac{x_n^2}{a_n + \lambda} = 1 \]

in n-dimensional euclidean space, where \( a_1, \ldots, a_n \) are distinct real numbers. Let \( P = (p_1, \ldots, p_n) \) be a real point, and \( p_i \neq 0 \) (i=1, \ldots, n).

Show that there are \( n-1 \) different real values for \( \lambda \) such that it is possible to draw a set of \( n \) mutually orthogonal tangents through \( P \).

[19(4) 1952, Nr. 133, D]

Let \( f_n \) be defined by

\[ f_1 = 1, f_n = \sum_{k=1}^{n-1} d_k f_k f_{n-k} \quad (n=2, 3, \ldots), \]

where \( d_k = a \) (k=1, 3, 5, \ldots), \( d_k = b \) (k=2, 4, 6, \ldots).

Find explicit expressions for \( f_{2n} \) and \( f_{2n+1} \).

[19(4) 1952, Nr. 134, D]
Find the asymptotic behaviour of \( f_{2n+1} \) and of \( f_{2n} \)
if it is assumed that \( a > 0 \), \( b > 0 \).

[19(4) 1952, Nr. 135, D]

Let \( \lambda \) be a positive constant. Define \( f \) by \( f(0)=f(1)=0 \), and
\[
f(x) = \exp \{ -x^{-\lambda} - (1-x)^{-\lambda} \} \quad (0<x<1).
\]
Show that we have for the \( n \)-th derivative
\[
|f^{(n)}(x)| \leq \{ C n^{\lambda+1} \}^n \quad (n=0,1,2,\ldots ; 0 \leq x \leq 1),
\]
where \( C \) depends on \( \lambda \) only.

[19(5) 1954. Nr. 168, D]

If \( \sigma_\lambda(n) \) represents the sum of the \( \lambda \)-th powers of the divisors of \( n \), then show that
\[
(\sigma_\lambda(n))^2 = \sum_{d|n} (n/d)^\lambda \sigma_\lambda(d^2).
\]

[19(5) 1954 Nr. 169, D]

A finite sequence of irreducible fractions \( p_k/q_k \)
\((q_k > 0, k=0,\ldots,m)\) is called a chain if
\[
|p_k q_{k-1} - q_k p_{k-1}| = 1 \quad (k=1,\ldots,m).
\]
Consider a second chain \( p'_k/q'_k \) \((k=0,\ldots,n)\) and assume that
\[
\frac{p_0}{q_0} < \frac{p'_0}{q'_0} < \frac{p_m}{q_m} < \frac{p'_n}{q'_n}
\]
Show that the two chains have a fraction in common.

[19(5) 1954. Nr. 170, D]
For which values of $\lambda$ do the roots of the equation $x^3 + 3ix^2 = \lambda$ form an equilateral triangle in the complex plane?

[20(1) 1955 Nr. 13, D]

A sequence $v_1, v_2, \ldots (v_n \geq 0)$ will be called a T-sequence if it has the property that the conditions (i) $a_1 + a_2 + \ldots$ is $C_1$-summable, and (ii) $|a_n| \leq v_n (n=1,2,\ldots)$, imply that $\Sigma a_n$ converges. It is known that

$$\lim_{b \to 0} \limsup_{N \to \infty} \sum_{n=N}^{N+b} v_n = 0$$

implies that $\{v_n\}$ is a T-sequence. Show that

a. If $\{v_n\}$ is a T-sequence, $\lim v_n = 0$, then (1) holds.

b. There exists a T-sequence that does not satisfy (1).

[20(1) 1955 Nr. 14]

It is known that the interval $(0,1)$ is not the union of a sequence of nowhere dense sets, nor the union of a sequence of sets of measure zero. Is it possible to dissect the interval into $S_1 \cup S_2 \cup \ldots$, such that each $S_i$ is the union of a nowhere dense set and a set of measure zero?

[20(1) 1955 Nr. 15]

If $f_1, f_2, \ldots$ is a sequence of measurable functions defined on $-\infty < x < \infty$ and $f_n(x) \to 0$ for all $x$, then a set $S$ is called regular whenever the convergence is uniform on $S$. A set $R$ is called a residue set whenever its complement $R'$ is the union of a countable number of regular sets. Egoroff's theorem shows that there exists a residue set of measure zero. Construct a sequence $f_1, f_2, f_3, \ldots$, satisfying the above conditions, such that every residue set is an everywhere dense set of power $c$ (the power of the continuum).

[20(1) 1955 Nr. 16]
Let $A_1, \ldots, A_m$ be impenetrable rigid bodies in $n$-dimensional euclidian space ($n > 1$). Assume that the bodies are convex and bounded. Show that it is possible to move the bodies such that $A_i$ tends to infinity whereas the others do not.

[20(1) 1955 Nr. 17]

If in the previous problem we moreover assume $n=2$, then there is at least one index $i$ such that $A_i$ can be moved to infinity without moving the others at all.

[20(1) 1955 Nr. 18]

A set $S$ of positive integers is called convex if $u \in S$, $w \in S$, $u/v$, $v/w$ always imply $v \in S$. Let $f$ and $g$ be functions defined on $S$. Show that the validity of one of the following relations for all $n \in S$ implies the validity of the other one for all $n \in S$:

$$f(n) = \sum_{d \in S} g(d), \quad g(n) = \sum_{d \in S} \mu(n/d) f(d).$$

Here $\mu$ indicates M"{o}bius' function.

[20(2) 1956 Nr. 53]

(For definitions see Nr. 53). Let $S$ be a finite convex set, consisting of the distinct positive integer $a_1, \ldots, a_n$. The matrix $T = (t_{ij})$ ($i, j = 1, \ldots, n$) be defined by $t_{ij} = 1$ if $a_j/a_i$, $t_{ij} = 0$ otherwise. Determine the determinant and the inverse of $T$.

[20(2) 1956 Nr. 54]
Let $S$ be a finite convex set consisting of the distinct positive integers $a_1, \ldots, a_n$. Let $f$ and $g$ be functions related according to Nr. 53. Let the matrix $M = (m_{ij})$ ($i, j = 1, \ldots, n$) be defined by $m_{ij} = f((a_i, a_j))$, where $(a_i, a_j)$ denotes the G.C.D. of $a_i$ and $a_j$. We define $f(a) = 0$ if $a$ is not in $S$.

Show that $\det M = g(a_1) \cdots g(a_n)$, and determine the inverse matrix of $M$.

[20(2)1956 Nr. 55]

Let $n$ and $k$ be integers, $1 \leq k \leq n$. Evaluate

$$\sum (-1)^{j_1 + j_2 + \cdots + j_k},$$

where the sum runs over all sets $j_1, \ldots, j_k$ which satisfy $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

[20(2)1956 Nr. 56]

Show that

$$\int_1^\infty e^{-x} \sum_{1 \leq k \leq [x]} \frac{x^n}{n!} \frac{dx}{x^2} = 1 - \gamma,$$

where $\gamma$ is Euler's constant.

[20(2)1956 Nr. 57]

Consider permutations $f(1), \ldots, f(n)$ of the elements $1, \ldots, n$ with the property that $f(k+1) - f(k) \neq 1$ ($k=1, \ldots, n-1$). Let $N(n,m)$ be the number of such permutations which moreover satisfy $f(1) = m$. Show that $N(n,2) = N(n,3) = \cdots = N(n,n)$.

[20(2) 1956 Nr. 58 jointly with P Erdős.]
13.

Show that the following statement is true for any sequence \( n_1, n_2, n_3, \ldots \) of integers \( > 1 \), if \( \lim_{k \to \infty} n_k = \infty \).

Let \( S \) be an infinite set, and assume that for any \( k (k=1,2,3,\ldots) \) there is a dissection \( S = \bigcup_{j=1}^{n_k} S_{kj} \) into non-empty disjoint subsets.

Show that it is possible to select, for each \( k \), an index \( m_k (1 \leq m_k \leq n_k) \) such that

\[
S = \bigcup_{k=1}^{\infty} S_{km_k}
\]

has infinitely many elements.

[20(2) 1956 Nr. 59 jointly with P. Erdős]

Show that the statement made in the preceding problem is false for any sequence \( n_1, n_2, \ldots \) that does not tend to infinity.

[20(2) 1956 Nr. 60 jointly with P. Erdős]

Let \( a_1, a_2, \ldots \) be a monotonic sequence of positive numbers, and let the sequence \( b_1, b_2, \ldots \) be a rearrangement of \( a_1, a_2, \ldots \). Assume that \( b_n + 1/b_n \) converges to a limit \( \ell \). Show that \( a_n + 1/a_n \) also tends to \( \ell \).

[20(3) 1957 Nr. 91]

Let \( n \) be a fixed positive integer, and abbreviate \( e^{2\pi i n x/n} = e(x) \).

Let the matrix \( (a_{ij}(t)) = A(t) \), for any complex number \( t \), be defined by

\[
a_{kj}(t) = n^{-1} \sum_{j=0}^{n-1} \{e(k+t)\}^j e(-kj) \quad (k, \ell=0,1,\ldots,n-1).
\]

Show that, for all \( s \) and \( t \), we have \( A(s)A(t) = A(s+t) \).

[20(3) 1957 Nr. 92]
If $A(t)$ is defined as in Nr. 92, then the mapping $t \to A(t)$ is a representation of the additive group of all complex numbers. Show that this representation is irreducible, and, if $n > 1$, also faithful.

[20(3)1957 Nr. 93]

If $|x| < 1$, show that

$$
\sum_{n=1}^{\infty} \frac{x^{n+1}}{1-x^n} = 0.
$$

[20(4)1958 Nr. 131]

If $|x| < 1$, show that

$$
\sum_{n=1}^{\infty} \frac{x^{n(n+1)}}{(1-x^n)} = \frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^6}{1-x^6} + \ldots
$$

[20(4)1958 Nr. 132]

Let $S_k$ be the $k$-th elementary symmetrical function of the variables $e^{2\pi it_1}, \ldots, e^{2\pi it_n}$ $(0 \leq k \leq n)$. Show that

$$
\int_{0}^{1} \ldots \int_{0}^{1} |S_k|^2 \cdot \prod_{1 \leq h < j \leq n} \sin^2 \pi(t_h - t_j) \cdot dt_1 \ldots dt_n = 2^{-n(n-1)} n!.
$$

[20(4)1958 Nr. 133]
Let the real sequence $a_1, a_2, a_3, \ldots$ have the property that to any $\varepsilon > 0$ there exists a number $k(\varepsilon)$ such that $k(\varepsilon) < n < m$ always implies $a_n - a_m < \varepsilon$. Show that $a_n$ can be written in the form $a_n = b_n + c_n$, where $b_n \to 0$, $b_n \to 0$, and $c_1 \leq c_2 \leq c_3 \leq \ldots$

[20(4)1958 Nr. 134]

Show that for sequences $\{\theta_n\}$, with $0 < \theta_n < 1$ ($n=1, 2, \ldots$), the following two conditions are equivalent:

1. There exists a real number $s$ such that $\sum_{n=1}^{\infty} e^{-s \lambda_n}$ converges

   (where $\lambda_n = (1 - \theta_n)^{-1}$)

2. If $a_1, a_2, \ldots$ are positive and such that $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \theta_n a_n$ converges as well.

[20(4)1958 Nr. 135 jointly with H. Freudenthal.]

Evaluate

$$\int_{-\infty}^{1} dx \int_{0}^{\infty} \frac{(1-x)^{\alpha + 2 \beta - 1} dy}{(2-x+\sqrt{x^2+y^2})^\gamma} \sqrt{x^2+y^2} \quad (\beta > 0, \alpha > -1, \alpha + 2\beta < \gamma).$$

[20(5)1959 Nr. 170]
In n-dimensional euclidean space $\mathbb{R}^n (n \geq 2)$ we take a two-dimensional plane $V$, which we consider as the complex plane. The unit of length in $\mathbb{R}^n$ equals the unit of length in the complex plane (thus, if $z \in V$, $z_2 \in V$, the distance of $z_1$ and $z_2$ is $|z_1 - z_2|$).

Let $\mathcal{O}$ be the zero point of $V$, and let $A_1, \ldots, A_n$ be points of $\mathbb{R}^n$ such that the vectors $OA_1, \ldots, OA_n$ form an orthonormal base for $\mathbb{R}^n$. The orthogonal projections of $A_1, \ldots, A_n$ onto $V$ produce the complex numbers $z_1, \ldots, z_n$. Show that $\sum_{k=1}^{n} z_k^2 = 0$, $\sum_{k=1}^{n} |z_k|^2 = 2$.

[21(1) 1960, Nr. 9, D]

As in the previous problem, let the complex plane $V$ be embedded in $\mathbb{R}^n (n \geq 2)$. Let $A_1, \ldots, A_{n+1}$ be the vertices of a regular simplex, and let the complex numbers $w_1, \ldots, w_{n+1}$ be their orthogonal projections on $V$. Show that

$$\frac{1}{n+1} \sum_{k=1}^{n+1} w_k^2 = \left( \frac{1}{n+1} \sum_{k=1}^{n+1} w_k \right)^2.$$

[21(1) 1960, Nr. 10, D]

In a non-equilateral triangle $ABC$ let $O$ be the orthocenter and $G$ the center of gravity. Let $P$ be a point on $OG$ with $\overline{OG} = p$, $\overline{OG}$ where $p$ is a real number. Denote the directions (i.e. the points at infinity) of $BC, CA, AB$ and $CP$ by $a, b, c$, and $l_p (a, b; c)$ respectively ($l_p$ only depends on the directions of $BC, CA$ and $AB$). Now prove:
If \( l_p(a,b;d) = c \) then \( l_p(a,b;\phi=d) = c \).

Remark. If \( p=\infty \) this is a theorem of P. Zeeman:

Each one of the four directions \( a, b, c, d \) is the direction of Euler's line in the triangles having sides parallel to the other three as soon as this is the case for one of them.

\[ 21(1)1960, \text{Nr. 11, D} \]

If \( a, \ldots, a_n \) are real, \( z_1, \ldots, z_n \) complex, then show that

\[
|\Sigma_{k=1}^{n} a_k z_k|^2 \leq \Sigma_{k=1}^{n} a_k^2 \cdot \left( \Sigma_{k=1}^{n} |z_k|^2 + |\Sigma_{k=1}^{n} z_k|^2 \right).
\]

\[ 21(1)1960, \text{Nr. 12, L} \]

Let \( p \) be a positive constant, and let \( c_n \) denote the coefficient of \( x^n \) in the power series expansion of \( e^{px} (x+1)(x+2) \ldots (x+n-1) (n=1,2,\ldots) \). Show that the series \( \Sigma_{n=1}^{\infty} c_n z^n \) has a positive radius of convergence \( \rho \), and that \( \rho \) is the smallest positive solution of the equation \( w^{-1} + \log w = p+1 \).

\[ 21(1)1960, \text{Nr. 13, L} \]

Let \( p \) be a prime of the form \( p=4v+1 \). Show that \( v^v \equiv 1 \pmod{p} \).

\[ 21(2)1961 \text{ Nr. 62, D} \]
Consider 2n distinct points on a circle. We want to split this set into n pairs, such that the n chords connecting the pairs do not intersect inside the circle. In how many ways is this possible?

[21(3)1962, Nr. 105, D]

Let a \(<\) b, and let f be a real function in the interval \(a \leq x \leq b\).

We assume that

\[
\limsup_{\delta \downarrow 0} f(x-\delta) \leq f(x) \quad (a < x \leq b),
\]

\[
0 \leq \limsup_{\delta \downarrow 0} \frac{f(x+\delta)-f(x)}{\delta} \leq \infty \quad (a \leq x < b).
\]

Show that \(f(a) \leq f(b)\).

[21(3)1962, Nr. 106]

Let \(a_1, a_2, a_3, \ldots\) be a sequence of positive numbers, with \(\lim_{n \to \infty} a_n = 0\). Let S be the set of all positive integers n with the property

\[
\prod_{p \mid n} p^n > n^{a_n}
\]

(p runs through the primes dividing n). Let \(\sigma(x)\) be the number of elements of S which do not exceed x. Show that \(\sigma(x)/x \to 1\) if \(x \to \infty\).

[21(3)1962, Nr. 107]
Let $C$ be a closed continuously differentiable Jordan curve in a plane, and let $F_1$ and $F_2$ be points inside $C$.

For every point $P$ on $C$ we consider the angle $\phi$ between the inside normal and the bisectrix of the angle $F_1 PF_2$. Show that

$$\int_C (r_1 r_2)^{\frac{1}{2}} \cos \phi \, d\sigma = 2\pi,$$

if $r_1 = PF_1$, $r_2 = PF_2$, and if $d\sigma$ denotes the line element of $C$.

[21(4)1963, Nr. 131, D]

Show by a counter-example that the following theorem is not true:

"Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be different norms in a linear space $M$. If a sequence $x_1, x_2, \ldots$ converges in both norms then the limits are equal".

[21(4)1963, Nr. 132, jointly with G.W. Veltkamp]

Evaluate

$$\sum_{n=1}^{\infty} \frac{\{J_n'(nx)\}^2}{x^n} \quad (0 \leq x < 1).$$

[21(5)1964, Nr. 168, jointly with C.J. Bouwkamp]
Let $k$ be a positive integer, and let $x_1, \ldots, x_k$ be real numbers satisfying $|x_1| + \ldots + |x_k| < \pi$. Show that
\[
\sum_{n=1}^{\infty} (\prod_{j=1}^{k} \sin nx_j \sin nx_k \ldots \sin nx_n)\]
has the value 0 if $k-j$ is an even integer with $0 < j < k$, and has the value $\frac{1}{2}x_1 \ldots x_k$ if $j=k$.

[21(5) 1964 Nr. 175]

Let $p$ be an odd prime. We want to construct an infinite sequence of integers $x_1, x_2, x_3, \ldots$ such that $x_1^2 \not\equiv 1 \pmod{p}$ and such that $2x_{n+1}x_n \equiv x_n^2 + 1 \pmod{p}$ for $n=1, 2, 3, \ldots$ Show that this construction is possible if and only if $p$ is not a Fermat prime.

[21(5) 1964 Nr. 176]

Let the function $\varphi$ satisfy
\[
\varphi(x) = 0 \quad (x < 0),
\quad \varphi(x) = x \quad (0 \leq x \leq 1),
\quad \varphi(x) - \varphi(x-1) = x\varphi(x) \quad (x > 0).
\]
Show that $\varphi(x)$ tends to a positive limit if $x \to \infty$.

[21(5) 1964 Nr. 177]

Show that the limit mentioned in the previous problem equals $e^\gamma$, where $\gamma$ is Euler's constant.

[21(5) 1964 Nr. 178]
Let $P$ be a polynomial of degree $n$. Show that for all $x$
\[ P(x) = \sum_{l} \sum_{k} (-1)^{n+l+k} \frac{x^{n}}{n} + l + k, \]
where the summation extends over all $l$, $k$ with $l \geq 0$, $k \geq 0$, $l + k \leq n$.

[22 (1) 1965, Nr. 8]

For each rational number $r$ in the interval $0 < r < 1$ we form
\[ f(r) = a^{-2} b^{-2}, \]
if $a/b$ is the irreducible fraction representing $r$. Show that the sum of all these $f(r)$'s equals $\frac{1}{4}$.

[22 (1) 1965, Nr. 9]

Let $\alpha(1), \alpha(2), \ldots$ be a sequence of complex numbers, and let
\[ A(n) = n^{-1} \sum_{d \mid n} \alpha(d) \]
(d runs through the divisors of $n$). Assume that $\sum_{n=1}^{\infty} \alpha(n)$ converges with sum $0$ and $\sum_{n=1}^{\infty} \alpha(n) \log n$ converges with sum $s$. Show that $\sum_{n=1}^{\infty} A(n)$ converges with sum $-s$.

[22 (1) 1965, Nr. 10]

Let $m$ be an integer $\geq 2$ and let $K$ be a finite field. Consider the $m$-dimensional projective geometry over $K$. Show that there exists a projective transformation $T$ with the property that both the set of all points and the set of all $(m-1)$-dimensional hyperplanes are permuted cyclically by $T$.

[22 (1) 1965, Nr. 11]
Let \( \varphi, \psi, \chi \) be bounded continuous functions of \( x, y, z \)
\((-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty)\). Show that the system
\[
\begin{align*}
x(x^2 + y^2 + z^2) &= \varphi(x, y, z) \\
y(x^2 + y^2 + z^2) &= \psi(x, y, z) \\
z(x^2 + y^2 + z^2) &= \chi(x, y, z)
\end{align*}
\]
has a solution.

[22 (1) 1965, Nr. 12]

Show that the equation
\[
|z|^5 - 2z^5 + z^4 - \text{Re } z = \alpha
\]
has at least one solution, for every complex number \( \alpha \).

[22 (1) 1965, Nr. 13]

Let \( T \) be the \( nxn \) matrix with first row 0100...0, second row
0010...0, ..., \((n-1)\)th row 00...01, and \( n \)th row
\((-1)^{n-1}\binom{n}{0}, (-1)^{n-2}\binom{n}{1}, ..., (-1)^0\binom{n}{n-1}\).
Determine the products \( T^k \) \((k=0, \pm 1, \pm 2, \ldots)\).

[22 (1) 1965, Nr. 14]

Find the Jordan canonical normal form of the matrix \( T \) of the
previous problem, and construct a matrix \( V \) that transforms \( T \)
into that normal form (by means of \( T \rightarrow V^{-1}TV \)).

[22 (1) 1965, Nr. 15]
Consider the complete graph of order \(n\), i.e. the graph with vertices \(P_1, \ldots, P_n\) connected in all possible ways (so there are \(\binom{n}{2}(n-1)\) edges). A subset \(T\) of the set of all edges is called a complete tree if it forms a tree with vertices \(P_1, \ldots, P_n\). According to Cayley there are \(n^{n-2}\) complete trees. Show that for each \(k\) (\(1 \leq k \leq n-1\)) there are \(\binom{n-2}{k-1}(n-1)^{n-k-1}\) complete trees with the property that the number of edges meeting at \(P_1\) equals \(k\).

[22 (1) 1965, Nr. 16]

Let \(p\) and \(q\) be real numbers, each chosen at random from the interval \((0,N)\). We define the sequence \(a_0, a_1, \ldots\) by \(a_0 = 1\), \(a_{n+1} = \exp(p \exp(qa_n))\). Let \(P_N\) be the probability that the sequence \(\{a_n\}\) converges. Show that \(P_N = 2K_2(2)N^{-2} + O(e^{-N^2})\) as \(N \to \infty\), where \(K\) stands for the modified Bessel function of the second kind.

[22 (1) 1965, Nr. 17]

Let the real function \(\psi\) be positive and increasing for \(0 < h < 1\), and \(\lim \psi(h) = 0\) if \(h \to 0\). Construct a real function \(f\), continuous on \((-\infty, \infty)\), such that there does not exist an \(x\) with the property that

\[
(f(x + h) - f(x))(\psi(h))^{-1}
\]

is bounded on \(0 < h < 1\).

[22 (1) 1965, Nr. 18]
Let the real function \( f \) be increasing and continuously differentiable on the interval \([0, \infty)\). Assume that \( f - \log f' \) is a concave function in that interval. Show that \( f \) itself is concave.

Let \( f \) and \( g \) be non-negative measurable functions on measure spaces \( X \) and \( Y \), respectively, and assume that \( \int_X f(x) \, dx = \int_Y g(y) \, dy \). Let \( C_{f,g} \) denote the following condition: for each measurable set \( E \subset Y \) we have

\[
\int_X \min(f, \mu(E)) \, dx \geq \int_E g(y) \, dy.
\]

\( C_{g,f} \) is obtained from \( C_{f,g} \) by interchanging the roles of \( X \) and \( Y \) and of \( f \) and \( g \). Show that \( C_{f,g} \) and \( C_{g,f} \) are equivalent.

Show the following asymptotic equivalence for \( x \to \infty \):

\[
\sum_{n=1}^{\infty} (-1)^{n-1} n^n - \log t - \log \log t \text{ tends to Euler's constant as } t \to \infty.
\]
Find positive constants $c_0$, $c_1$, $c_2$, ... such that

$$(\sum_{n=0}^{\infty} c_n x^n) / \Gamma(x)$$

tends to 1 if $x > 0$, $x \to \infty$. 

[22 (1) 1965, Nr. 23]
From 1965 onward the "Wiskundige opgaven met de oplossingen" ceased to exist as a separate journal.

Both the following problems and their solutions were published in the "Nieuw Archief voor Wiskunde". The references at the end of the problem indicate problem number, volume, year and page of publication of the problem, and possibly, volume, year and page of publication of the solution.

A function $\varphi$ is called "tame" if it is defined and positive for $x > 0$ and if

$$\lim_{\lambda \to 0} \lim_{x \to \infty} \sup_{x \leq t \leq \lambda x} \varphi(t) = 1.$$ 

Let $f$ be positive for $x > 0$ and integrable over every finite interval $0 < x < A$. Put $L(x) = x^{-1} \int_0^x f(t) \, dt$. Assume that

$$\lim_{x \to \infty} L(\lambda x)/L(x) = 1$$

for every $\lambda > 0$, and that either $f$ or $1/f$ is tame. Show that

$$f(x) \sim L(x) \quad (x \to \infty).$$

As usual in number theory $(m,n)$ denotes the g.o.d. of $m$ and $n$, $\mu$ denotes the Möbius function, $\delta_{i,j}$ the Kronecker delta. If $a,b,c$ are positive integers, put

$$S(a,b;c) = \sum_{d \mid a, (bd,c) = d \mu(d)}$$

(the summation extends over all divisors $d$ of $a$ which satisfy $(bd,c) = d$). Show that

$$S(a,b;c) = \mu((ab,c)\delta_{a,c},(b,c)).$$

Let the real function $K(x,y)$ be given for $0 \leq x < 1$, $0 < y < 1$, and assume that $K$, $\partial K/\partial x$, $\partial K/\partial y$, $\partial^2 K/\partial x \partial y$ are continuous. Moreover assume that $K$ is a non-negative kernel, i.e. $K(x,y) = K(y,x)$, and

$$\int_0^1 \int_0^1 K(x,y)f(x)f(y) \, dx \, dy \geq 0$$

for all real continuous functions $f$. Show that

$$K(\partial^2 K/\partial x \partial y) - (\partial K/\partial x)(\partial K/\partial y)$$

is also non-negative definite.


Let $\phi(x)$ be a twice differentiable real function satisfying $\phi(x) > 0$, $\phi''(x) \geq 0$ ($-\infty < x < \infty$). Let $y_1$ and $y_2$ be real solutions of the second order differential equation

$$y'' + (\phi(x))^{-2}y = 0.$$ 

Show that if $a < b$, $y_1(a) = y_1(b) = 0$, then $y_2$ has at least one zero in the interval $a < x < b$.


Show that, if $t \to +\infty$,

$$\Sigma_{n=1}^{\infty} \frac{(-t)^n}{n!} \log n - \log \log t$$

has the asymptotic expansion

$$-\gamma'(1) - \Sigma_{k=1}^{\infty} \Gamma(k) (1) (\log t)^{-k}.$$ 

Assume $a_1 = 1, 0 \leq a_k < 1$ ($k = 2, 3, \ldots$), $\sum_{k=1}^{\infty} a_k < \infty$. Put $S_1(n) = \sum_{v \geq n} a_v (n = 1, 2, \ldots)$, and define $S_2(n), S_3(n), \ldots$ by induction:

$$S_{k+1}(n) = \sum_{v \geq n} a_v S_k(v).$$

Determine $\lim_{k \to \infty} S_k(1)$.

[Nr. 64, 13 (1965), p. 120; 14 (1966), p. 66 - 67].

If $b$ is a fixed positive number, show that

$$\int_0^1 (1 - y)^n e^{-by} dy \sim n^{\frac{1}{2}} \left( \frac{\sqrt{\pi}}{\sqrt{b}} \right) e^{-b} - \frac{1}{2}(nb) \quad (n \to \infty).$$


Let $C$ be the space of all continuous real functions on the interval $(-\infty, \infty)$. If $f \in C$ we define its norm $\|f\|$ by

$$\|f\| = \sup_{x \in (-\infty, \infty)} |f(x + n)|.$$

Let $B$ be the set of all $f \in C$ with finite norm. Show that $B$ is a Banach space.


The space $S$ consists of all real functions $f$ defined on $-\infty < x < \infty$ with the property that $f$ has at most finitely many discontinuities, and that $f$ is constant in every interval where it is continuous. In $S$ we take as a norm

$$\|f\| = \max_{-\infty < x < \infty} |f(x)|.$$

Determine all bounded linear functionals on $S$.

Let \( m \) and \( n \) be relatively prime positive integers. Let, for each \( i (i = 0, \ldots, m-1) \), \( A_i \) be a cyclic \( n \times n \) matrix. Show that the partitioned matrix

\[
\begin{pmatrix}
A_0 & A_1 & \cdots & A_{m-1} \\
A_{m-1} & A_0 & A_1 & \cdots & A_{m-2} \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
A_1 & A_2 & \cdots & A_0
\end{pmatrix}
\]

can be written in the form \( CPC^{-1} \), where \( C \) is an \( mn \times mn \) cyclic matrix and \( P \) is a permutation matrix.

\[ [N r. 82, 13 (1965), p. 233; 14 (1966), p. 149-150]. \]

If \( N \geq 1 \) we put \( S_k = \sum_{n=0}^{N-1} e^{-\frac{n}{N} k} \) \((k = 1, 2, \ldots)\). Show that

\[
\sum_{k=1}^{\infty} k |S_k - S_{k-1}| = O(N).
\]

\[ [N r. 83, 13 (1965), p. 233; 14 (1966), 150-151]. \]

Derive the following with the aid of the result of the previous problem: If \( a_0, a_1, \ldots \) is a sequence of real numbers with \( \lim_{N \to \infty} (N-1) \sum_{n=0}^{N-1} a_n = A \) and if \( b_n = \sum_{k=0}^{\infty} e^{-\frac{n}{N} k} a_k / k! \), then we have

\[
\lim_{N \to \infty} (N-1) \sum_{n=0}^{N-1} b_n = A.
\]

\[ [N r. 84, 13 (1965), p. 234; 14 (1966), p. 151-152]. \]

If \( x \) runs through the real numbers, \( x \geq 1 \), show that

\[
\sum_{n=1}^{\infty} n < n^{-1} \log(1 + nx^{-1}) = \frac{\pi^2}{12} + O(x^{-1}).
\]

If \( \varphi \) denotes Euler's indicator, show that
\[
\sum_{k=1}^{N} k^{-2} \varphi(k) \log(1 + k/N) = \frac{1}{2} + O(N^{-1} \log N) \quad (N \to \infty).
\]


Assume \( a > 0, 0 < \lambda < 1 \). If \( 0 < b < a \) we define \( N_b(t) \) on the interval \( 0 < t < b \) by
\[
N_b(t) = (a - b)^{1-\lambda} (b - t)^{\lambda-1} (a - t)^{-1}.
\]

Let \( f \) be an integrable function on the interval \( 0 < x < a \), and assume that the left-hand limit \( f(a - 0) \) exists. Show that
\[
\lim_{b \to a} \int_{0}^{b} f(t) \, N_b(t) \, dt = \pi (\sin n\lambda)^{-1} f(a - 0).
\]


Let \( p \) be a positive constant and let \( a_1, a_2, \ldots \) be a sequence of real numbers such that \( \lim_{N \to \infty} N^{-1} \sum_{k=1}^{N} a_k \) exists. For each \( n \geq 1 \) we define \( b_n \) as the average of \( a_{k'} \) taken over all \( k \) for which \( n - pn^{\frac{1}{2}} \leq k \leq n + pn^{\frac{1}{2}} \).

Show that \( \lim_{N \to \infty} N^{-1} \sum_{k=1}^{N} b_k \) exists.


Assume \( c > 0, 0 < \lambda < 1 \). Let the real function \( f \) be defined on the interval \( 0 < x < c \). Put \( f(x) = x^{-\lambda} g(x) \), assume that \( g(x) \) has bounded variation on \( 0 < x < c \) and that \( g(0^+) = 0 \). Define \( F \) by \( F(0) = 0 \) and
\[
F(x) = \int_{0}^{x} f(t) (x-t)^{\lambda-1} \, dt \quad (0 < x < c).
\]

Show that on \( 0 < x < c \) the function \( F \) is continuous and of bounded variation.

(Continuation of no. 165). Show (with the aid of no. 165 that for \(0 < x < c\) we have for the left-hand limit

\[ f(x-0) = \pi^{-1} \sin n\lambda \int_{0}^{x} (x-y)^{-\lambda} dF(y), \]

where the integral is an improper Stieltjes integral (to be taken in the sense of \(\lim_{b \to x} \int_{0}^{b} \))

If in no. 165 the assumption is added that \(g\) is continuous on \(0 < x < c\) and continuously differentiable on \(0 < x < c\), then show that \(F\) is continuously differentiable \((0 < x < c)\), that \(F(0) = 0\) and that

\[ f(x) = \pi^{-1} \sin n\lambda \int_{0}^{x} (x-y)^{-\lambda} F'(y) \, dy \quad (0 < x < c). \]

Let \(g(x)\) be absolutely continuous \((0 < x < c)\), and \(g(0) = 0\). Put \(x^{-\lambda} g(x) = f(x)\). Show that Abel's integral equation

\[ \int_{0}^{x} (x-y)^{-\lambda} \varphi(y) \, dy = f(x) \]

has a solution \(\varphi\) which is integrable on \([0, c]\), and even such that for every \(x(0 < x < c)\) the function \((x-y)^{-\lambda} \varphi(y)\) is an integrable function of \(y\) on \(0 < y < x\).

Let \(B_n\) be the number of equivalence relationships that can be defined on a set of \(n\) points. (So \(B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, B_7 = 877, \ldots\).) Show that \(B_n\) is even if and only if \(n \equiv 2 \pmod{3}\).
Let \( n \) be an even natural number. Evaluate the determinant of the following \( n \times n \) matrix

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & -1 & 1 \\
-1 & 0 & 1 & \ldots & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & \ldots & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & -1 & 0 & 1 & 1 \\
1 & 0 & \ldots & 0 & -1 & 0 & 1 \\
-1 & -1 & \ldots & -1 & -1 & 0 & 0
\end{pmatrix}
\]

(the matrix formed by the first \( n-1 \) rows and \( n-1 \) columns is cyclic).

[Nr. 188, 16 (1968), p.125; 17(1969)p. 75-77,D].

A conic \( K \) passes through the points \( P_1, \ldots, P_4, Q_1, \ldots, Q_4 \). The pencil of conics through \( P_1, \ldots, P_4 \) contains a conic touching both \( Q_1 Q_2 \) and \( Q_3 Q_4 \). Show that this pencil also contains a conic touching both \( Q_1 Q_3 \) and \( Q_2 Q_4 \), as well as a conic touching both \( Q_1 Q_4 \) and \( Q_2 Q_3 \). The six points of contact lie on a single straight line.

[Nr. 189, 16 (1968), p.125, 17 (1969) p.77-78,D].

Let \( n \) be a positive integer, and let \( f \in L_2(\mathbb{R}^n) \). Let \( g \) be its Fourier transform, defined by

\[
\| g - g_T \| \to 0 \quad \text{as} \quad (T \to \infty)
\]

where

\[
g_T(x_1, \ldots, x_n) = \int_{-T}^{T} \cdots \int_{-T}^{T} \exp(-2\pi i (x_1 y_1 + \ldots + x_n y_n)) \times f(y_1, \ldots, y_n) \, dy_1 \cdots dy_n.
\]
Show that

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1^2 + \cdots + x_n^2) \left( |f(x_1, \ldots, x_n)|^2 + |g(x_1, \ldots, x_n)|^2 \right) \cdot dx_1 \cdots dx_n \geq \frac{n}{2\pi} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_n)|^2 dx_1 \cdots dx_n.
\]

When does the equality sign hold?


Prove the following n-dimensional extension of Weyl's uncertainty relation (using the notation of problem No. 221):

\[
\Delta(f) \Delta(g) \geq n(4\pi)^{-1}
\]

with

\[
\Delta(f) := \left( \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1^2 + \cdots + x_n^2) |f(x_1, \ldots, x_n)|^2 dx_1 \cdots dx_n}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_n)|^2 dx_1 \cdots dx_n} \right)^{\frac{1}{2}}
\]

and a similar definition for \( \Delta(g) \). When does the equality sign hold?


Let \( S_n \) be the closed unit sphere in n-space (n > 1). Assume \( S_n \subset C_1 \cup \cdots \cup C_k \), where \( C_1, \ldots, C_k \) are measurable cylinders. These measurable cylinders \( C_i \) are obtained from any measurable set \( B_i \) in any \((n-2)\)-dimensional hyperplane, if we take all points \( x + tb_i \), where \( b_i \) is a unit
vector perpendicular to that hyperplane, \( x \in B_i \) and \( t \) runs through the reals. By "cross section" of that cylinder we denote the \((n-1)\)-dimensional volume of \( B_i \).

Show that the sum of the cross sections of \( C_1, \ldots, C_k \) is not less than the \((n-1)\)-dimensional volume of \( S_{n-1} \).

\[ [\text{Nr. 232, } 17(1969) \text{ p. 225;} \]
\[ 19(1971), \text{ p. 79-80}.] \]

Let \( X \) and \( Y \) be topological spaces, and let \( X \) be compact. Let \( U \) be a subset of the cartesian product \( X \times Y \). We put

\[
V(x) := \{ y \in Y \mid (x,y) \in U \} \quad (x \in X),
\]

\[
W(y) := \{ x \in X \mid (x,y) \in U \} \quad (y \in Y).
\]

Assume that \( V(x) \) is an upper semi-continuous function of \( x \). (This means that for every \( x_0 \in X \) and every open set \( Q \) with \( V(x_0) \subset Q \subset Y \) there exists an open set \( P \) with \( x_0 \in P \subset X \) such that \( V(x) \subset Q \) for all \( x \in P \).) Moreover assume that for every \( x \in X \) and for every \( y \in Y \) either \( y \in V(x) \) or there exist disjoint open sets \( Q_1, Q_2 \) with \( V(x) \subset Q_1, y \in Q_2 \).

Show that \( W(y) \) is an upper semi-continuous function of \( y \), and that \( W(y) \) is closed for every \( y \in Y \). (The special case that \( Y \) is Hausdorff, and that \( U \) is the graph of a continuous injection of \( X \) into \( Y \), is a well-known theorem).

\[ [\text{Nr. 263, } 18(1970) \text{ p. 295;} \]
\[ 19(1971), \text{ p. 167-168}.] \]

From an urn containing \( k \) black balls and \( k \) red balls, all \( 2k \) balls are drawn out, one at a time, without replacement. Each time, before drawing a ball, one guesses the colour of the ball to be drawn. We want to optimize the number of correct guesses. The best strategy is: always take as a guess the colour which has the majority in the urn, guessing at random if there are as many black balls as red balls. Determine the expectation of the number of correct guesses under this strategy.

\[ [\text{Nr. 273, } 19(1971) \text{ p. 78;} \]
\[ 19(1971), \text{ p. 229-232, D}]. \]
Let $p$ and $q$ be integers, both $\neq 0$, and assume $p$ and $q$ to be relatively prime. Let $C_{p,q}$ denote the curve in 3-space, given by:

$$\begin{align*}
x &= (2 - \cos pt) \cos qt \\
y &= (2 - \cos pt) \sin qt \\
z &= \sin pt
\end{align*}$$

(0 \leq t \leq 2\pi).

Show that $C_{p,q}$ and $C_{q,p}$ belong to the same isotopy type. (Note that $C_{3,2}$ and $C_{3,-2}$ are clover leaf knots).


Let $I$ be an open interval on the real line, and let $f$ be a real function on $I$. Show that the following two conditions are equivalent:

(i) For all $u,v,w \in I$ we have

$$|\Delta_{u,v,w}| \leq |(u-v)(v-w)(w-u)|,$$

where

$$\Delta_{u,v,w} := \begin{vmatrix} f(u) & f(v) & f(w) \\ u & v & w \end{vmatrix}.$$  

(ii) $f$ is differentiable on $I$, and for all $u,v \in I$ we have

$$|f'(u) - f'(v)| \leq 2|u-v|.$$  

[Nr. 337, 21 (1973) p.97; 21 (1973) p.228-290].

Players P and Q play a game, of which the rules are determined by positive integers $k, l, m$. There is a countable set of markers, labelled 1, 2, 3, ... P and Q move alternately; P moves first. Each move of P consists of taking $k$ markers, and each move of Q consists of taking $l$ markers. P has won as soon as his set of markers contains a sequence of $m$ consecutive integers. Determine all cases $(k,l,m)$ where P has a winning strategy.

Let $u$ be a continuous function on $[0, \infty)$. Put
\[ p(x) := \int_0^x u(t) \, dt \quad u(x). \]
We assume that $x^{-1} \int_0^x u(t) \, dt \to \infty$, and that $\lambda$ is a real number, $\lambda < 1$, with $p(x) = 0(x^\lambda)$, $(x \to \infty)$. Show $u(x) = 0(x^{\lambda-1})$, $(x \to \infty)$.

Let $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ be sequences of elements of $\mathbb{N} = \{1, 2, 3, \ldots\}$. Show the existence of $a_1, a_2, \ldots \in \mathbb{N}$ such that every $k \in \mathbb{N}$ there are exactly $n_k$ values of $i \in \mathbb{N}$ with $a_i = k$ and exactly $m_k$ values of $j \in \mathbb{N}$ with $|a_{i+1} - a_i| = k$. (The special case that $n_1 = n_2 = \ldots = 1$ is due to P.J. Slater and W.Y. Vélez, Pacific J. Math. 71 (1977) 193-196.)

Nr. 515, 26 (1978) p. 463;
Let $u$ be a continuous function on $[0, \infty)$. Put

$$p(x) := \int_0^x u(t) \, dt - xu(x).$$

We assume that $x^{-1} \int_0^x u(t) \, dt \to 0 \; (x \to \infty)$, and that $\lambda$ is a real number, $\lambda < 1$, with $p(x) = O(x^{\lambda-1})$, $(x \to \infty)$. Show that $u(x) = O(x^{\lambda})$, $(x \to \infty)$.


Let $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ be sequences of elements of $\mathbb{N} (= \{1, 2, 3, \ldots\})$. Show the existence of $a_1, a_2, \ldots \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there are exactly $n_k$ values of $i \in \mathbb{N}$ with $a_i = k$ and exactly $m_k$ values of $j \in \mathbb{N}$ with $|a_{j+1} - a_j| = k$.

(The special case that $n_1 = n_2 = \ldots = 1$ is due to P.J. Slater and W.Y. Velez, Pacific J. Math. 71 (1977) 193-196).