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by

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Theorem on the existence of solutions of quasi-static moving boundary problems

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Using the theory of conformal mappings, we show that two-dimensional quasi-static moving boundary problems can be described by a non-linear Löwner-Kufarev-equation and a functional relation $F$ between the shape of the boundary and the velocity at the boundary. Together with the initial data this leads to an initial value problem (i.v.p.). Assuming that $F$ satisfies certain conditions, we prove a theorem stating that this i.v.p. has a local solution in time. The proof is based on some straightforward estimates on solutions of Löwner-Kufarev-equations and an iteration technique.

1. Introduction

Consider a clump of matter in the two-dimensional plane which is moving according to its hydrodynamical velocity $v$. The geometric shape evolution is determined by the normal component $v_n$ of this velocity. Assume that the domain $G_t$ occupied by the matter at time $t$ is simply connected and includes $0$ for all time $t$. Then, the unique conformal mapping $f(\cdot, t)$, mapping the open unit disc $D \subset C$ to $G_t$ and normalized by:

$$f(0, t) = 0; \quad f'(0, t) > 0$$

satisfies the Löwner-Kufarev-equation ([1], [2]):

$$\dot{f}(\zeta, t) = f'(\zeta, t)h(\zeta, t)$$

(with: $\cdot = \frac{\partial}{\partial t}, \quad ' = \frac{\partial}{\partial \zeta}$). The regular function $h$ on $D$ hereby is defined by the relations:

$$\operatorname{Re} h(\zeta, t) \bigg|_{\partial D} = v_n(f(\zeta, t))/|f'(\zeta, t)| \bigg|_{\partial D}, \quad \operatorname{Im} h(0, t) = 0.$$  

The relation (2) can best be understood by identifying $f$ and $f = (f_1, f_2)$:

$$\dot{f}(\zeta, t) \cdot n(f(\zeta, t)) = \operatorname{Re} \frac{\dot{f}(\zeta, t)f'(\zeta, t)\overline{\zeta}}{|f'(\zeta, t)|} = \operatorname{Re} f'(\zeta, t) \cdot h(\zeta, t) = v_n(f(\zeta, t)).$$

So to speak, the velocity $v$ is replaced by a “regular” velocity with a normal component that equals $v_n$ – assuring that the shape evolution is not modified –
and such that its pull-back under \( f \) is a regular function.
There are several problems in physics ([3], [4], [5]) of the following form: the hydrodynamical velocity of a clump of matter occupying a simply connected domain \( G \) is determined by the geometric shape of this domain \( G \). Given the domain \( G_0 \) at time \( t = 0 \), what is the domain \( G_t \) at time \( t > 0 \)? We call such problems quasi-static moving boundary problems. Formulating the problem mathematically, there is a mapping \( \mathcal{F} \) from the space \( S \) of univalent functions on \( D \) to the space of regular functions on \( D \):

\[
\mathcal{F} : h \in S \rightarrow \mathcal{F}(h) = f[h]
\]

(5)

(see also the relations (3)). We are interested under which conditions on \( \mathcal{F} \) the i.v.p. given by:

\[
\begin{align*}
\dot{f}(\zeta, t) &= f'(\zeta, t) h[f(\zeta, t)](\zeta) \\
f(\zeta, 0) &= f_0(\zeta)
\end{align*}
\]

(6a)

(6b)
can be solved (with \( f_0 : D \rightarrow G_0 \) conformal). In section 2 we will prove some estimates on the solution \( f \) of equation (2). These estimates are used to prove a theorem in section 3 where sufficient conditions on \( \mathcal{F} \) are stated for the i.v.p. (6) to have a local solution.

2. Preliminary Results

First we recapitulate an old result on the Löwner–Kufarev equation (2). We deal with functions \( h \) on \( D \times [0, \infty) \) with the following properties:

- \( h \) is continuous on its domain
- \( h \) is regular with respect to its first (complex) variable
- \( \Re h(\zeta, t) \leq 0 \); \( \Im h(0, t) = 0 \) (for all \( \zeta \in D \) and all \( t \geq 0 \)).

So we only consider problems where the velocity at the boundary is pointing inwards.

Theorem. Let the function \( h \) have the above stated properties. The i.v.p. given by the equations (2) and (6b) has an unique solution. This function \( h(\cdot, t) \) is univalent for all \( t \geq 0 \).

Proof. We will not prove this theorem (see [6], [7], [8]) but only show the main ideas. In the i.v.p. given by:

\[
\begin{align*}
\dot{\varphi}(\zeta, t) &= -\varphi(\zeta, t) h(\varphi(\zeta, t), t) \\
\varphi(\zeta, 0) &= \zeta
\end{align*}
\]

(7)

the \( \zeta \in D \) can be viewed as a parameter. Applying Picard's theorem ([9]) in its simplest form, one shows that the problem (7) has a unique maximal solution. One shows that for each \( t \geq 0 \) the function \( \varphi(\cdot, t) \) is regular (as it can be shown that it is the limit of a sequence of regular functions -- coming
from Picard's iteration - which converge uniformly on compacta). Moreover, as \( \varphi(\cdot,t) \) is univalent (as trajectories do not cross each other), there is a regular, univalent inverse \( \varphi^{-1}(\cdot,t) \) such that:

\[ \varphi^{-1}(\varphi(\zeta,t),t) = \zeta. \]  

(8)

To show that this so defined inverse function \( \varphi^{-1} \) has a domain \( D \times [0,\infty) \), one considers the time-inversed problem; i.e. for each \( T \geq 0 \) one considers the i.v.p.:

\[
\begin{align*}
\dot{\psi}_T(\zeta,t) &= \psi_T(\zeta,t)h(\psi_T(\zeta,t),T-t) \\
\psi_T(\zeta,0) &= \zeta.
\end{align*}
\]  

(9)

It is easy to show that the modulus of the solution \( \psi_T(\zeta,t) \) is monotonously non-increasing with time. It follows that \( \psi_T(\cdot,t) \) has a domain \( D \). Finally one establishes the relation:

\[ \varphi^{-1}(\zeta,t) = \psi_t(\zeta,t) \]  

(10)

and checks by differentiation that:

\[ f(\zeta,t) = f_0(\varphi^{-1}(\zeta,t)) \]  

(11)

satisfies the i.v.p.

\[ \square \]

We proceed by making some estimates on the solution \( f \). First of all, it is remarked that the solution \( \psi_T \) of the i.v.p. (9) can be written as:

\[ \psi_T(\zeta,t) = \zeta + \int_0^t \psi_T(\zeta,\tau)h(\psi_T(\zeta,\tau),T-\tau)d\tau. \]  

(12)

This leads immediately to:

**Lemma 1.** If:
- \( f_0 \) has a bounded derivative
- \( h \) is bounded on \( D \times [0,t] \) for each \( t \geq 0 \),

then the solution \( f \) of the i.v.p. satisfies:

\[
\sup_{\zeta \in D} |f(\zeta,t) - f_0(\zeta)| \leq t \sup_{\zeta \in D} |f'_0(\zeta)| \max_{\tau \in [0,t]} \{ \sup_{\zeta \in D} |h(\zeta,\tau)| \}. \]  

(13)

**Proof.** Straightforward from the relations (10),(11) and (12).

\[ \square \]

Next we remark that the real quantity \( |\psi_T(\zeta,t)| \) can be bounded as follows. Let \( M_0 \) be a continuous function on \([0,1] \times [0,\infty)\) which is Lipschitz-continuous with respect to its first variable and is such that:

\[
\sup_{|k|<r} \text{Re} h(\zeta,t) \leq M_0(r,t) \leq 0. \]  

(14)
Let $r_T$ for a fixed, arbitrary $T \geq 0$ on $[0, T]$ be defined as the solution of the following i.v.p.:

$$
\begin{align*}
\dot{r}_T(t) &= r_T(t)M_0(r_T(t), T-t) \\
r_T(0) &= 1.
\end{align*}
$$

(15)

We define: $r(t) = r_1(t)$.

**Lemma 2.** For all $T \geq 0$, $t \in [0, T]$ and all $\zeta \in D$, the function $\psi_T$ satisfies:

$$
|\psi_T(\zeta, t)| \leq r_T(t) \leq 1.
$$

(16)

**Proof.** Assuming the statement is false, one easily derives a contradiction using the mean value theorem. \qed

Next we prove a lemma that is interesting in its own.

**Lemma 3.** Under the conditions that:

- $f_0$ has a bounded derivative
- there exists a continuous function $M_1$ on $[0, 1] \times [0, \infty)$ such that:

$$
\sup_{|\zeta|<r} \text{Re}\{h(\zeta, t) + h'(\zeta, t)\zeta\} \leq M_1(r, t),
$$

the derivative $f'$ of the solution $f$ of the i.v.p. is bounded:

$$
\sup_{\zeta \in D} |f'(\zeta, t)| \leq \sup_{|\zeta|<r} |f_0(\zeta)|e^a.
$$

(17)

(18)

**Proof.** Differentiating the relations (9) with respect to $\zeta$ we get:

$$
\begin{align*}
\psi_T(\zeta, t) &= \psi_T'(\zeta, t)\{h(\psi_T(\zeta, t), T-t) + h'(\psi_T(\zeta, t), T-t)\psi_T(\zeta, t)\} \\
\psi_T(\zeta, 0) &= 1.
\end{align*}
$$

(19)

Regarding $\psi_T$ as a known function and regarding $T$ as a parameter, one checks that the solution of this i.v.p. (19) can be written as:

$$
\psi_T(\zeta, t) = e^a 
$$

(20)

Using the previous lemma we get:

$$
\sup_{\zeta \in D} |\psi_T''(\zeta, t)| \leq e^a.
$$

(21)

Substituting this result for $T = t$ in the relation (11), inequality (18) follows. \qed
Remarks. It is clear from the proof that the conditions on $M_1$ can be weakened; it is already sufficient that $M_1(1, \cdot)$ is locally integrable for $f'(\cdot, t)$ to be bounded.

Starting with differentiating the relations (9) $k$ times to $\zeta$, one generalizes this result easily by deducing conditions on $h$ and $f_0$ for $f^{(k)}$ to be bounded (together with the theorem of Kellog-Warschanski ([10]) such results are practical for applications).

Now we are able to prove a lemma which generalizes lemma 1.

**Lemma 4.** Under the conditions stated in the lemma's 1 and 3, the solution $f$ of the i.v.p. satisfies for all $t_1 \geq t_2 \geq 0$:

$$
\sup_{\zeta \in D} |f(\zeta, t_1) - f(\zeta, t_2)| \leq \int_0^{t_2} M_1(\tau, t - \tau) d\tau \cdot \max_{\tau \in [t_2, t_1]} \left\{ \sup_{\zeta \in D} |h(\zeta, \tau)| \right\}.
$$

(22)

**Proof.** It follows from the theory of ordinary differential equations that:

$$
\psi_T - \psi_T(\zeta, t, \tau) = \psi_T(\zeta, t + \tau)
$$

(23)

for all $\zeta \in D$ and all $T, t, \tau \geq 0$ such that $t + \tau \leq T$. Using this relation for the case: $T = t_1$, $t = t_1 - t_2$ and $\tau = t_2$, we find:

$$
\sup_{\zeta \in D} |\psi_{t_1}(\zeta, t_1) - \psi_{t_2}(\zeta, t_2)| = \sup_{\zeta \in D} |\psi_{t_2}(\psi_{t_1}(\zeta, t_1 - t_2), t_2) - \psi_{t_2}(\zeta, t_2)| = \sup_{\zeta \in D} \left| \int_{t_1}^{t_2} \psi'_2(\zeta, t_2) d\zeta \right| \leq \sup_{\zeta \in D} \left| \psi_{t_1}(\zeta, t_1 - t_2) - \zeta \right| \cdot e^0 \int_0^{t_2} M_1(\tau, t - \tau) d\tau
$$

(24)

where we used lemma 3. The first term of the right-hand side can be estimated by (see relation (12)):

$$
\sup_{\zeta \in D} |\psi_{t_1}(\zeta, t_1 - t_2) - \zeta| \leq (t_1 - t_2) \cdot \max_{\tau \in [t_2, t_1]} \left\{ \sup_{\zeta \in D} |h(\zeta, \tau)| \right\}.
$$

(25)

So, we conclude:

$$
\sup_{\zeta \in D} |f(\zeta, t_1) - f(\zeta, t_2)| \leq \sup_{\zeta \in D} |f_0(\psi_{t_1}(\zeta, t_1)) - f_0(\psi_{t_2}(\zeta, t_2))| \leq \int_0^{t_2} M_1(\tau, t - \tau) d\tau \cdot \max_{\tau \in [t_2, t_1]} \left\{ \sup_{\zeta \in D} |f(\zeta, \tau)| \right\}.
$$

(26)
We end this section with a lemma on the solutions \( f_1 \) and \( f_2 \) of the i.v.p.'s given by:

\[
\begin{align*}
\dot{f}_i(\zeta, t) &= f'_i(\zeta, t)h_i(\zeta, t)\zeta \\
f_i(\zeta, 0) &= f_0(\zeta)
\end{align*}
\]  

(27)

where both \( h_1 \) and \( h_2 \) satisfy the conditions mentioned in the beginning of this section. The corresponding functions \( \psi_T \) (see the relations (9)) are denoted by \( \psi_{1/2}, \ i = 1, 2 \).

**Lemma 5.** Under the conditions that

- \( f_0 \) has a bounded derivative
- \( h_1 \) and \( h_2 \) are bounded on \( D \times [0, t] \) for each \( t \geq 0 \),

the solutions \( f_{1,2} \) of the i.v.p.'s (27) satisfy:

\[
\sup_{\zeta \in D} |f_1(\zeta, t) - f_2(\zeta, t)| \leq t \sup_{\zeta \in D} |f'_0(\zeta)| \cdot \max_{r \in [0,t]} \left\{ \sup_{\zeta \in D} |h_1(\zeta, r) - h_2(\zeta, r)| \right\} \cdot \left( \max_{r \in [0,t]} \left\{ \sup_{\zeta \in D} |h_1(\zeta, r)| + \sup_{\zeta \in D} |h_2(\zeta, r)| \right\} \right)
\]

(28)

**Proof.**

\[
\begin{align*}
\sup_{\zeta \in D} |\psi_{1/2}(\zeta, t) - \psi_{1/2}(\zeta, t)| &= \\
\sup_{\zeta \in D} \left| \int_0^t \left\{ \psi_{1/2}(\zeta, \tau)h_1(\psi_{1/2}(\zeta, \tau), T - \tau) - \psi_{1/2}(\zeta, \tau)h_1(\psi_{1/2}(\zeta, \tau), T - \tau) \right\} d\tau \right| \leq \\
\int_0^t \sup_{\zeta \in D} |\psi_{1/2}(\zeta, \tau)| \cdot |h_1(\psi_{1/2}(\zeta, \tau), T - \tau) - h_1(\psi_{1/2}(\zeta, \tau), T - \tau)| d\tau + \\
\int_0^t \sup_{\zeta \in D} |\psi_{1/2}(\zeta, \tau)| \cdot |h_1(\psi_{1/2}(\zeta, \tau), T - \tau) - h_2(\psi_{1/2}(\zeta, \tau), T - \tau)| d\tau + \\
\int_0^t \sup_{\zeta \in D} |\psi_{1/2}(\zeta, \tau) - \psi_{1/2}(\zeta, \tau)| \cdot |h_2(\psi_{1/2}(\zeta, \tau), T - \tau)| d\tau \leq \\
\int_0^t \sup_{\zeta \in D} |\psi_{1/2}(\zeta, \tau) - \psi_{1/2}(\zeta, \tau)| \cdot \left( \sup_{\zeta \in D} |h'_1(\zeta, T - \tau)| + \sup_{\zeta \in D} |h_2(\zeta, T - \tau)| \right) d\tau + \\
\int_0^t \sup_{\zeta \in D} |h_1(\zeta, T - \tau) - h_2(\zeta, T - \tau)| d\tau .
\end{align*}
\]

(29)
As the last term can be estimated by:

\[
\int_0^1 \sup_{\zeta \in D} |h_1(\zeta, T - \tau) - h_2(\zeta, T - \tau)|d\tau \leq T \max_{\tau \in [0, T]} \left( \sup_{\zeta \in D} |h_1(\zeta, \tau) - h_2(\zeta, \tau)| \right),
\]

we get from inequality (29) and the lemma of Grönwall:

\[
\sup_{\zeta \in D} |\psi_f^1(\zeta, t) - \psi_f^2(\zeta, t)| \leq T \max_{\tau \in [0, T]} \left\{ \sup_{\zeta \in D} |h_1(\zeta, \tau) - h_2(\zeta, \tau)| \right\} \ast \tau \left( \max_{\tau \in [0, T]} \left( \sup_{\zeta \in D} |h_1'(\zeta, \tau)| + \sup_{\zeta \in D} |h_2'(\zeta, \tau)| \right) \right\}.
\]

By substituting this result in the relation:

\[
f_1(\zeta, t) - f_2(\zeta, t) = \int_{\psi_f^1(\zeta, t)}^{\psi_f^2(\zeta, t)} f_0'(z) \, dz.
\]

inequality (28) follows.

3. Final Result

We return to the i.v.p. (6). Let \( S \) as in section 1 denote the space of all conformal mappings on \( D \), and let \( T \) be the space of regular functions \( h \) on \( D \) such that:

\[
\text{Re } h \leq 0 \quad ; \quad \text{Im } h(0) = 0 \quad ; \quad h' \text{ is bounded}.
\]

A mapping \( \mathcal{F} : S \rightarrow T \) is called Lipschitz-continuous if there exists a \( K \) such that for all \( f_1, f_2 \in S \):

\[
\sup_{\zeta \in D} |h_{[f_1]}(\zeta) - h_{[f_2]}(\zeta)| \leq K \sup_{\zeta \in D} |f_1(\zeta) - f_2(\zeta)|.
\]

A mapping \( \mathcal{F} : S \rightarrow T \) is called continuous with respect to the derivative if the functional

\[
\mathcal{F}' : f \in S \rightarrow \sup_{\zeta \in D} |h_{[f]}'(\zeta)|
\]

is continuous.

**Theorem.** Let the mapping \( \mathcal{F} : S \rightarrow T \) have the above stated properties in a neighbourhood of a function \( f_0 \in S \) which has a bounded derivative. Then the i.v.p. (6) has a local solution. The function \( f(\cdot, t) \) is univalent.

**Proof.** Step 1. At this stage we do not bother whether the functions \( f_n \) to be defined are in a neighbourhood of \( f_0 \). Let \( f_n \) for each \( n \in \mathbb{N} \) be defined by:

\[
\begin{align*}
  f_0(\zeta, t) &= f_0(\zeta) \\
  f_{n+1}(\zeta, t) &= f_{n+1}(\zeta, t) h_{[f_n]}(\zeta)(\zeta) \\
  f_{n+1}(\zeta, 0) &= f_0(\zeta).
\end{align*}
\]
It follows from the theorem stated in section 2 that the functions $f_n$ are properly defined on $D \times [0, \infty)$ if the functions $h_n(\zeta, t) := h_{y,n}(\zeta)\zeta$ have the following properties:

- $h_n$ is continuous
- $\sup_{\zeta \in D} |h'_n(\zeta, t)|$ is a continuous function of $t$.

We show by induction that $h_n$ has indeed this properties. As $f_0$ does not depend on $t$, neither does $h_0$ and therefore $h_0$ has these properties. Now assume that $h_n$ have the mentioned properties. This implies that $f_{n+1}$ is properly defined, i.e. exists and is unique. Moreover, it follows from lemma 4 that for all finite $t_1, t_2 \geq 0$ there is a constant $k$ such that:

$$\sup_{\zeta \in D} |f_{n+1}(\zeta, t_1) - f_{n+1}(\zeta, t_2)| < k |t_1 - t_2| .$$

(37)

One checks by some analysis that this inequality and the continuity of $h_{n+1}$ with respect to the first variable (so: for fixed $t$), together with inequality (34) implies the continuity of $h_{n+1}$. The second property ($\sup_{\zeta \in D} |h'_n|$ continuous) follows from inequality (37) and the continuity of $F'$.

Step 2. We now show that for every $d > 0$, there is a $T > 0$ such that for all non-negative $t < T$ and all $n \in \mathbb{N}$:

$$\sup_{\zeta \in D} |f_n(\zeta, t) - f_0(\zeta)| < d .$$

(38)

Define:

$$M = \sup_{\zeta \in D} |h_0(\zeta)| ; \quad C = \sup_{\zeta \in D} |f'_0(\zeta)| ; \quad T = \frac{d}{C M + K d} .$$

(39)

We prove the induction step that for all nonnegative $t < T$ and all $n \in \mathbb{N}$:

$$\sup_{\zeta \in D} |f_n(\zeta, t) - f_0(\zeta)| \leq \frac{M}{K} \left( \sum_{k=1}^{n} (CKt)^k \right) .$$

(40)

Assume that this inequality holds for a certain $n \in \mathbb{N}$. Applying lemma 1, we find that for all $t < T$:

$$\sup_{\zeta \in D} |f_{n+1}(\zeta, t) - f_0(\zeta)| \leq C t \max_{\tau \in [0, t]} \{ \sup_{\zeta \in D} |h_n(\zeta, \tau)| \} \leq C t \max_{\tau \in [0, t]} \{ \sup_{\zeta \in D} |h_n(\zeta, \tau)| \} + M \leq C t \max_{\tau \in [0, t]} \{ K \sup_{\zeta \in D} |f_n(\zeta, \tau) - f_0(\zeta)| + M \} \leq C t \max_{\tau \in [0, t]} \{ M \left( \sum_{k=0}^{n} (CKt)^k \right) \} \leq \frac{M}{K} \left( \sum_{k=1}^{n+1} (CKt)^k \right) .$$

(41)
Then it is remarked that the inequalities \( t < T \) and inequality (40) imply inequality (38) and:

\[
\sup_{\zeta \in D} |f_n(\zeta, t)| < M + K d \quad \text{for } t < T .
\]  

(42)

Step 3. It follows from the result deduced in step 2 and from the continuity of \( \mathcal{F} \) that there exists a \( T > 0 \) and numbers \( K_2, K_3 \) such that for all \( n \in \mathbb{N} \) and all non-negative \( t < T \):

\[
\sup_{\zeta \in D} |h_n(\zeta, t)| < K_2 , \quad \sup_{\zeta \in D} |h_n(\zeta, t)| < K_3 .
\]  

(43)

Defining:

\[
L = C e^{T(K_2+K_3)}
\]  

(44)

and using lemma 5, we find for arbitrary \( n \in \mathbb{N} \) and all \( t < T \):

\[
\sup_{\zeta \in D} |f_{n+1}(\zeta, t) - f_n(\zeta, t)| \leq e^{T(K_2+K_3)t} \max_{\zeta \in \partial D} \left\{ \sup_{\tau \in [0,t]} |h_n(\zeta, \tau) - h_{n+1}(\zeta, \tau)| \right\} \leq
\]

(45)

One then shows in the standard way that \( f_n(\cdot, t) \) is a Cauchy–sequence in \( S \) with the sup–norm for all non-negative \( t < \overline{T} := \min\{T, (LK)^{-1}\} \). Therefore, \( f_n(\cdot, t) \) is a sequence of univalent functions converging uniformly to a regular function \( f(\cdot, t) \) on \( D \). One easily shows that the function \( f(\cdot, t) \) cannot be constant and therefore is univalent ([11]). One shows by standard techniques that the function \( f \) on \( D \times [0, \overline{T}) \) thus defined is a solution of the i.v.p. (6). \( \Box \)

4. Concluding Remarks

Our main result is the theorem formulated in section 3 which gives sufficient conditions on the mapping \( \mathcal{F} : S \rightarrow T \) for the i.v.p. (6) to be locally solvable. Direct applications of this result for standard physical problems are restricted for two reasons. The first reason is that \( \mathcal{F} \) maps into the space of functions \( h \) which have negative real part. This corresponds to moving boundary problems where the domain is shrinking. If one wants to generalize the theorem such that \( \mathcal{F} \) maps into a larger space containing also functions \( h \) which have real parts that are not purely negative, the same methods only apply if those functions \( h \) can be extended regularly outside \( D \). The second reason is that the conditions on \( \mathcal{F} \) can be formulated as continuity conditions on how the normal component \( v_n \) of the velocity depends on the shape of the boundary (see section 1, in particular the relations (3)). For standard problems (for instance the Hele-Shaw problem, [12]), it is non-trivial, and it may even be impossible, to show that these conditions are indeed satisfied.
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