On the NP hardness of the purely complex computation, analysis/synthesis, and some related problems in multidimensional systems

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On the $\mathcal{NP}$-hardness of the purely complex $\mu$ computation, analysis/synthesis, and some related problems in multidimensional systems

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Abstract
In this paper, it is shown that for a given complex matrix $M$, and a purely complex uncertainty structure $\Delta$, the problem of checking whether the inequality $\mu_\Delta(M) < 1$ holds, is $\mathcal{NP}$-hard. It is also shown that the problem of checking whether the frequency domain $\mu$-norm, $\|M(s)\|_\mu$, of an LTI system, $M(s)$, is less than 1, and the problem of checking whether the best achievable $\mu$-norm, inf$_{\mathcal{H}_\infty}$ $\|\mathcal{F}(T, Q)\|_\mu$, of an LFT, $\mathcal{F}(T, Q)$, is less than one, are both $\mathcal{NP}$-hard problems, namely purely complex $\mu$ computation, analysis/synthesis are all $\mathcal{NP}$-hard. It is already known that the computation of $\mu$ is $\mathcal{NP}$-hard for purely real [22] and mixture real/complex [7] uncertainty structures, but these results do not give much information about the computational complexity of the purely complex $\mu$ problem, nor they imply much about complexity of the $\mu$ analysis/synthesis problems. Although general $\mathcal{H}_\infty$ norm computation, analysis/synthesis have a well established theory for LTI systems, there is no known non-conservative polynomial time procedure for purely complex $\mu$ computation, analysis/synthesis problems. The main result of this paper is to provide proofs of the $\mathcal{NP}$-hardness of these problems. These results imply that it is rather unlikely to find non-conservative polynomial time procedures for the purely complex $\mu$ computation, analysis/synthesis problem, contrary to the standard $\mathcal{H}_\infty$ problems. As independent results, it is also shown that the problem of checking the stability and the problem of computing the $\mathcal{H}_\infty$ norm, are both $\mathcal{NP}$-hard problems for multidimensional systems. These results imply that it is rather unlikely to find a simple analogue of the Schur-Cohn test for checking the stability of the purely complex systems, in the context of multidimensional systems.

1. Introduction
In this paper, it is shown that the purely complex $\mu$ problem is $\mathcal{NP}$-hard, namely given a matrix $M$, and a purely complex uncertainty structure $\Delta$, the decision problem "is $\mu_\Delta(M) < 1"$ is $\mathcal{NP}$-hard. It is also shown that the problem of checking whether the $\mu$-norm of a (linear time invariant) LTI system is less than one, and the problem of checking whether the best achievable $\mu$-norm of a (linear fractional transformation) LFT is less than one, are also $\mathcal{NP}$-hard. Namely, purely complex $\mu$ computation, analysis/synthesis problems are all $\mathcal{NP}$-hard.

$\mathcal{NP}$-hardness of a problem implies that it is rather unlikely¹ to find a polynomial time algorithm for a solution of that problem [15]. There are several known $\mathcal{NP}$-hard problems and none of them has a known polynomial time solution procedure [15]. For example the traveling salesman problem, minimization of boolean expressions, graph isometry problem, quadratic programming, and integer programming are $\mathcal{NP}$-hard problems with no known polynomial time solution procedure. But for all of these problems, there are several known conservative approaches. For example, although the problem of minimizing boolean expressions is $\mathcal{NP}$-hard, there are several known techniques for obtaining a simpler boolean expression from a given complicated one. Similarly, for the purely complex $\mu$ computation and analysis problems, there are known conservative upper bound tests which require less computational effort. Recently, it was shown that these conservative upper bound tests actually correspond to non-conservative robust stability tests for a family of time varying structured perturbations [23, 26]. Furthermore, for the $\mu$ synthesis problem, $D - K$ iteration [10, 21] is a conservative design approach. But the $\mathcal{NP}$-hardness of these problems imply that it is rather unlikely to find polynomial time non-conservative tests for the purely complex $\mu$ computation, analysis/synthesis problems. It is known that both the real [22] and the mixed real/complex $\mu$ problems are $\mathcal{NP}$-hard. Furthermore, the mixed real/complex $\mu$ problem remains $\mathcal{NP}$-hard even for the family of problems which depend on problem data continuously [3, 29, 7, 24]. But these results do not give much information about the computational complexity of the purely complex $\mu$ problem. Because, the complexity of decision problems can change drastically when the underlying field is changed. For example, consider the problem of checking whether a given multivariable polynomial equation $p(t_1, \ldots, t_n) = 0$ has a solution. If the variables $t_1, \ldots, t_n$ are allowed to be complex, then by Hilbert Nullstellensatz [16], the above solvability test reduces to checking whether the polynomial is equal to a non-zero constant or not. On the other hand, if the variables $t_1, \ldots, t_n$ are restricted to be real, then the problem becomes $\mathcal{NP}$-hard (see [15] and references therein). By Tarski's theorem [27, 16] both the above solvability problem and the purely complex $\mu$ problem can be answered in doubly exponential amount of time, and therefore $\mathcal{NP}$-hardness of these problems do not imply their unsolvability. Nevertheless, as long as computation time is considered, we are naturally forced to work with conservative alternatives for both of these problems, instead of the inefficient² non-conservative ones.

¹In this paper, all of the statements of the form "it is rather unlikely ..." can be replaced by "it is impossible ..." under the assumption $\mathcal{P} \neq \mathcal{NP}$.

²In this paper, "efficient" method, procedure or algorithm means a solution procedure which requires at most polynomial amount of time.
The standard robust stability and performance tests of the $\mu$-analysis [10, 21, 12] are against purely complex perturbations, and these tests are based on the purely complex $\mu$ rather than the real or the mixed real/complex $\mu$. Therefore, $NP$-hardness results of [22, 7] do not give much information about the complexity of the problem of checking the robust stability of an LTI system against structured complex perturbations ($\mu$-analysis), and the problem of checking whether it is possible to design an LTI controller such that the closed loop system has robust stability against a given complex perturbation structure ($\mu$ synthesis). The main results of this paper are the $NP$-hardness of these problems, which are discussed in Section 2.

In Section 3, it is shown that, checking the stability and computing the $\mathcal{H}_\infty$ norm are $NP$-hard problems for multidimensional systems. Finally, in Section 4, some concluding remarks are made, and in the appendix the definitions of the complexity classes $\mathcal{P}$, $\mathcal{NP}$, $\mathcal{NP}$-complete and $\mathcal{NP}$-hard, are given.

2. Purely complex $\mu$ problem
In this section, the purely complex $\mu$ problem is defined and shown that the complex programming polynomially reduces to the purely complex $\mu$ problem. This result, together with the $NP$-hardness of the complex programming (which is proved in Section 3) shows that the purely complex $\mu$ problem is $NP$-hard.

Definition 1 (Purely complex $\mu$ problem):
Given a matrix $M \in \mathbb{C}^{n \times n}$ and a purely complex uncertainty structure $\Delta$, determine whether the inequality $\mu_\Delta(M) < 1$ holds.

Consider the structure defined by

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad \Delta z \in \mathcal{B}(\Delta z)$$

By using the results on LFTs given in [12], one can transform this structure to a standard form,

$$\Delta_1 = \{ \text{diag}[\delta_1 I_2, \ldots, \delta_n I_2] : \delta_1, \ldots, \delta_n \in \mathbb{C} \}, \quad \Delta_1 \in \mathcal{B}(\Delta_1),$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

$$M_{11} = P \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} P^T \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}, \quad M_{12} = P \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M_{21} = E^T \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad M_{22} = M_{n \times n},$$

$$P = [e_1 \ldots e_{n-1} e_2 \ldots e_{2n}]$$

and $e_k = [0 \ldots 1 \ldots 0]^T \in \mathbb{R}^{2n}$.

In other words, with the above notation, we have

$$E^T \Delta z A \Delta z E = M_{22} + M_{21} \Delta_1 (I - M_{11} \Delta_1)^{-1} M_{12}.$$

Define

$$\Delta_2 = \{ \delta : \delta \in \mathbb{C} \},$$

$$\Delta = \{ \Delta : \Delta = [\text{diag}[\delta_1 I_2, \ldots, \delta_n I_2, \delta_{n+1}] : \delta_1, \ldots, \delta_{n+1} \in \mathbb{C} \} \}.$$
therefore
\[ \sup_{\mathbf{R}(s) \geq 0} \mu_\Delta(M(s)) < 1 \iff \mu_\Delta(M) < 1. \]

By the \(N\mathcal{P}\)-hardness of the purely complex \(\mu\) problem, it follows that testing the robust stability of LTI systems against structured complex perturbations, is \(N\mathcal{P}\)-hard. Note that, the problem of checking whether the \(\mathcal{H}_\infty\) norm, \(\|M(s)\|_{\mathcal{H}_\infty}\), of an LTI system, \(M(s)\), is less than one, can be solved in polynomial time using the results of [6], but the \(\mu\) analysis problem is \(N\mathcal{P}\)-hard.

Corollary 2 (\(\mu\)-synthesis): The problem of checking whether it is possible to design an LTI controller such that the closed loop system has robust stability against a given complex perturbation structure, is \(N\mathcal{P}\)-hard. Because, using the notation of [10], consider the problem of checking whether \(\mu_{\text{opt}} < 1\), where
\[ \mu_{\text{opt}} = \inf_{Q \in \mathcal{E}^{\mathcal{D}}} \|F(T, Q)\|_\mu, \]
and \(T_{11}, T_{12}, T_{21} \in \mathbb{R}^{\mathcal{D}, \infty}, T_{12}^* T_{12} = I, T_{21} T_{21}^* = I, T_{22} = 0\). This corresponds to checking whether there exists a controller \(Q\), such that the closed loop system is internally stable and the transfer function has \(\mu\) norm less than one.

Define
\[ w(s) = 1, \quad m_1(s) = \frac{s-2}{s+2}, \quad m_2(s) = \frac{s-3}{s+3}, \]
\[ \gamma_0 = \inf_{Q \in \mathcal{E}^{\mathcal{D}}} \|w + m_1 m_2 Q\|_\infty. \]

Now, for a given \(M \in \mathbb{R}^{n \times n}\), and a purely complex perturbation structure \(\Delta\), define
\[ \Delta_e = \left\{ \left[ \begin{array}{cc} \delta & 0 \\ 0 & \Delta \end{array} \right] : \delta \in \mathbb{C}, \Delta \in \Delta \right\}, \]
\[ T_{11}(s) = \left[ \begin{array}{cc} k w(s) & 0 \\ 0 & M \end{array} \right], \quad T_{12}(s) = \left[ \begin{array}{c} m_1(s) \\ \vdots \\ 0 \end{array} \right], \]
\[ T_{21}(s) = \left[ \begin{array}{c} m_2(s) \\ \vdots \\ 0 \end{array} \right], \quad T_{22} = 0, \]
and choose \(k\) as a rational number with \(|k| < \frac{1}{\gamma_0} \rho(M)\). Then,
\[ F(T, Q) = T_{11} + T_{12} Q T_{21} = \left[ \begin{array}{cc} k w - m_1 m_2 Q & 0 \\ 0 & M \end{array} \right] \]
and
\[ \|F(T, Q)\|_\mu \leq \sup_{w \in \mathbb{R}} \mu_\Delta \left( \left[ \begin{array}{cc} k w - m_1 m_2 Q & 0 \\ 0 & M \end{array} \right] \right) \]
\[ = \max \left\{ \|k w + m_1 m_2 Q\|_\infty, \mu_\Delta(M) \right\}. \]

Since \(|k| < \frac{1}{\gamma_0} \rho(M) < \frac{1}{\gamma_0} \mu_\Delta(M)\), we obtain
\[ \inf_{Q \in \mathcal{E}^{\mathcal{D}}} \|k w + m_1 m_2 Q\|_\infty < \mu_\Delta(M). \]
This implies that,
\[ \inf_{Q \in \mathcal{E}^{\mathcal{D}}} \|F(T, Q)\|_\mu \leq \mu_\Delta(M). \]

Therefore
\[ \inf_{Q \in \mathcal{E}^{\mathcal{D}}} \|F(T, Q)\|_\mu \leq \mu_\Delta(M) < 1. \]

By the \(N\mathcal{P}\)-hardness of the purely complex \(\mu\) problem, it follows that the problem of checking whether the best achievable \(\mu\)-norm of an LFT is less than one, and hence the problem of checking whether there exists an LTI controller such that the closed loop system has robust stability against a given perturbation structure, are \(N\mathcal{P}\)-hard problems. Note that, for the general \(\mathcal{H}_\infty\) design problem, there are effective procedures, see [10, 14, 11] and references therein, but the \(\mu\) synthesis problem is \(N\mathcal{P}\)-hard.

3. Some observations for multidimensional systems

In this section, it is shown that the problem of checking the stability and the problem of computing the \(\mathcal{H}_\infty\) norm are \(N\mathcal{P}\)-hard for multidimensional systems.

3.1. Stability

In this section, it is shown that checking the stability of a multidimensional system is an NP-hard problem.

Consider an \(n\)-dimensional causal filter with transfer function
\[ H(z_1, \ldots, z_n) = \frac{N(z_1, \ldots, z_n)}{D(z_1, \ldots, z_n)}, \]
where \(N, D \in \mathbb{C}[z_1, \ldots, z_n]\). If \(N\) and \(D\) do not have a common root in \(\mathbb{C}^n\), equivalentlly if there exist \(X, Y \in \mathbb{C}[z_1, \ldots, z_n]\) such that \(X N + Y D = 1\), then according to the stability definition of [9], \(H(z_1, \ldots, z_n)\) is stable iff \(D(z_1, \ldots, z_n)\) has no root in \(\mathbb{D}^n\). Therefore, stability of \(1/D(z_1, \ldots, z_n)\) is equivalent to \(D(z_1, \ldots, z_n)\) having no root in \(\mathbb{D}^n\).

There are several known results related with the stability theory of \(n\)-dimensional systems, see for example [1, 4, 5, 9, 17, 18, 26, 30, 31] and references therein. Both efficient but conservative tests, and non-conservative but inefficient tests for checking the stability of multidimensional systems are known. But according the authors' knowledge, there is no known efficient and non-conservative stability test for multidimensional systems. In this context, Theorem 3 implies that the problem of checking the stability of a multidimensional system is \(N\mathcal{P}\)-hard, and hence explains this situation (See [28] for a proof of Theorem 2).

Theorem 2:
(a) For a given polynomial \(p(z_1, \ldots, z_n) \in \mathbb{R}[z_1, \ldots, z_n]\), it is \(N\mathcal{P}\)-hard to check whether it has a root in \(\mathbb{D}^n\).
(b) For a given polynomial \(p(z_1, \ldots, z_n) \in \mathbb{R}[z_1, \ldots, z_n]\), it is \(N\mathcal{P}\)-hard to check whether it has a root in \(\mathbb{D}^n\).

Theorem 2 implies that, it is rather unlikely to find a simple analogue of the Schur-Cohn test for multidimensional systems.
3.2. $H^\infty$ norm

In this section, it is shown that computing the $H^\infty$ norm of a stable transfer function is $NP$-hard for multidimensional systems. The $H^\infty$ norm of a multidimensional system is defined as

$$\|H(z_1, \ldots, z_n)\|_{H^\infty} := \sup_{z_1, \ldots, z_n \in \mathbb{C}} \|H(z_1, \ldots, z_n)\|.$$ 

If $H$ is stable, then by the maximum modulus theorem, supremum over $\mathbb{C}^n$ can be replaced by supremum over $\mathbb{T}^n$.

**Theorem 3:**

(a) For a given stable transfer function $H(z_1, \ldots, z_n)$, and a given $\gamma_L$, it is $NP$ hard to check whether

$$\|H(z_1, \ldots, z_n)\|_{H^\infty} > \gamma_L.$$ 

(b) For a given stable transfer function $H(z_1, \ldots, z_n)$, and a given $\gamma_L$, it is $NP$ hard to check whether

$$\|H(z_1, \ldots, z_n)\|_{H^\infty} < \gamma_L.$$ 

For a proof of the above theorem, see [28].

This result shows that it is rather unlikely to find an efficient generalization of the $\gamma$ iteration procedure of [6] for multidimensional systems. This also answers a question related with a recent result of B. Bamieh et al. [2]. In [2] it is shown that, the robust performance problem can be transformed to the problem of computing the $H^\infty$ norm of a multidimensional transfer function, and it is mentioned that efficient methods for computing the $H^\infty$ norms of such functions are still lacking. $NP$-hardness results of Theorem 4 explains this situation.

4. Concluding remarks

In this paper, it was shown that the purely complex $\mu$ computation, analysis/synthesis are $NP$-hard problems. The proof presented here was based on two technical lemmas on complex valued functions and complex programming, and the $NP$-hardness of the Knapsack problem [15, 19]. The $NP$-hardness of the complex $\mu$ problem implies that it is rather unlikely to find a polynomial time non-conservative test for the purely complex $\mu$ computation, analysis/synthesis problems. As an important remark about the results presented in this paper, all of the statements of the form "it is rather unlikely that ..." can be replaced by "it is impossible that ...", under the assumption $P \neq NP$. Although, this is still an open question, based on the current evidence it is rather unlikely that $P$ and $NP$ are equal, and therefore it is rather unlikely that $NP$-hard problems have polynomial time solution procedures. Nevertheless, like many other $NP$-hard problems, several conservative approaches are possible, for example upper bound tests are conservative alternative approaches which require less computational effort. In [23, 25], it was shown that these upper bound tests actually correspond to non-conservative robust stability tests against a family of time varying structured perturbations. Similarly, $D-K$ iteration [10, 21] is a conservative approach for the $\mu$ synthesis problem. The main results of this paper show once more the importance of the results of [10, 21, 23, 25] on the upper bounds. In this paper, it was also shown that the problem of checking the stability and the problem of computing the $H^\infty$ norm are both $NP$-hard problems for multidimensional systems. These results imply that it is rather unlikely to find a simple analogue of the Schur-Cohn test for checking the stability and an efficient generalization of bisection method of [6] for computing the $H^\infty$ norm, in the context of multidimensional systems.

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**Appendix**

The complexity class $P$ denotes the class of decision problems (languages) that can be solved (recognized) by a deterministic Turing machine in polynomial time. On the other hand, the complexity class $NP$ is defined as the class of decision problem (languages) that can be solved (recognized) by a nondeterministic Turing machine in polynomial time. Although it is still an open question whether the complexity classes $P$ and $NP$ are equal or not, current evidence shows that they are not. A problem is said to be $NP$-complete if it is in the complexity class $NP$ and the existence of a polynomial time solution procedure for that problem implies the equality of the complexity classes $P$ and $NP$. Finally, a problem is said to be $NP$-hard if an $NP$-complete problem can be polynomially reduced to that problem, but the $NP$ membership is not required. Therefore, $NP$-hard problems can be harder than the $NP$-complete ones. Based on the current evidence, it is rather unlikely that the complexity classes $P$ and $NP$ are equal, and the existence of a polynomial time solution procedure for an $NP$-complete or $NP$-hard problem implies the equality of these two complexity classes. Therefore it is rather unlikely that $NP$-complete and $NP$-hard problems have polynomial time solution procedures. The first $NP$-hardness result was on the satisfiability problem of boolean expressions [8], which also implies the $NP$-hardness of the problem of minimizing boolean expressions. Later in [19], it was shown that several other combinatorial optimization problems are $NP$-hard too, including the Knapsack problem. This problem implies the $NP$-hardness of some important continuous optimization problems [15]. For an extensive list of $NP$-hard problems and related references, the reader is referred to [15].

**References**
