LINEAR MULTIVARIABLE SYSTEMS

Preliminary Problems in Mathematical Description, Modelling and Identification.

by

A.K. Hajdasiński
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Abstract

This report contains a partial knowledge about linear multivariable systems. It starts with very simple concepts from multivariable system theory, and closes with some proposals of further research in the field of MIMO systems identification.

Selected subjects were mainly discussed, however, forming a comprehensive set. The choice was certainly subjective, but presented methods were either applied with good experience or convincing records about their application were found.

There is, however, an exception which still needs further research, namely the Akaike FPE method, which intuitively is quite obvious, but practically never well explained.

This work deals with subjects to be found in generally available literature, but also (this is a subjective feeling) with subjects which are presented in an artificially complicated way (e.g. innovation approach) or which are mainly authors' studies (e.g. Markov parameters, order tests for MIMO systems).
1. INTRODUCTORY INFORMATION

The notion of the multivariable dynamical system has appeared in literature and in practice as a natural evolution of the scalar dynamical system being the very first approximation of real processes. R.W. Brocket and H.H. Rosenbrock in their foreword to the series "Studies in Dynamical Systems" have written: ...... "During the last twenty years there has been a progressive increase in the complexity and degree of interconnection of systems of all kinds. The reasons are clear: recent progress in communication data processing, and control have made possible a much greater degree of coordination between the parts of a system than ever before."(*) Such a development demanded new techniques, new mathematical models and methods suitable for handling more complex and intercorrelated tasks of the aggregated systems. A quick development of the multi input - multi output systems theory had to go in line with a very advanced mathematical apparatus application and unavoidable incorporation of digital computers and numerical methods. In this study we will try to give a comprehensive description of selected problems being of particular interest for a system designer. For the rest of already tremendously imposing material we will refer to an extended bibliography.

1.1. Definition of a multivariable dynamical system

The definition of the multivariable (multi input - multi output system) dynamical system as proposed by Wolovich (1974), Niederliński (1974) and Rosenbrock (1970) is as follows:

**Definition** 1  The multivariable system is the system having more than one input and one output and inputs to this system may influence more than one output at a time.

(*) Published by Thomas Nelson, London.
The block diagram of such a system is shown in Fig. 1.

Further we will consider only a relatively simple class of multivariable dynamical systems, namely those which are linear, time invariant and finite dimensional.

1.1.1. Preliminaries - definitions of some important notions

Definition 2 For the multivariable dynamical, linear, time invariant and finite dimensional system having \( p \) inputs \( u_1(k) \ldots u_p(k) \) (forming the input vector \( \mathbf{u}(k) \)) and \( q \) outputs \( y_1(k) \ldots y_q(k) \) (forming the output vector \( \mathbf{y}(k) \)). Here is defined the \( q \times p \) matrix \( \mathbf{K}(z) \), called the transfer matrix (being considered the rational matrix of the argument \( z \)) fulfilling the following condition:

\[
\mathbf{y}(z) = \mathbf{K}(z)\mathbf{u}(z)
\]

(1)

where

\[
\mathbf{y}(z) = \begin{bmatrix}
    y_1(z) \\
    \vdots \\
    y_q(z)
\end{bmatrix}
\]

\[
\mathbf{u}(z) = \begin{bmatrix}
    u_1(z) \\
    \vdots \\
    u_p(z)
\end{bmatrix}
\]
and $y(z)$, $u(z)$ are the "z" transforms of $y(k)$ and $u(k)$ respectively, under zero initial conditions. (see also Zadeh L.A., C.A. Desoer (1963), Eykhoff P. (1974), Rosenbrock H.H. (1970), Niederliński (1974), Schwarz H. (1971), Wolovich (1974)).

The coordinates of the $u(k)$ vector can be both control variables and disturbances, while the coordinates of the $y(k)$ vector are the output variables. Further there will be considered only such linear systems, for which outputs are linearly independent i.e. the outputs cannot be described as a linear combination of remaining ones. This simply means that the inputs and the outputs must fulfil the following condition:

$$q \leq p$$

(2)

This condition is always fulfilled while the rank of the $K(z)$ matrix is equal to $q$.

$$\{\text{rank } K(z)\} = q \iff \{q \leq p\}$$

(3)

**Definition 3** The characteristic polynomial $w(z)$ of the strictly proper or proper transfer matrix $K(z)$ is defined as the least Common Denominator of all minors $\text{min}_z K(z)$, having by the greatest power of "z" the coefficient equal to one. (see also Wolovich (1974), Schwarz, (1971), Rosenbrock (1970)).

Proceeding further with definitions we have to define the degree of a parametric (polynomial) matrix and the state representation of the multivariable dynamical system.

**Definition 4** The degree $\delta[K(z)]$ of the strictly proper or proper transfer matrix $K(z)$ is defined as the degree of its characteristic polynomial.
(Practically it is the smallest number of shifting elements necessary to model the dynamics of this system).

1. A multivariable system is called "a proper system" or its transfer matrix is called "a proper transfer matrix"
   if
   \[ \lim_{z \to \infty} K(z) \neq 0 \]

2. A multivariable system is called "a strictly proper system" or its transfer matrix is called "a strictly proper transfer matrix"
   if
   \[ \lim_{z \to \infty} K(z) = 0 \]

3. A multivariable system is called "an improper system" if at least for one component of a transfer matrix it holds that the degree of a nominator is greater than that of a denominator.

**Definition 5** For the multivariable, linear, time invariant, dynamical system, the state of the system of an arbitrary time instant \( k = k_0 \) is defined as a minimal set of such numbers \( X_1(k_0), X_2(k_0), \ldots, X_n(k_0) \) the knowledge of which, together with the knowledge of the system model and inputs for \( k \geq k_0 \) is sufficient for determination of the system behaviour for \( k \geq k_0 \).

\[
X(k_0) = \begin{bmatrix}
X_1(k_0) \\
X_2(k_0) \\
\vdots \\
X_n(k_0)
\end{bmatrix}
\]

is called the state vector, and members \( X_1(k_0), \ldots, X_n(k_0) \) are called state variables.

**Definition 6** The set of difference equations

\[ x(k + 1) = \begin{bmatrix} A & B \end{bmatrix} x(k) + B u(k) \]  

(4)

where \( x(k + 1) \) is a \((n \times 1)\) state vector

\( u(k) \) is a \((p \times 1)\) input vector

is called the state equation, while the set of difference equation

\[ y(k) = C x(k) + D u(k) \]  

(5)

where \( y(k) \) is a \((q \times 1)\) output vector

is called the output equation.

**Definition 7** The triplet of matrices \( \{A, B, C\} \) is defined as the realization of the dynamical, linear, time invariant, multivariable system.

**Definition 8** The number of state variables "n" in the state equation is defined as the dimension of the state vector or the state space and also denoted as the dimension of the complete system.

**Definition 9** Any polynomial \( f(z) \)

\[ f(z) = z^k + C_1 z^{k-1} + \ldots + C_{k-2} z^2 + C_{k-1} z + C_k \]  

(6)

for which holds

\[ f(A) = A^k + C_1 A^{k-1} + \ldots + C_{k-2} A^2 + C_{k-1} A + C_k = \phi \]  

(7)
is called the annihilating polynomial of the $A$ matrix.

Investigating various properties of multivariable system the following Lemma drawn from the Cayley-Hamilton Theorem can be of great help.

**Lemma 1** The characteristic polynomial of the $A$ matrix $- \tilde{W}_A(z)$ is one of the annihilating polynomials of the $A$ matrix.

**Definition 10** The polynomial $f(z)$ of the smallest, nonequal zero, degree $k$, fulfilling definition 9, is called the minimal polynomial of the $A$ matrix.

**Definition 11** The matrix coefficient $M_k = C A^k B$ for $k = 0, 1, 2, \ldots$ is referred to as the $k$-th Markov Parameter of the system defined by the realization $\{A, B, C\}$. (see also Ho B.L., Kalman R.E. (1966), Schwarz (1971), Gerth, W. (1971), Tether A.J. (1970), Hajdasiński A.K. (1976, 1978), Hajdasiński A.K., Damen A.A.H. (1979)).

**Definition 12** The following description of the multivariable dynamical system is referred to as the Hankel model (H - model) of this system.

$$Y = M_H U + H_k \beta$$
$$u(i) = 0 \text{ for } i < 0$$

where

$$Y = \begin{bmatrix}
    Y(1) \\
    Y(2) \\
    Y(3) \\
    \cdot \\
    \cdot 
\end{bmatrix}$$

$$U = \begin{bmatrix}
    u(0) \\
    u(1) \\
    u(2) \\
    \cdot \\
    \cdot 
\end{bmatrix}$$

$\beta$-properly dimensioned block vector containing initial conditions
and $M_k$ for $k = 0, 1, 2, \ldots$ are the Markov parameters of the considered system.


In aiming for equivalency conditions for different types of models of multivariable systems, we must pass through two fundamental theorems and the definition of the order of the multivariable dynamical system.

**Theorem 1** The sequence of Markov parameters $\{M_k\}$ for $k = 0, 1, 2, 3, \ldots$ has a finite dimensional realization $\{A, B, C\}$ if and only if there are an integer $r$ and constants $a_i$ such that:

$$M_r + j = \sum_{i=1}^{r} a(i) M_r + j - i \quad \text{for all } j \geq 0$$

(9)

where $r$ is the degree of the minimal polynomial of the state matrix $A$ (assuming we consider only minimal realizations).
Remark: Theorem 1 is called the realisability criterion and the $r$ is called the realisability index.


Theorem 2 If the Markov parameters sequence $\{M_k\}$ for $k = 0, 1, 2, \ldots$ has a finite dimensional realization $\{A, B, C\}$, with realisability index $r$, then the minimal dimension $n_0$ in the state space (also of the realization) for this realization fulfills

$$\text{rank} \left[ H_r \right] = n_0$$

(10)

where $n_0$ - minimal state space dimension

$$[M_i] - (q \times p) \text{ matrix }$$

$$n_o \leq r \times \min(p, q)$$

(11)

and

$$H_r = \begin{bmatrix}
M_0 & M_1 & \cdots & M_{r-1} \\
M_1 & M_2 & \cdots & M_r \\
\vdots & \vdots & \ddots & \vdots \\
M_{r-1} & M_r & \cdots & M_{2r-2}
\end{bmatrix}$$

- the Hankel Matrix (finite)

The most correct proof of this theorem is to be found in Schwarz H., (1971).

Remark: From linear dependence of Markov parameters it follows that

$$\text{rank} \left[ H_r + N \right] = \text{rank} \left[ H_r \right] = n_0$$

(12)

for all $N \geq 0$
Discussion of the Chapter 1.1.

With the aid of these 12 definitions, 1 Lemma and 2 Theorems it is possible now to find a link between different types of multivariable system descriptions. There will be no rigorous mathematical derivation presented here. To visualize this link we will draw a block scheme showing interdependence of different type models. The arrows in this scheme show only possible direct links (fig. 2).

From this scheme we learn that while from the state space description there is a straightforward way to get the transfer matrix $K(z)$, the reverse procedure must be completed employing the realization theory, which is a lot more complicated.
On the contrary, knowing Markov parameters, it is equally easy to get any required form of description. For the sake of modelling, Markov parameters can be derived as easily from the state space description as from the transfer matrix. Obviously Markov parameters are also used in the H-model.

**Example 1** Let us consider a simple two input - two output system described in the following transfer matrix $K(z)$:

\[
K(z) = \begin{bmatrix}
-1.0 & z - 0.5 \\
(z - 0.8) & 2(z - 0.2)(z - 0.8)
\end{bmatrix}
\begin{bmatrix}
-1.0 & 1.0 \\
(z - 0.4) & (z - 0.4)
\end{bmatrix}
\]

the characteristic polynominal $w(z)$ of the strictly proper matrix $K(z)$ is:

\[
w(z) = (z - 0.2)(z - 0.4)(z - 0.8)
\]

the degree of the $K(z)$ is:

\[
\delta [K(z)] = 3
\]

Markov parameters for this system are:

\[
M_0 = \begin{bmatrix}
-1.0 & 0.0 \\
1.0 & 1.0
\end{bmatrix} ;
M_1 = \begin{bmatrix}
-0.8 & 0.6 \\
0.4 & 0.4
\end{bmatrix} ;
M_2 = \begin{bmatrix}
-0.64 & 0.6 \\
0.4 & 0.4
\end{bmatrix}
\]
\[
M_3 = \begin{bmatrix}
-0.512 & 0.504 \\
0.064 & 0.064
\end{bmatrix} \quad \text{and so on.}
\]

\[
M_4 = C A^i B
\]

One of the possible realizations of this system is:

\[
A = \begin{bmatrix}
0.2 & 0.0 & 0.0 \\
0.0 & 0.4 & 0.0 \\
0.0 & 0.0 & 0.8
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.0 & -1.0 \\
1.0 & 1.0 \\
1.0 & 1.0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1.0 & 0.0 & -1.0 \\
0.0 & 1.0 & 0.0
\end{bmatrix}
\]

\[
H_1 = \begin{bmatrix}
-1.0 & 0.0 \\
1.0 & 1.0
\end{bmatrix} = M_0 \quad \text{rank } H_1 = 2
\]

\[
H_2 = \begin{bmatrix}
-1.0 & 0.0 & -0.8 & 0.6 \\
1.0 & 1.0 & 0.4 & 0.4 \\
-0.8 & 0.6 & -0.64 & 0.6 \\
0.4 & 0.4 & 0.4 & 0.4
\end{bmatrix} \quad \text{rank } H_2 = 3
\]
rank $H_3 = rank H_2 = rank H_2 + N \quad N > 2$

Thus $r$ - realizability index = 2

$n_o$ - dimension of the realization = 3

Coefficients of the minimal polynomial are: $a_1 = -1; \quad a_2 = 0.16$

$$f_{\min}(A) = A^2 - A + 0.16 I = \phi$$

Remark: It is made evident now that not every annihilating polynomial of $A$ (viz. $W(A) = (A - 0.2)(A - 0.4)(A - 0.8)$) must be of minimal order. In this case the characteristic polynomial of $A$, being one of the annihilating polynomials for $A$, is of the 3-rd order while the minimal order is 2.

$$M_{-2+j} = M_{-2+j-1} - 0.16 M_{-2+j-2} \quad \text{for } j \geq 0$$

1.2. Mathematical models commonly used for the multivariable dynamical system description

Solving problems in multivariable dynamical systems requires implementation of quite a huge and advanced mathematics: theory of sets, matrix algebra and analysis with special attention payed to polynomial matrices and functions of matrices, theory of linear spaces, theory of limiting processes, advanced mathematical analysis, some topics in functional analysis, theory of differential equations, complex analysis, Laplace and "Z" transform techniques and many supplementary topics from related disciplines.

It is not possible to give a review of even selected problems and the only possibility is to direct a reader to references.

Unfortunately there does not exist any comprehensive publication treating a
subject of "Mathematical methods" for multivariable systems entirely. Moreover, the best references are with literature concerning the control problems, because the control engineering science was stimulating the development of certain mathematical disciplines.

In this report we will attempt to give the most intuitive and simple description of multivariable systems.

The practical applications show that sometimes simpler models may better serve the control tasks than very sophisticated ones. This always is a compromise between achievable accuracy, "common sense" and a scientifically formal approach.

Extensive references for further readings will also be given here.

1.2.1. Transfer function matrix models

The main interest will be focused on the discrete-time systems. However, it seems to be useful to start with the continuous-time, linear systems and generalize derived results using the "Z" transform concept.

Assuming that there are given: $K(s)$ - the transfer function matrix and $u(s)$ - the Laplace's transform of the input vector, it is always possible to find the output vector $y(t)$ with the zero initial conditions.

$$ y(t) = L^{-1}[K(s)u(s)] $$  \hspace{1cm} (13)

and using the convolution integral:

$$ y(t) = \int_0^t K(t - \tau)u(\tau)d\tau $$  \hspace{1cm} (14)

where

$$ K(t) = L^{-1}[K(s)] $$

is the (q x p) "weighing matrix" (see Zadeh L.A., Polak E. (1969), Schwarz H.
The weighing matrix has an interesting physical interpretation for columns of this matrix can be interpreted as impulse responses to the separately applied input Dirac pulses.

If

$$u_{i\delta}(\tau) = \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_{i-1}(\tau) \\ \delta_i(\tau) \\ u_{i+1}(\tau) \\ \vdots \\ u_p(\tau) \end{bmatrix}$$

then

$$\delta_i(\tau) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$$

where

$$u_k(\tau) = \begin{cases} 1, & \text{if } k = i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\gamma(t) = k_{\delta_i(t) \delta_1(t) \cdots \delta_p(t)}$$

$$= \left\{ \begin{array}{c} k_{11}(t-\tau)k_{12}(t-\tau)\cdots k_{1i}(t-\tau)\cdots k_{1p}(t-\tau) \\ k_{21}(t-\tau)k_{22}(t-\tau)\cdots k_{2i}(t-\tau)\cdots k_{2p}(t-\tau) \\ \vdots \\ k_{p1}(t-\tau)k_{p2}(t-\tau)\cdots k_{pi}(t-\tau)\cdots k_{pp}(t-\tau) \end{array} \right\} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta_i(t) \\ \delta_i(t) \\ \delta_i(t) \\ \vdots \\ \delta_i(t) \end{bmatrix}$$

$$\int_0^t \delta_1(\tau)d\tau = k_{i1}(t) - i - \text{column of } K(t)$$
Example 2:

As an example to start with, we can consider a simple two input and two output system of the level control.

The task is to maintain levels \( h_1 \) and \( h_2 \) at a certain range, manipulating valves \( v_1 \) and \( v_2 \) such that the volumetric flows \( Q_1 \) and \( Q_2 \) can be properly adjusted. In order to propose a control algorithm, we have to find a model
of the given system, having the following block diagram:

Writing linearized mass balance equations

\[
\begin{align*}
\dot{\Delta Q}_1 - \Delta Q_{11} &= A_1 \frac{d\Delta h_1}{dt} \\
\dot{\Delta Q}_2 - \Delta Q_{21} &= A_2 \frac{d\Delta h_2}{dt} \\
\dot{\Delta Q}_{11} + \dot{\Delta Q}_{21} - \Delta Q &= A \frac{d\Delta h}{dt}
\end{align*}
\]

where \( \Delta Q_1, \Delta Q_2, \Delta Q_{11}, \Delta Q_{12}, \Delta Q, \Delta h_1, \Delta h_2, \Delta h \) are small deviations of variables around the working points.

assuming

\[
\begin{align*}
Q &= f(h) \\
\frac{\partial f(h)}{\partial h} = a_0 \Delta h \\
\frac{\partial f(h)}{\partial h} = a_0 \Delta h
\end{align*}
\]

\[
\Delta \dot{Q} = a_o \Delta h
\]

also

\[
\begin{align*}
\Delta Q_{11} &= \beta_1 \Delta h \\
\Delta Q_{21} &= \beta_2 \Delta h
\end{align*}
\]
finally, after applying the Laplace transform:

\[ \Delta Q_1(s) = A_1 s \Delta h_1(s) + \beta_{10} \Delta h(s) \]

\[ \Delta Q_2(s) = A_2 s \Delta h_2(s) + \beta_{20} \Delta h(s) \]

\[ \Delta Q(s) = \beta_{10} \Delta h(s) + \beta_{20} \Delta h_2(s) - A_s \Delta h(s) \]

\[ A_s \Delta h(s) + A_s \Delta h(s) = \beta_{10} \Delta h(s) + \beta_{20} \Delta h_2(s) \]

The block diagram of this dynamical process is following:
and

\[
\begin{bmatrix}
\Delta h(s) \\
\Delta h_2(s)
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\alpha_o} \\
\frac{\beta_{20}}{\alpha_o} \\
0.0
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
0.0
\end{bmatrix}
\begin{bmatrix}
\beta_{10} \\
\beta_{20} \\
\frac{1}{\beta_{20}} s + 1
\end{bmatrix}
\begin{bmatrix}
\Delta Q_1(s) \\
\Delta Q_2(s)
\end{bmatrix}
\]

introducing

\[
\begin{align*}
k_{11} &= \frac{1}{\alpha_o} ; \\
k_{12} &= \frac{\beta_{20}}{\alpha_o} ; \\
k_{22} &= \frac{1}{\beta_{20}}; \\
\end{align*}
\]

\[
\begin{align*}
T_1 &= \frac{A_1}{\beta_{10}} ; \\
T_2 &= \frac{A_2}{\beta_{20}} ; \\
T &= \frac{A}{\alpha_o} ;
\end{align*}
\]

we get

\[
Y(s) = K(s) U(s)
\]

where

\[
K(s) =
\begin{bmatrix}
k_{11} & k_{12} \\
\frac{(1 + sT_1)(1 + sT)}{1} & \frac{(1 + sT_2)(1 + sT)}{1}
\end{bmatrix}
\]

is the transfer matrix, and

\[
K(t) =
\begin{bmatrix}
k_{11} e^{-\frac{T}{T_1}} (T - T_1) e^{-\frac{T}{T_1}} & k_{12} e^{-\frac{T}{T_2}} (T - T_2) e^{-\frac{T}{T_2}} \\
0.0 & k e^{-\frac{T}{T_2}}
\end{bmatrix}
\]
is the weighting matrix of the dynamical system

\[ W_{\text{K}(s)} = (1 + sT_1)(1 + sT_2)(1 + sT) \]

\[ \delta_{\text{K}(s)} = 3 \]

1.2.2. Decomposition of the transfer matrix and classification of the multivariable dynamical systems

Following Niederliński (1974) we will decompose the input vector \( u(s) \) into q - control inputs and p - q - disturbing inputs:

\[
\begin{bmatrix}
  u_1(s) \\
  u_2(s) \\
  \vdots \\
  u_q(s) \\
  u_{q+1}(s) \\
  \vdots \\
  u_p(s)
\end{bmatrix} = \begin{bmatrix}
  y(s) \\
  z(s)
\end{bmatrix} = \begin{bmatrix}
  u_1(s) \\
  \vdots \\
  u_q(s) \\
  \vdots \\
  u_{p}(s)
\end{bmatrix} \quad (p - q \times 1)
\]

which leads to the following relation:

\[
y(s) = \begin{bmatrix}
P(s) & Q(s)
\end{bmatrix} \begin{bmatrix}
  y(s) \\
  z(s)
\end{bmatrix} = P(s)y(s) + Q(s)z(s) \quad (16)
\]

\[ P(s) \rightarrow q \times q \text{ transfer matrix of control inputs} \]

\[ Q(s) \rightarrow q \times (p - q) \text{ transfer matrix of disturbances} \]

Following Niederliński (1974) and Isermann (1977) we will classify the multivariable dynamical systems in the following way:

1. stable multivariable dynamical systems - i.e. all those systems for which...
all poles of the transfer function matrices lie in the left half plane of
the complex variable "s" and there are no poles on the imaginary axis of the
plane.

2. nonstable multivariable dynamical systems - i.e. all those not fulfilling
the stability definition.

3. minimum-phase m.d.s. - i.e. all those for which all zeros of the
determinant \( \det(P(s)) \) are in the left half plane of the complex variable "s".

4. non-minimum phase m.d.s. - if at least one zero of the \( \det(P(s)) \) appears
in the right half plane of the complex variable "s".

It will be noticed that the multivariable system may be non-minimal phase.
However, each of its components \( P_{ij}(s) \) (\( i,j = 1,2, \ldots, q \)) is the minimal
phase object. The non-minimal phase objects are much more difficult for
handling than minimal phase ones.

Another classification due to Niederländer (1974) and Isermann (1977) is made
according the internal couplings of the multivariable dynamical systems:

I. m.d.s. with a negative coupling - i.e. such a system for which

\[
1 - \frac{\det P(o)}{\prod_{i=1}^{q} P_{ii}(o)} < 0 \tag{17}
\]

II. m.d.s. with a positive coupling - i.e. such a system for which

\[
1 - \frac{\det P(o)}{\prod_{i=1}^{q} P_{ii}(o)} > 0 \tag{18}
\]
III. \textbf{m.d.s. with a zero coupling} - i.e. such a system for which
\begin{equation}
\det P(o) = 0
\end{equation}
\begin{equation}
1 - \prod_{i=1}^{q} P_{ii}(o) = 0
\end{equation}
M.d.s. with a positive coupling resemble very much simple input-output systems with a positive feedback, inclining to monotonic instability.

Another classification, credited to MacFarlane (1970), Ostrowski (1952) and Rosenbrock (1970), is suggested according to a very convenient stability criterion by MacFarlane (1970):

a) \textbf{m.d.s. with a dominant main diagonal} - i.e. systems for which
\begin{equation}
\left| P_{ii}(j\omega) \right| > \sum_{k=1, k\neq i}^{q} \left| P_{ik}(j\omega) \right| \quad i = 1, 2, \ldots, q
\end{equation}
\begin{equation}
\left| P_{ii}(j\omega) \right| > \sum_{k=1}^{q} \left| P_{ki}(j\omega) \right| \quad i = 1, 2, \ldots, q
\end{equation}

b) \textbf{m.d.s. without a dominant main diagonal} - i.e. all those for which (20) or (21) does not hold.


1.2.3. \textbf{State space representations}
Let us start with a definition.

Definition 5 in the chapter 1.1.1. is valid for both the continuous and discrete time systems. The physical meaning of this definition is that the dynamical
system has a certain "memory" which contains information about influence of past events on the present and the future. For the continuous time systems definition 6 changes into the set of equations

$$\frac{d}{dt} \mathbf{x}(t) = A \mathbf{x}(t) + B \mathbf{u}(t)$$

$$\mathbf{y}(t) = C \mathbf{x}(t) + D \mathbf{u}(t)$$

assuming $$\mathbf{x}(t_0) = \mathbf{x}(0)$$ - initial conditions.

A solution to this problem is

$$\mathbf{x}(t) = e^{A(t - t_0)} \mathbf{x}(0) + \int_{t_0}^{t} e^{A(t - \tau)} B \mathbf{u}(\tau) d\tau$$

(23)

$$\mathbf{y}(t) = C e^{A(t - t_0)} \mathbf{x}(0) + C \int_{t_0}^{t} e^{A(t - \tau)} B \mathbf{u}(\tau) d\tau + D \mathbf{u}(t)$$

(24)

For the discrete time systems following definition 6 we have:

$$\mathbf{x}(k + 1) = A \mathbf{x}(k) + B \mathbf{u}(k)$$

(25)

$$\mathbf{y}(k) = C \mathbf{x}(k) + D \mathbf{u}(k)$$

(26)

where matrices $$\{A, B, C, D\}$$ from eq. (22) are not the same as from eq. (25) (26).

Solutions to (25) and (26) are:

$$\mathbf{x}(N) = A^N \mathbf{x}(0) + \sum_{k=0}^{N-1} A^{N-k-1} B \mathbf{u}(k)$$

(27)

$$\mathbf{y}(N) = C A^N \mathbf{x}(0) + C \sum_{k=0}^{N-1} A^{N-k-1} B \mathbf{u}(k) + D \mathbf{u}(N)$$

(28)
Using equations (22) and (25), (26) it is easy to find relations between state equations and transfer functions for both types of systems:

a) **continuous-time m.d.s.**

applying the "Laplace" transform and combining two operator equations:

assuming

\[ y(s) = C(sI - A)^{-1} B u(s) + C(sI - A)^{-1} x(o) + D u(s) \]  \hspace{1cm} (29)

for the "zero" initial conditions are required for the transfer function definition, \( x(o) = 0 \)

\[ K(s) = C(sI - A)^{-1} B + D, \text{ where } I = \begin{bmatrix} \text{Unitary matrix} \end{bmatrix} \text{ of dimension } (n \times n) \] \hspace{1cm} (30)

b) **discrete-time m.d.s.**

applying the "Z" transform and combining eq. (25)(26)

\[ y(z) = C(zI - A)^{-1} B u(z) + z C(zI - A)^{-1} x(o) + D u(z) \] \hspace{1cm} (31)

and again for the "zero" initial condition

\[ K(z) = C(zI - A)^{-1} B + D \] \hspace{1cm} (32)
The block scheme of the multivariable dynamical system described in terms of state equations is as follows:

Example 3: Reconsidering example 2, we can write the following set of equations:

\[
\begin{align*}
\dot{\Delta V}_1 &= \Delta Q_1 - \Delta Q_{11} ; \\
\Delta V_1 &= \Delta h_1 \\
\dot{\Delta V}_2 &= \Delta Q_2 - \Delta Q_{21} ; \\
\Delta V_2 &= \Delta h_2 \\
\dot{V} &= \Delta Q_{11} + \Delta Q_2 - \Delta Q ; \\
\Delta V &= \Delta h \\
\end{align*}
\]

choosing \( \Delta V_1, \Delta V_2 \) and \( \Delta V_3 \) as state variables and writing

\[
\begin{align*}
\dot{\Delta Q} &= \alpha \frac{\Delta V}{\Delta} \\
\Delta Q_{11} &= \beta \frac{\Delta V_1}{\Delta} \\
\end{align*}
\]
we get:

\[
\begin{align*}
\frac{d\Delta V_1}{dt} &= -\frac{\beta_{10}}{A_1} \Delta V_1 + \Delta Q_1 \\
\frac{d\Delta V_2}{dt} &= -\frac{\beta_{20}}{A_2} \Delta V_2 + \Delta Q_2 \\
\frac{d\Delta V}{dt} &= \frac{\beta_{10}}{A_1} \Delta V_1 + \frac{\beta_{20}}{A_2} \Delta V_2 - \frac{\alpha_0}{A} \Delta V \\
\Delta h &= \frac{\Delta V}{A} \\
\Delta h_2 &= \frac{\Delta V_2}{A_2}
\end{align*}
\]

\[
\begin{bmatrix}
\frac{d\Delta V_1}{dt} \\
\frac{d\Delta V_2}{dt} \\
\frac{d\Delta V}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\frac{\beta_{10}}{A_1} & 0 & 0 \\
0 & -\frac{\beta_{20}}{A_2} & 0 \\
\frac{\beta_{10}}{A_1} & \frac{\beta_{20}}{A_2} & -\frac{\alpha_0}{A}
\end{bmatrix}
\begin{bmatrix}
\Delta V_1 \\
\Delta V_2 \\
\Delta V
\end{bmatrix} +
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta Q_1 \\
\Delta Q_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta h \\
\Delta h_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta V_1 \\
\Delta V_2 \\
\Delta V
\end{bmatrix}
\]

\[
A =
\begin{bmatrix}
-\frac{\beta_{10}}{A_1} & 0 & 0 \\
0 & -\frac{\beta_{20}}{A_2} & 0 \\
-\frac{\beta_{10}}{A_1} & \frac{\beta_{20}}{A_2} & -\frac{\alpha_0}{A}
\end{bmatrix},
B =
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
C =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
D = 0
\]
Assuming that first q inputs to the system are the control quantities and remaining p - q are the disturbing quantities, we get the following state equation.

\[
\frac{dx(t)}{dt} = A \cdot x(t) + \begin{bmatrix} B_p & B_Q \end{bmatrix} \begin{bmatrix} v(t) \\ z(t) \end{bmatrix} \\
= A \cdot x(t) + B_p \cdot v(t) + B_Q \cdot z(t)
\]

(33)

\[
v(t) = C \cdot x(t) + \begin{bmatrix} D_p & D_Q \end{bmatrix} \begin{bmatrix} v(t) \\ z(t) \end{bmatrix} \\
= C \cdot x(t) + D_p \cdot v(t) + D_Q \cdot z(t)
\]

(34)

\(B_p(n \times q), \ B_Q(n \times (p - q)), \ D_p(q \times q), \ D_Q(q \times (p - q))\) which corresponds to the transfer matrix of control inputs

\[
P(s) = C(\mathbf{I}s - A)^{-1} \cdot B_p + D_p
\]

(35)

and transfer matrix of disturbances

\[
Q(s) = C(\mathbf{I}s - A)^{-1} \cdot B_Q + D_Q
\]

(36)

The matrix \(\mathbf{I}s - A\) is called the characteristic matrix of \(A\) and the polynomial

\[
W_A(s) = \det(s\mathbf{I} - A)
\]

(37)

is the characteristic polynomial of \(A\).

Example 4

For the dynamical system from examples 2 and 3 we have

\[
W(s) = (s + \frac{\beta_1}{A_1})(s + \frac{\beta_2}{A_2})(s + \frac{\alpha_0}{A})
\]
\( W_A(s) \) - calculated as the Characteristic Polynomial of \( A \)

\[
W_A(s) = (s + \frac{B_{1p}}{A_1})(s + \frac{B_{2p}}{A_2})(s + \frac{C_0}{A})
\]

\( W(s) = W_A(s) \)

Discussion

The degrees of the \( W(s) \) and \( W_A(s) \) are the invariants of the multivariable dynamical system. The degree of the \( W_A(s) \) is equal to the dimension of the state space (see def. 8). The degree of \( W(s) \) is the degree of the transfer matrix \( K(s) \) (see def. 4). In general, \( W(s) \neq W_A(s) \) and \( W_A(s) \neq W(s) \). This will be discussed in the sequel.

1.2.4. Nonuniqueness of the state space equations

It is sometimes difficult to find a state space description which has a physical interpretation. This happens while estimating the state space equations basing an experimental input-output data without a prior knowledge of the physical structure of the considered object. This is due to nonuniqueness of the state space equations.

It can be proven that for a m.d.s. it is possible to find and infinite number of state equations.

Let

\[
X(t) = T \hat{X}(t)
\]

where \( T \) (\( n \times n \)) is any nonsingular matrix. Thus equations (22) or (25)(26) will result in

\[
\frac{d \hat{X}(t)}{dt} = T^{-1}AT \hat{X}(t) + T^{-1}Bu(t)
\]
This new set of state equations results in the same transfer function matrix as equations (22):

\[
\begin{align*}
\mathbf{C} \mathbf{T} (\mathbf{I}s - \mathbf{A}^{-1} \mathbf{T})^{-1} \mathbf{T}^{-1} \mathbf{B} + \mathbf{D} &= \mathbf{C} \mathbf{T} \left[ (\mathbf{T}s - \mathbf{A}^{-1} \mathbf{T})^{-1} \mathbf{T}^{-1} \mathbf{B} + \mathbf{D} \right] \\
&= \mathbf{C} \mathbf{T} \left[ (\mathbf{I}s - \mathbf{A})^{-1} \mathbf{T}^{T} \mathbf{B} + \mathbf{D} \right] \\
&= \mathbf{C} (\mathbf{I}s - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D} = \mathbf{K}(s)
\end{align*}
\] (41)

The relation \( \mathbf{A} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T} \) is called the "similarity relation" and there is a special type equivalence between matrices \( \mathbf{A} \) and \( \mathbf{A} \). For more details see: DeRusso, P.M., Roy R.J., Close Ch. M. (1965), Rosenbrock H.H. (1970), Wolovich W.A. (1974), Birkhoff G., MacLane S. (1965), Gantmacher (1959) and many others.

1.3. Controllability and observability in multivariable dynamical systems

The multivariable dynamical system described by means of the state equation

\[
\dot{x}(t) = \mathbf{A} x(t) + \mathbf{B}_p v(t)
\] (42)

is said to be COMPLETELY CONTROLLABLE in Kalman's sense if given any initial state \( x(t_0) \) there exist such the control vector \( v(t) \), which will drive the m.d.s. to the final state \( x(t_f) = \phi \) for finite \( (t_f - t_0) \).

**Theorem 3**

The necessary and sufficient condition for the multivariable dynamical system to be controllable in Kalman's sense is that the block matrix (n x qk)

\[
\begin{bmatrix}
B_p & A B_p & A^2 B_p & \cdots & A^k B_p
\end{bmatrix}
\]

where \( K \leq n \) is the degree of the minimal polynomial of \( A \), has the rank equal to \( n \).


From condition (43) it follows that the complete controllability in Kalman's sense is the property of the pair of matrices \((A, B_p)\) and does not depend on the way outputs are produced by the system.

The multivariable dynamical system described by means of the state equation (42) and the output equation

\[
\chi(t) = C x(t)
\]

is said to be completely observable in Kalman's sense if given any input \( \chi(t) \) and output \( \chi(t) \) for \( t_o \leq t \leq t_f \) is sufficient to determine the initial state \( x(t_o) \) for a finite interval \([t_o, t_f]\).

The necessary and sufficient conditions for complete observability in Kalman's sense are usually formulated in the following way.

**Theorem 4**

The necessary and sufficient condition for the multivariable dynamical system to be completely observable in Kalman's sense is that the block matrix (n x qk)

\[
\begin{bmatrix}
C^T & A C^T & (A^T)^2 C^T & \cdots & (A^T)^k C^T
\end{bmatrix}
\]

(45)
where \( k \leq n \) is the degree of the minimal polynomial of \( A \), has the rank equal \( n \).
(Alternative conditions are also discussed by Kuo B.C. (1970), Paul. C.R. and Kuo Y.L. (1971)).

In the classical analysis of control systems, transfer functions are often used for the modelling of linear time-invariant systems. Although controllability and observability are concepts of modern control theory, they are closely related to the properties of the transfer function. The following theorem gives the relationship between controllability and observability and the pole zero cancellation of a transfer function.

Theorem 5
If the input-output transfer function of a linear system has pole-zero cancellation, the system will be either not state controllable or unobservable, depending on how the state variables are defined. If the transfer function of a linear system does not have pole-zeros cancellation, the system can always be represented by dynamic equations as a completely controllable and observable system. Kuo B.C. (1975). (This is an excellent reference for further readings for everyone who wants to gain more information about dynamical systems being considered in a very physical way).

Concepts of controllability and observability in the Kalman's sense are rather useless while dealing with noisy systems. Also for some theoretical systems those concepts can cause misunderstandings. An example will be given while discussing a closely related concept of the multivariable system order.
2. Basic structures of the multivariable dynamical systems and canonical forms

Three main types of models incorporated in multivariable dynamical systems design, analysis and identification have been described in brief. According to very specific properties of each of these models, slightly deeper insight must be done into their structural properties and utility possibilities for different types of tasks. The concept of the multivariable dynamical system order will be of an essential importance for the following part of the text.

2.1. Definition of the order of the multivariable dynamical system

In trying to model the reality, one has to answer first a question: what type of applications for this model is considered?, thus facing the problem of the "structure" choice for this model. This structure presents a desired type of relations between inputs and outputs. However, two additional steps must be performed - those are: demarcation of the "degree of complexity" for this model (corresponds to the order determination) and parametric estimation of the chosen model being of the pre-estimated degree of complexity


This procedure in most practical cases is an iterative seeking for the model order and incorporated set of parameters, matching it to real data according to a given optimality criterion and comparing with results of previous runs.
Definition 13  The order of the multivariable dynamical system will be defined as the minimal number of Markov-parameters necessary and sufficient to reconstruct the entire realizable sequence of Markov-parameters. Hajdasiński, Damen (1979).
Remark It means that the system order is equal to the realizability index "r" - see relation (9) Theorem 1.

This definition favours the Hankel model description and it really is the intention for this type of model to give the most general possibilities of descriptiveness without causing any ambiguity. Having a properly described model complexity it is very easy to generate proper structures in the state space and a properly structured transfer function matrix.

For example for the state space description, the multivariable system order can be defined as the degree of the minimal polynomial. For the transfer function matrix description, however, the order definition in the general case is not possible.

Definition 14 The dimension of the multivariable dynamical system is defined as the number "n" being equal to the rank of the \( H_r \) - Hankel matrix for this system, where "r" is the order of the system.

Remark Compare def. 14 with the theorem 2, relations (10) and (11).

Alternatively for the state space description it is the dimension of the state matrix A. And again for the transfer matrix description there does not exist a unique definition of the system dimension. Only in the cases when all poles in elements of the transfer matrix are either different or "equal and common" the dimension can be determined as the degree of this matrix (see definition 4.)

To illustrate this, the following example shows the case of equal but distinct or non-common poles. Hajdasinski - Damen (1979).

\[
K(z) = \begin{bmatrix}
1 & 0.0 \\
\frac{1}{(z - 0.8)(z - 0.25)(z - 0.5)} & 0.0 \\
0.0 & \frac{1}{(z - 0.8)(z - 0.6)(z - 0.01)}
\end{bmatrix}
\]
Dimension of this system is $n = 6$, but degree $\delta k(z) = 5$ because the equal poles $z = 0.8$ are noncommon and they refer to different state variables. However, in practical cases it will be seldom that distinct poles have exactly the same value i.e. are equal. Nevertheless, when poles are given up to a certain accuracy (for example numerically evaluated poles) it may be difficult to decide whether they are really distinct or not. This problem is also one of the drawbacks of the transfer matrix description and one more argument for the state space and Hankel description, where this ambiguity never arises.

2.1.1. Advantages and disadvantages of the transfer function matrix models

The following features of transfer function matrix models are worth noticing.

I. the transfer matrix description is a unique description of the multi-variable dynamical system given a unique ordering of inputs and outputs. It means that there exists one and only one transfer function matrix $k(z)$ for a given order of inputs and outputs.

II. the transfer function matrix has a very easy physical interpretation for elements $K_{ij}(z)$ of $K(z)$ are transmitances between $y_i(k)$ outputs and $u_j(k)$ inputs of the considered system.

III. the transfer function matrix description is not very economical for the analog computer modelling.

IV. the transfer function matrix is very inconvenient for digital modelling.

V. the transfer function matrix may not encounter all dynamical properties of the system (see 2.1.).

VI. from knowledge of the transfer function matrix it is difficult to derive state space equations, but it is fairly easy to derive the Hankel model.
2.1.2. Advantages and disadvantages of the state space description

The following features of the state space models are worth noticing:

I. the state space model is a non-unique description of the multivariable dynamical system given an unique ordering of inputs and outputs (see 1.2.4.)

II. in a general case there is a lack of physical interpretation for the state space model.

III. the state space model is more economical for analog modelling than the transfer function matrix model.

IV. the state space model is very convenient for digital modelling

V. the state space model encounters all dynamical properties of the system being modelled.

VI. the state space model provides equally easy transformation into the transfer function matrix as into the Hankel model.
2.2. Observable and controllable canonical forms for the state space models

This chapter will start with the phase canonical forms for single input - single output controllable systems. The generalization for multi input - multi output systems will be then easier. The major results in the field of canonical forms are due to Kalman R.E. (1963), Luenberger G.D. (1966), Mayne D.Q. (1972a, 1972b) Popov V.M. (1972).

Assuming that there is given the transfer function $K(z)$:

$$K(z) = \frac{y(k)}{u(k)} = \frac{b_0 + b_1 z + \ldots + b_{m-1} z^{m-1}}{a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + z^n}$$

for $m \leq n$, and there is no pole-zero cancellation, the corresponding state space equations in the minimal canonical form may take the following form:

$$\begin{bmatrix}
x_1(k + 1) \\
x_2(k + 2) \\
\vdots \\
x_{n-1}(k + 1) \\
x_n(k + 1)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0 & -a_1 & -a_2 & \ldots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix}
x_1(k) \\
x_2(k) \\
\vdots \\
x_{n-1}(k) \\
x_n(k)
\end{bmatrix} +
\begin{bmatrix}
x_1(k) \\
x_2(k) \\
\vdots \\
x_{n-1}(k) \\
x_n(k)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & \ldots & b_{m-1} & 0 & \ldots & 0 \end{bmatrix}
\begin{bmatrix}
x_1(k) \\
x_2(k) \\
\vdots \\
x_{n-1}(k) \\
x_n(k)
\end{bmatrix}$$

Equations (47) are referred to as the "phase-variable canonical form" and
state variables $x_i(u)$ are called the "phase variables".

The state $A$ is called the "Frobenius matrix" or the "companion matrix of the polyno-
minal $a_0 + a z + \ldots + a_{u-1} z^{u-1} + z^n$.

Another phase canonical form will be

\[
\begin{bmatrix}
\hat{x}_1(k+1) \\
\hat{x}_2(k+1) \\
\vdots \\
\hat{x}_{n-1}(k+1) \\
\hat{x}_n(k+1)
\end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 - a_0 \\
1 & 0 & \cdots & 0 & 0 - a_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 - a_{n-2} \\
0 & 0 & \cdots & 0 & 1 - a_{n-1}
\end{bmatrix} \begin{bmatrix}
\hat{x}_1(k) \\
\hat{x}_2(k) \\
\vdots \\
\hat{x}_{n-1}(k) \\
\hat{x}_n(k)
\end{bmatrix} + \begin{bmatrix} b_0 \\
b_1 \\
\vdots \\
b_{m-1} \\
0
\end{bmatrix} u(k)
\]

(48)

\[
y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix}
\hat{x}_1(k) \\
\hat{x}_2(k) \\
\vdots \\
\hat{x}_{n-1}(k) \\
\hat{x}_n(k)
\end{bmatrix}
\]

The development of phase canonical form for single output - single input system was an attractive area of research for two main basic reasons:

(1) simplicity of derivation

(2) a convenient starting point for certain control design problems.

The canonical forms for MIMO systems are even more important than for SISO systems. The canonical form will be defined as trans-
formation of the state vector to a new coordinate system in which the system equations take a particular simple form". (see Niederliński - Hajdasiński (1979)).

Unlike the SISO case, the corresponding canonical forms for multivariable systems are not unique. Among the most used canonical forms, the canonically observable and controllable forms are of the greatest importance.

2.2.1. **Canonically observable form**

Consider the discrete completely observable multivariable system represented by state equations:

\[ x(k+1) = F x(k) + G u(k) \]
\[ y(k) = H x(k) \]  \hspace{1cm} (49)

Let
\[ H = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_q^T \end{bmatrix} \]  \hspace{1cm} (50)

Constructing the vector sequences
\[ h_1, (F^T h_1)^2, \ldots, h_q, (F^T h_q)^2, \ldots \]  \hspace{1cm} (51)

and selecting them in the following order
\[ h_1, h_2, \ldots, h_q, (F^T h_1)^2, \ldots, (F^T h_q)^2, \ldots \]  \hspace{1cm} (52)

retaining a vector \((F^T)^s h_1\) in (52) if and only if it is independent from
all previously selected ones and all the vectors \((F^T)^j_{h_j}\) \((0 < j < s)\) have already been selected. Let \(v_1, v_2, v_3 \ldots v_q\) be the numbers of vectors selected from the first, second and \(\ldots\) q-th sequence in (51);

\[
\begin{align*}
&h_1, (F^T)^2_{h_1}, (F^T)^3_{h_1} \ldots h_q, (F^T)^2_{h_2}, (F^T)^3_{h_2}, \ldots
\end{align*}
\]

Thus the vectors \((F^T)^j_{h_j}\) are therefore linearly dependent on previously selected ones (because of the choosing procedure and because \(v_j\) is counted from zero).

The complete observability of the system implies that

\[
v_1 + v_2 + \ldots + v_q = n
\]

where \(n\) is the dimension of the considered system.

Let us now construct the nonsingular matrix \( T\):

\[
T = \begin{bmatrix}
h_1, F^T h_1, \ldots (F^T)^{(v_1-1)} h_1 & h_q, \ldots (F^T)^{(v_q-1)} h_q
\end{bmatrix}
\]

and use the \(T\) as a transformation matrix for the similarity transformation (see 1.2.4.) providing a new state vector \(\hat{x}\):

\[
\hat{x} = Tx
\]

Remark: The \(T\) is a nonsingular \((n \times n)\) matrix which consists of linearly nondependent columns, each product \((F^T)^i_{h_j}\) is the vector and

\[
v_1 + v_2 \ldots v_q = n\] Luemberger (1967).

The state equations are:

\[
\begin{align*}
\dot{x}(k+1) &= A \hat{x}(k) + B u(k) \\
x(k) &= C \hat{x}(k)
\end{align*}
\]
where matrices $A, B,$ and $C$ are of the following form

$$A = T F T^{-1} = \{A_{ij}\} \quad i, j = 1, 2, \ldots, q$$

and $(v_i \times v_i)$ matrices $A_{ii}$ show the structure

$$A_{ii} = \begin{bmatrix} 0 \\ \vdots \\ I_{v_i-1} \\ \vdots \\ 0 \\ a_{ii,1} \ldots a_{ii,v_i} \end{bmatrix}$$

which is exactly as already discussed for SISO systems Frobenius matrix.

Because of the order followed in the selection of vectors in (51) and of the consequent structure of $T$, in every matrix $A_{ij}$ at most $v_i + 1$ elements are non-identically zero if $j < i$ and $v_i$ if $j > i$. Thus the number $v_{ij}$ is given by

$$v_{ij} = \begin{cases} 1 + \min(v_i, v_j - 1) & \text{for } j < i \\ \min(v_i, v_j) & \text{for } j > i \end{cases}$$

and the $(v_i \times v_j)$ matrices $A_{ij}$ are of the type

$$A_{ij} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ a_{ij,1} & \ldots & a_{ij,v_{ij}} & \ldots & 0 \end{bmatrix}$$

Further more

$$C = HT^{-1} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & (v + 1) & \ldots & v_{i} & (v_i + \ldots + v_{q-1} + 1) \end{bmatrix}$$
\[ B = T G = \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1p} \\
  b_{21} & b_{22} & \cdots & b_{2p} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{np}
\end{bmatrix} \]

does not possess any special structure \((62)\)

It should be noticed that the structure of the canonically observable couple \((A, C)\) is completely determined by \(v_i\) - indices which are called \textit{Kronecker invariants}. These are \textit{structural invariants} of the dynamical system. (see Luenberger D.G. (1967), Guidorzi R.P. (1973), Popov V.M. (1972), Niederliński, Hajdasiński (1979)). A single-output, single-input system has only one structural invariant, namely the system order \(r\) or the system dimension \(n\) (which are related to each other by the relation \(n = \text{rank } H_r\) - see Theorem 2, relation (10)).

The transformation \(T\) decomposes the original system \((F, H)\) into \(q\) interconnected subsystems having a structure guaranteeing the complete observability of the \(j\)th subsystem from the \(j\)th output component. This properly justifies the name "canonically observable form". (Niederliński - Hajdasiński (1979)).

\textbf{Remark:} The dependent vectors \((F^T)^i v_i h_j\) are a linear combination, with coefficients given by the \(i\)th significant row of \(A\) matrix, of previously selected ones i.e. as demonstrates Popov (1972).

\[ (F^T)^i v_i h_j = \sum_{j=1}^{i-1} \sum_{k=1}^{\min(v_i, v_j) - 1} a_{ij,k} (F^T)^k h_j + \sum_{j=1}^q \sum_{k=1}^{\min(v_i, v_j)} a_{ijk} (F^T)^k h_j \quad (63) \]

(See also Niederliński - Hajdasiński (1979)).

\textbf{Remark:} Given \textit{Kronecker invariants} \(v_i\), the number of \(a_{ij,k}\) invariants is well determined with the following properties (Niederliński Hajdasiński (1979)).

\[(1)\] the set of \(v_i\) and \(a_{ij,k}\) invariants are independent, i.e. for any private integers \(v_i\) satisfying \(v_1 + v_2 + \ldots + v_q = n\)
and any numbers aij,k eR with properly ranging indices (i,j,k), there exists a pair of matrices (F,H) whose invariants are precisely the above integers vi and numbers aij,k.

(2) the set of invariants vi and aij,k are complete, i.e. if for two pairs of matrices (F₁,₁) and (F₂,₂) of the same dimension the invariants are respectively equal, there exists a nonsingular matrix \( T \in \mathbb{R}^{n \times n} \) such that \( \hat{F} = T F T^{-1} \) and \( \hat{H} = H T^{-1} \).

(3) the set of vi and aij,k invariants in the smallest set of parameters determining a canonical form. It is impossible to construct a canonical form having the universality properly with less parameters than the number of aij,k parameters, with the vi parameters determining their position in the canonical form.

2.2.2. **Canonically controllable form**

For the controllable couple (G,F) from equations (49), (50) also exists a canonically controllable form derived by writing the G matrix in the form:

\[
G = \begin{bmatrix} G_1 & G_2 & \cdots & G_p \end{bmatrix}
\]  

and constructing the sequence

\[
E_1, E_2, \ldots, E_p, F_{g_1}, F_{g_2}, \ldots, F_{g_p}, F^2_{g_1}, F^2_{g_2}, \ldots, F^2_{g_p}, \ldots
\]

Similarly, as before, the choice of linearly independent vectors \( F^s_{g_j} \) can be made and again \( n_1, n_2, \ldots, n_p \) are the numbers of vectors selected from sequences.

\[
\begin{align*}
E_1, & F_{g_1}, F^2_{g_1}, \ldots, \\
E_2, & F_{g_2}, F^2_{g_2}, \ldots, \\
\vdots, & \vdots, \vdots, \ldots, \\
E_p, & F_{g_p}, F^2_{g_p}, \ldots
\end{align*}
\]
Then the vectors $F_{n_1}^{n_2}g_i$ are linear combinations of previously chosen $F_{n_1}^{n_2}g_j$ and following relations hold:

$$n_1 + n_2 + \ldots + n_p = n \quad (67)$$

($n_i$ are Kronecker invariants of the controllable form).

$$F_{n_i}^{n_k}g_i = \sum_{j=1}^{i-1} \sum_{k=0}^{\text{min}(n_i,n_j-1)} \tilde{a}_{ij,k} F_{n_i}^{k}g_i + \sum_{j=1}^{i} \sum_{k=0}^{\text{min}(n_i,n_j-1)} \tilde{a}_{ij,k} F_{n_i}^{k}g_i \quad (68)$$

where $\tilde{a}_{ij,k}$ are invariant parameters corresponding to the $a_{ij,k}$. The sets of the $\{n_i\}$ and $\{\tilde{a}_{ij,k}\}$ invariants share all properties of the sets of $\{v_i\}$ and $\{a_{ij,k}\}$ invariants.

Introducing the new state vector $z = R x$, with

$$R^{-1} = \begin{bmatrix} g_1 & F_{n_1}^{n_2}g_1 & F_{n_1}^{n_2}g_1 & \ldots & F_{n_1}^{n_p}g_1 \\ \end{bmatrix} \quad (69)$$

the system equations (49) (50) become

$$z(k+1) = A z(u) + B u(k) \quad (70)$$

$$y(k) = C z(k) \quad (71)$$

where $A = R F R^{-1} = \{A_{ij}\}$, $i,j = 1,2, \ldots, p \quad (72)$

with $n_i \times n_i$ matrices $A_{ii} = \begin{bmatrix} 0 & \ldots & 0 & \tilde{a}_{ii} & \ldots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & 1_{n_i} - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \tilde{a}_{ii}, n_i & \ldots & \tilde{a}_{ii} & 0 & \ldots \\ \end{bmatrix} \quad (73)$

and $n_i \times n_j$ matrices $A_{ij} = \begin{bmatrix} 0 & \ldots & 0 & \tilde{a}_{ij,1} & \ldots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \tilde{a}_{ij,2} & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \tilde{a}_{ij,n_i} & \ldots & \tilde{a}_{ij,1} & \tilde{a}_{ij,2} & \ldots \\ \end{bmatrix} \quad (74)$
and
\[ n_{ij} = \begin{cases} 
1 + \min(n_i, n_j - 1) & \text{for } j < i \\
\min(n_i, n_j) & \text{for } j > i 
\end{cases} \tag{75} \]

Further
\[
B = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
0 & 1 & \ldots & 0 \\
\ldots & \ldots & \ddots & \ldots \\
0 & 0 & \ldots & 1
\end{bmatrix} \quad \text{or } n_1 + 1
\tag{76}
\]

and \( C = H R^{-1} \) does not reflect any specific structure
\[
C = \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1u} \\
c_{q_1} & c_{q_2} & \cdots & c_{qu}
\end{bmatrix}
\tag{77}
\]

It can be noticed that the transformation (69) has decomposed the original couple \((F, G)\) into \(p\) interconnected subsystems having a structure guaranteeing the complete controllability of the \(i\)th subsystem by the \(i\)th input component.

This property justifies the name "canonically controllable form". (see Niederliński, Hajdasiński (1979)).

Remark: The observable and controllable canonical forms are only two among many possible canonical forms, but also very important ones.
As it will be discussed later, these two canonical forms play an important role in determination of the multivariable system structure.

For further reading in the subject of canonical forms see Denham (1974), Mayne (1972a,b), Niederliński, Hajdasiński (1979).

2.3. Innovation state space models

The essence of the innovation approach to the state space modelling can be expressed in words of Kailath T. (1968):

"...... the innovation approach is first to convert the observed process to a white-noise process, to be called the innovation process, by means of a causal and causally invertible linear transformations. The point is that the estimation problem is very easy to solve with white-noise observations. The solution to this simplified problem can then be re-expressed in terms of the original observations by means of the inverse of the original "whitening" filter ....."

The given observations are recorded in the form

\[ y(k) = z(k) + v(k) \quad k = 0,1,2 \ldots \quad (79) \]

\[ E\{v(k)\} = 0 \quad \text{- expected value of the } v(k) \quad (80) \]

\[ E\{v(k)v^T(l)\} = R(k) \delta_{kl} \quad \text{- covariance matrix for } v(k) \quad (81) \]

\[ \delta_{kl} = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases} \quad \text{Kronecker delta} \]

+ The external properties of a physical system can frequently be characterized by an operator relation of the form

\[ g = T f \quad (78) \]
\{z(k)\} a zero-mean finite-variance process, i.e.

\[
\text{trace } E\{z(k)z^T(k)\} < \infty
\]  \hspace{1cm} (82)

Let us introduce further the linear least-squares estimate of

\[ z(k), \hat{z}(k|k-1) \] given \( y(1), 0 \leq 1 < k - 1 \)

what is the meaning of the last, it is necessary to solve the problem of the optimum estimation and conditional expectation. Also, for the sake of the innovation derivation we will need a theorem of the orthogonal projection.

To solve the final problem, we will refer to sub-chapters 2.3.1. and 2.3.2.

2.3.1. Optimum estimation and conditional expectation

The innovation approach to multivariable dynamical systems plays a still more and more important role in modelling identification and the state reconstruction.

\[ \text{ in which } g \text{ and } f \text{ are real } n\text{-vector valued functions of } t, \text{ such that } g^Tf \]

represents the total instantaneous power entering the system. (For example an electromechanical system, the jth component of \( f \) might be voltage or force at jth accessible point of the system, in which case \( g \) would be the corresponding n-vector of currents or velocities. The notions of energy and causality play a central role in the study of physical systems. Let us agree to say that \( T \) is causal if for every \( \delta > -\infty \), \( Tf_1 \) and \( Tf_2 \) agree on the interval \( (-\infty, \delta) \) whenever the n-vector-valued functions \( f_1 \) and \( f_2 \) are permissible inputs to the system \( T \) and \( f_1 \) and \( f_2 \) agree on \( (-\infty, \delta) \). Let us say also that \( T \) is passive if \[ \int_{-\infty}^{\delta} (Tf)^Tf \ dt \geq 0 \] for all real \( x \) and permissible \( f \). It has been proven that under some weak assumptions, if an operator \( T \) is linear and passive, it is also causal.
For many other reasons, such as simplification of derivations, equivalence to the Kalman filter without necessity to solve the Riccatti equation, easier implementation of the identifiability notion, to mention but a few, we will show a very formal derivation of the innovation state space model. This requires more complicated mathematics and a few notions which may seem to be complicated at first sight. However, it will profit in deeper understanding of the more general properties of the state space models and related to it realizations.

Let \( \{y(0), y(1), \ldots, y(k-1)\} \) be a set of measured random variables of \( y(i) \). We can determine in principle, the probability of \( z_i(j) \) (where \( z_i(j) \) is the \( j \)th component of the \( z(j) \)) assuming value \( \xi_i \) or smaller for \( i = 1, 2, \ldots, n \).

This follows from the definition of the conditional probability function.

Considering a random variable \( z \) of a random process, given the actual value \( \eta \) with which the random variable \( y \) has occurred, the conditional probability function of \( z \) given \( y \) is denoted by \( p(\xi|\eta) \), and is defined by the following relationship:

\[
p_x = \int_{\xi < z < \xi+} d\xi | y = \eta = p(\xi|\eta) d\xi \tag{83}
\]

In general, the conditional probability density of \( z \), given

\[
y(0) = \eta_0, y(1) = \eta_1, \ldots, y(k-1) = \eta_{k-1}
\]

is denoted by \( p(\xi|\eta_0, \eta_1, \ldots, \eta_{k-1}) \).

The conditional probability distribution function of \( z \) given

\[
y(0) = \eta_0, \ldots, y(k-1) = \eta_{k-1}
\]
is written

\[ P(\xi | \eta_0, \eta_1, \ldots, \eta_{k-1}) = \frac{\int_{-\infty}^{\xi} P(x|\eta_0, \eta_1, \ldots, \eta_{k-1}) \, dx}{P(\eta_0, \eta_1, \ldots, \eta_{k-1})} \]  \hspace{1cm} (84)

Defining the joint probability distribution function of the random variables \( z(j), z(j), \ldots, z(j) \) as the probability of the simultaneous occurrence of:

\[ z_1(j) \leq \xi_1, z_2(j) \leq \xi_2, \ldots, z_n(j) \leq \xi_n \]  \hspace{1cm} (85)

and denoting it \( P(\xi_1, \xi_2, \ldots, \xi_n) \), and defining the marginal probability distribution function of random variables \( z, y(0), y(1), \ldots, y(k-1) \)

\[ P(z \leq \xi, y(0) \leq \eta_0, \ldots, y(k-1) \leq \eta_{k-1}) = P(\xi, \eta_0, \eta_1, \ldots, \eta_{k-1}) \]  \hspace{1cm} (86)

the conditional probability distribution function is written

\[ P(\xi | \eta_0, \eta_1, \ldots, \eta_{k-1}) = \frac{P(\xi, \eta_0, \eta_1, \ldots, \eta_{k-1})}{P(\eta_0, \eta_1, \ldots, \eta_{k-1})} \]  \hspace{1cm} (87)

which is nothing but a Bayesian relation.

Let us use the expression

\[ \mathcal{Z}(j | k - 1) = P(r \{ z_1(j) \leq \xi_1, \ldots, z_n(j) \leq \xi_n | y(0), \ldots, y(k-1) \} \]  \hspace{1cm} (88)

to describe the probability that the random variable \( z_i(j) \) will be less than or equal to some value \( \xi_i \) simultaneously for \( i = 1, 2, \ldots, n \), at the time \( j \), given measurements \( y(0), y(1), \ldots, y(k-1) \). This expression of the conditional expectation distribution function is clearly also a statistical estimate of the
random variable $x(j)$, given information on $y(j)$. This statistical estimate is represented by $\hat{x}(j|k-1)$, which is assumed to be a fixed vector whose elements are known whenever $y(0), y(1), \ldots, y(k-1)$ are known.

The general estimate $\hat{x}(j|k-1)$ will be different from the actual random variable $z(j)$, which is unknown. For the purpose of comparison, it is desirable to define an error function vector $e$,

$$ e(j) = -\hat{x}(j|k-1) + z(j) \quad (89) $$

Let us define, according to the statement made, that we are looking for the linear least-squares estimate of $z(j)$, the loss function as $L(e) = e^T(j)e(j)$. We want to show that minimisation of the average or expected value of $L(e)$ is equivalent to minimising conditional expectation. Given the random variable vector $y(j) = z(j) + v(j)$, where $z(j)$ is the actual signal vector and $v(j)$ is the noise vector and $y(0), y(1), \ldots, y(k-1)$ are k given measured random vectors.

It is desired to find the best estimate of $z(j)$ at $j \geq k$ or $j < k$ from knowledge of the measured random variables. ($j > k$ - smoothing (interpolation), $j = k$ - filtering, $j > k$ - predicting).

The problem is to find the optimum estimate $\hat{x}_{opt}(j|k-1)$, given the measurement on $y(0), \ldots, y(k-1)$, so that the expected value of the loss function $L(e)$ is minimized.

$$ E[e^T(j)e(j) | y(0), y(1), \ldots, y(k-1)] = $$$$ E[(z(j) - \hat{x}(j|k-1))^T(z(j) - \hat{x}(j|k-1)) | y(0), y(1), \ldots, y(k-1)] = \min \quad (90) $$

Performing operations under the expectation operator we get

$$ E[e^T(j)e(j) | y(0), y(1), \ldots, y(k-1)] = E[z^T(j)z(j) | y(0), y(1), \ldots, y(k-1)] - $$ $$ - E[z^T(j|k-1)z(j) | y(0), y(1), \ldots, y(k-1)] - E[z^T(j)\hat{x}(j|k-1) | y(0), y(1), \ldots, y(k-1)] + $$ $$ + E[\hat{x}^T(j|k-1)\hat{x}(j|k-1) | y(0), y(1), \ldots, y(k-1)] \quad (91) $$
Since \( \hat{z}(j|k-1) \) is a known constant vector when \( y(0), y(1) \ldots y(k-1) \) are given, we have:

\[
E\{z^T(j)e(j)|y(0), \ldots, y(k-1)\} = E\{z^T(j)z(j)|y(0), \ldots, y(k-1)\}
\]

\[
- \hat{z}(j|k-1)E\{z(j)|y(0), \ldots, y(k-1)\} - E\{z^T(j)|y(0), y(1), \ldots, y(k-1)\} +
\]

\[
+ \hat{z}^T(j|k-1)\hat{z}(j|k-1)
\]  

(92)

Taking the partial derivative of the last equation with respect to \( \hat{z}(j|k-1) \) and set it equal to 0.

\[
\frac{\partial}{\partial \hat{z}(j|k-1)} A \begin{bmatrix}
\frac{\partial}{\partial \hat{z}_1(j|k-1)}
\frac{\partial}{\partial \hat{z}_2(j|k-1)}
\vdots
\frac{\partial}{\partial \hat{z}_n(j|k-1)}
\end{bmatrix}
\]

(93)

\[
\frac{\partial}{\partial \hat{z}(j|k-1)} \left\{ \hat{z}^T(j|k-1) E\{z(j)|y(0), y(1), \ldots, y(k-1)\} \right\} = E\{z(j)|y(0), \ldots, y(k-1)\}
\]

(94)

\[
\frac{\partial}{\partial \hat{z}(j|k-1)} \left\{ E\{z^T(j)|y(0), y(1), \ldots, y(k-1)|\hat{z}(j|k-1)\} \right\} = E\{z(j)|y(0), \ldots, y(k-1)\}
\]

(95)

\[
\frac{\partial}{\partial \hat{z}(j|k-1)} \left\{ \hat{z}^T(j|k-1)\hat{z}(j|k-1) \right\} = \frac{\partial}{\partial \hat{z}(j|k-1)} \sum_{i=1}^{n} \hat{z}_{i}^2(j|k-1) = 2 \hat{z}(j|k-1)
\]

(96)

\[
-2E\{z(j)|y(0), \ldots, y(k-1)\} + 2\hat{z}_{opt}(j|k-1) = 0
\]

(97)
and

\[ \tilde{z}_{\text{opt}}(j|k-1) = \mathbb{E}\{z(j), y(0), y(1), \ldots, y(k-1)\} \]  

(98)

This shows that the optimum estimate of \( z(j) \) in the sense of the minimum least squares error is the conditional expectation of \( z(j) \).

2.3.2. Optimum estimation and orthogonal projection

Consider the set of random vectors \( \{z(j)\} \), \( j = 0, 1, 2, \ldots \), where \( z(j) \) is an \( n \)-vector. The set of all linear combinations of these random vectors of the form:

\[ \sum_{i=0}^{j} A_{i} z(i) \]  

(99)

(where \( A_{i} \) is an \( n \times n \) matrix with constant coefficients) forms a vector space which is denoted \( Z(j) \).

Similarly, we will call the vector space generated by the set of vectors

\[ \sum_{i=0}^{k-1} B_{i} y(i) - Y(k-1) \]  

(100)

The dimension of the matrix \( B_{i} \) with constant elements is \( n \times q \). If \( k - 1 \leq j \), the vector space \( Y(k-1) \) is a subspace of the vector space \( Z(j) \).

Now let us consider that the random vector \( z(j) \) is composed of two components, that is:

\[ z(j) = \tilde{z}(j|k-1) + \tilde{z}(j | k-1) \]  

(101)

where \( \tilde{z}(j|k-1) \) is the orthogonal projection of \( z(j) \) on the subspace \( Y(k-1) \).
The vector \( \tilde{z}(j|k-1) \) is the component which is orthogonal to the subspace \( Y(k-1)(\tilde{z}(j|k-1) \perp Y(k-1) \), that is, orthogonal to every vector in \( Y(j) \). Now we shall prove the following theorem:

**Theorem 6** The orthogonal projection \( \tilde{z}(j|k-1) \) of \( \tilde{z}(j) \) is that vector in \( Y(k-1) \) which minimizes the loss function

\[
E[\tilde{e}^T(j)\hat{e}(j) | \chi(0), \chi(1) \ldots \chi(k-1)].
\]

**Proof:** Let \( \xi \) be any vector \( n \times 1 \) in the vector space \( Y(k-1) \).

Forming the conditional expectation

\[
E\left[ (\tilde{z}(j) - \xi)^T [\tilde{z}(j) - \xi] | \chi(0), \chi(1) \ldots \chi(k-1) \right]
\]

and substituting (101) into (102) we get

\[
E[\tilde{z}(j) - \xi^T [\tilde{z}(j) - \xi] | \chi(0), \chi(1) \ldots \chi(k-1)] =
\]

\[
E[\tilde{z}(j|k-1)\tilde{z}(j|k-1) + \tilde{z}(j|k-1)^T [\tilde{z}(j|k-1) - \xi] +
\]

\[
+ \left[ \tilde{z}(j|k-1) - \xi \right]^T [\tilde{z}(j|k-1) - \xi] | \chi(0), \chi(1) \ldots \chi(k-1)]
\]

(103)

Since \( \tilde{z}(j|k-1) - \xi \) is the vector which is in \( Y(k-1) \), and \( \tilde{z}(j|k-1) \) is orthogonal to \( Y(k-1) \) and therefore also to \( \tilde{z}(j|k-1) - \xi \), equation (103) can be written:

\[
E[\tilde{z}(j) - \xi^T [\tilde{z}(j) - \xi] | \chi(0), \chi(1) \ldots \chi(k-1)] =
\]

\[
E[\tilde{z}(j|k-1)\tilde{z}(j|k-1) | \chi(0), \chi(1) \ldots \chi(k-1)] +
\]

\[
E[\tilde{z}(j|k-1)^T [\tilde{z}(j|k-1) - \xi] | \chi(0), \chi(1) \ldots \chi(k-1)]
\]

(104)

Because the term \( E[\tilde{z}(j|k-1)\tilde{z}(j|k-1) | \chi(0), \chi(1) \ldots \chi(k-1)] \)

will remain always positive definite, it is apparent that to
minimize (104), we must have \( \xi \) equal to equal \( \hat{z}(j|k - 1) = \hat{z}_{\text{opt}}(j|k - 1) \), so that the minimum loss function is:

\[
E\left[ \left( z(j) - \hat{z}_{\text{opt}}(j|k - 1) \right)^T \left( z(j) - \hat{z}_{\text{opt}}(j|k - 1) \right) \right] \big| y(0), y(1) \ldots y(k-1) = \mathcal{N} \left( 0, \Sigma_{\text{opt}} \right)
\]

\[
= E\left[ \hat{z}^T(j|k - 1) \hat{z}(j|k - 1) \big| y(0), y(1) \ldots y(k-1) \right]
\]

Q.E.D;

Remark 1: Relation (101) is of a great importance for consideration. It shows that \( \hat{z}(j|k - 1) \), which is an optimal estimate of \( z(j) \), can be expressed as a difference between a fixed vector (for a measured \( y(0), y(1), \ldots y(k - 1) \) and the correcting term \( \hat{z}(j|k - 1) \)

\[
\hat{z}(j|k - 1) = z(j) - \hat{z}(j|k - 1)
\]

To find the correcting term \( \hat{z}(j|k - 1) \) is due to the Kalman filtering theory and the innovational approach. (106) is called a "correction equation".

Remark 2: Based on the discussion carried out in these last two sections, we can now summarize:

\[
\hat{z}_{\text{opt}}(j|k - 1) = \text{OPTIMAL ESTIMATE OF } z(j) \text{ GIVEN } y(0), \ldots y(k - 1) = E\{z(j) \mid y(0), y(1), \ldots y(k - 1)\}
\]

Kalman filter approach \( \longrightarrow \)

Innovation approach \( \longrightarrow \)
2.3.3. The discrete-time innovation problem

Let us return to equations (79)(80)(81)(82).

\[ y(k) = z(k) + v(k) \quad \text{for } k = 0, 1, 2, \ldots \]  
\[ \mathbb{E}\{v(k)\} = 0 \quad \mathbb{E}\{z(k)\} = 0 \]  
\[ \mathbb{E}\{v(k)v^T(k)\} = R(k)\delta_k \]  
\[ \text{tr } \mathbb{E}\{z(k)z^T(k)\} < \infty \]

Let us define the innovation process by

\[ \mu(k) = y(k) - \hat{z}(k|k - 1) \]

where as \( \hat{z}(k|k - 1) = \hat{z}_{\text{opt}}(k|k - 1) \) - the linear least squares estimate of \( z(k) \) given \{\( y(l) \) 0≤\( l \)≤\( k - 1 \)\}

Thus we can calculate

\[ \mathbb{E}\{\mu(k)\} = \mathbb{E}\{y(k)\} - \mathbb{E}\{z(k|k - 1)\} = 0 \]

\[ \mu(k) = y(k) - \hat{z}(k|k - 1) = y(k) - \hat{z}(k|k - 1) = \]
\[ = z(k) + v(k) - \hat{z}(k|k - 1) = \hat{z}(k|k - 1) + \hat{z}(k|k - 1) - \]
\[ \hat{z}(k|k - 1) + v(k) = v(k) + \hat{z}(k|k - 1) \]

thus for \( k > 1 \)

\[ \mathbb{E}\{\mu(k)\mu^T(k)\} = \mathbb{E}\{y(k)v^T(k)\} + \mathbb{E}\{v(k)\hat{z}(k|k - 1)\} + \]
\[ + \mathbb{E}\{\hat{z}(k|k - 1)v^T(k)\} + \mathbb{E}\{\hat{z}(k|k - 1)v^T(k)\} = \]
\[ = \mathbb{E}\{v(k)v^T(k)\} + \mathbb{E}\{\hat{z}(k|k - 1)[v(k) + v(k)]^T\} = \]
\[ = \mathbb{E}\{v(k)v^T(k)\} \]

\[ \mathbb{E}\{\mu(k)\mu^T(k)\} = \mathbb{E}\{v(k)v^T(k)\} \quad k > \ell \]

the same can be shown for \( k < \ell \).
For \( k = l \) we have
\[
E\{v(k)u^T(k)\} = E\{v(k)v^T(k)\} + 2E\{v(k)\tilde{z}(k|k-1)\} + E\{\tilde{z}(k)\tilde{z}^T(k|k-1)\}
\]
\[
= E\{v(k)v^T(k)\} + E\{\tilde{z}(k|k-1)\tilde{z}^T(k|k-1)\}
\]
\[
\text{covariance of the covariance matrix of the error noise in the estimate } \tilde{z}(k|k-1)
\]
\[
E\{\tilde{z}(k|k-1)\tilde{z}^T(k|k-1)\} = E\{(z(k) - \tilde{z}(k|k-1)) [z(k) - \tilde{z}(k|k-1)]^T\}
\]
\[
= P_z(k)
\]
\[
E\{\tilde{z}(k|k-1)\tilde{z}^T(k|k-1)\} = \left[ P_z(k) + R(k) \right] \delta_{kk}
\]
(113)

So that \( A(k) \) like \( v(k) \) is white but with a different variance.

Remark: The continuous-time case can be approached by a limiting procedure in which \( R(k) \) becomes indefinitely large, while \( P_z(k) \) remains finite, so that the variance of \( A(k) \) and \( v(k) \) are the same. (see Kailath T. (1968).

Let us refer to our initial equation:
\[
\begin{align*}
x(k+1) &= A_x x(k) + B_u u(k) \\
y(k) &= C_x x(k) + v(k) \\
z(k) &= C_z x(k)
\end{align*}
\]
(114)

and let
\[
E\{u(k)u^T(k)\} = \Omega(k)\delta_{kk}; \quad E\{u(k)v^T(k)\} = W(k)\delta_{kk}
\]
(115)

Assuming that the matrix \( \left[ P_z(.) + R(.) \right]^{-1} \) exists, for \( \ell \leq k \) we get
\[
x(k+1) = x(k+1|k) + \tilde{x}(k+1|k), \text{ where } \tilde{x} \perp u
\]
(116)

Let an optimal estimate of \( x(k+1) \) be
\[
\hat{x}(k+1|k) = \sum_{\ell = 0}^{k} g_{\ell}(k)u(\ell)
\]
(117)
so that
\[ x(k + 1) - \hat{x}(k+1|k) = \sum_{\ell = 0}^{k} g_\ell(k)u(\ell) \]  
(118)

according to the projection theorem. Multiplying both sides of (118) by \( \mu^T(k) \) we get:
\[ x(k + 1)\mu^T(k) - \hat{x}(k + 1|k)\mu^T(k) = \sum_{\ell = 0}^{k} g_\ell(k)u(\ell)\mu^T(k) \]  
(119)

\[ E\{x(k + 1)\mu^T(k) - \hat{x}(k + 1|k)\mu^T(k)\} = E\{x(k + 1)\mu^T(k)\} \]  
(120)

according to the projection theorem:
\[ E\{x(k + 1)\mu^T(k)\} = \sum_{\ell = 0}^{k} g_\ell(k) E\{u(\ell)\mu^T(k)\} \]  
(121)

thus
\[ E\{x(k + 1)\mu^T(\ell)\} = \sum_{\ell = 0}^{k} g_\ell(k) \left[ P_z(\ell) + R(\ell) \right] \]  
(122)

and
\[ g_\ell(k) = E\{x(k + 1)\mu^T(\ell)\} \left[ P_z(\ell) + R(\ell) \right]^{-1} \]  
(123)

Putting (123) into (117) we get
\[ \hat{x}(k + 1|k) = \sum_{\ell = 0}^{k} E\{x(k + 1)\mu^T(\ell)\} \left[ P_z(\ell) + R(\ell) \right]^{-1} \mu(\ell) \]  
(124)

Rearranging (124) in the way
\[ \hat{x}(k + 1|k) = \sum_{\ell = 0}^{k-1} E\{x(k + 1)\mu^T(\ell)\} \left[ P_z(\ell) + R(\ell) \right]^{-1} \mu(\ell) + \]  
\[ + E\{x(k + 1)\mu^T(k)\} \left[ P_z(k) + R(k) \right]^{-1} \mu(k) \]  
(125)
and substituting

\[ K(k) \triangleq \mathbb{E}\{x(k + 1)\mu^T(k)\} \left[ p_z^z(k) + R(k) \right]^{-1} \]

(126)

and

\[ \sum_{\ell=0}^{k-1} \mathbb{E}\{x(k + 1)\mu^T(\ell)\} \left[ p_z^z(\ell) + R(\ell) \right]^{-1} \mu(\ell) = A \hat{x}(k|k - 1) \]

(127)

because

\[ \sum_{\ell=0}^{k-1} \mathbb{E}\{x(k + 1)\mu^T(\ell)\} \left[ p_z^z(\ell) + R(\ell) \right]^{-1} \mu(\ell) = \sum_{\ell=0}^{k-1} \mathbb{E}\{A x(k)\mu^T(\ell) + B u(k)\mu^T(\ell)\} \left[ p_z^z(\ell) + R(\ell) \right]^{-1} \mu(\ell) = \sum_{\ell=0}^{k-1} \mathbb{E}\{A x(k)\mu^T(\ell)\} \left[ p_z^z(\ell) + R(\ell) \right]^{-1} \mu(\ell) = A \sum_{\ell=0}^{k-1} \mathbb{E}\{x(k)\mu^T(\ell)\} \left[ p_z^z(\ell) + R(\ell) \right]^{-1} \mu(\ell) = A \hat{x}(k|k - 1) \]

we finally get

\[ \hat{x}(k + 1|k) = A \hat{x}(k|k - 1) + K(k)\mu(k) \]

(128)

form (107)

\[ \chi(k) = C \hat{x}(k|k - 1) + \mu(k) \]

(129)

\[ \mu(k) = \chi(k) - \hat{x}(k|k - 1) \quad - \text{innovation equation} \]

\[ \hat{x}(k|k - 1) = C \hat{x}(k|k - 1) \]

Equations (128), (129) constitute the innovation model of the state space.

\( K(k) \) is called the Kalman filter gain matrix, and to evaluate it we have to consider

\[ \mathbb{E}\{x(k)\mu^T(k)\} = \mathbb{E}\{A x(k) + B u(k)\} \left[ \hat{x}^T(k|k - 1) C^T + \nu^T(k) \right] = A \mathbb{E}\{x(k)\} \hat{x}^T(k|k - 1) C^T + B \mathbb{E}u(k) = A P(k) C^T + B \mathbb{E}u(k) \]

(130)
where

$$P(k) \triangleq E[\tilde{z}(k|k-1)\tilde{z}^T(k|k-1)]$$

(131)

using relation $\chi(k) = \tilde{z}(k) + \chi(k)$ and the orthogonal projection of
$\tilde{z}(t)$ it is straightforward to show

$$P_z(k) \triangleq E[\tilde{z}(k|k-1)\tilde{z}^T(k|k-1)] = P(k)C^T$$

(132)

so

$$K(k) = A[P(k)C^T + B w(k)][C P(k)C^T + R(k)]^{-1}$$

(133)

Finally it can be shown (see Kalman R.E. (1960), Falb P.L. (1967),
Kalman R.E., Bucy R.S. (1961) for example),

$$P(k+1) = A S(k)A^T + Q(k)$$

(134)

$$S(k) = P(k) - K(k) \left[ P_z(k) + R(k) \right] K^T(k)$$

(135)

Equations (128) and (134)(135) define the so called discrete-time Kalman filter.

An innovation approach to least squares estimation of the state has many
advantages and is broadly discussed and used in the modern system theory. Many
identification methods refer to this model, which incorporates less parameters
than ordinary state space model to identify from the same set of input-output
data. (it will be demonstrated in the next chapter). Also some order test
algorithms refer to this method. The complete derivation of the innovation
model was presented in order to introduce the reader to more advanced, however
intuitually quite simple, mathematical formalisms, which are not to be
avoided when studying literature on the multivariable system identification.

The following references are also warmly suggested to enlarge experience in
state space models handling: Jazwinski (1970), Kailath T. (1968), Kailath T.
2.4. Generation of canonical forms from Hankel matrices

State equations for the purely conceptual noiseless system deliver the already discussed transfer function matrix:

\[ K(z) = C(z I - A)^{-1}B \]  

(136)

where \( K(z) \) is assumed to be proper or strictly proper.

According to Schwarz (1971), Ho-Kalman (1966), Hajdasiński (1976) \( K(z) \) can be described in the form of the exponentional expansion:

\[ K(z) = \sum_{i=0}^{\infty} \frac{M_i}{z^{i+1}} \]  

(137)

where \( M_i \) are already known Markov parameter matrices

\[ M_i = C A^i B - (q \times p) \]  

(138)

see Ho-Kalman (1966), Gantmachev (1959).

Alternatively, applying the \( z \)-transform to the state and output equations, we may describe this system by means of the so-called "weighting sequence". (See Hajdasiński (1978)).

\[ x(k) = C A^k x(0) + \sum_{i=0}^{k-1} M_i u(k - i - 1) = \]

\[ = M_k B_0 + \sum_{i=0}^{k-1} M_i u(k - i - 1) \]  

(139)

with \( B_0 x_0 = x_0 \)

It is very easy and intuitively simple to define the equivalence of dynamical systems in terms of Markov-parameters.

**Definition 15** Two dynamical systems \( \Sigma_1(A, B, C) \in X \) and \( \Sigma_2(A_1, B_1, C_1) \in X \) are said to be equivalent if and only if their Markov-parameters fulfill the following condition (Ho-Kalman (1966)),

\[ C A^i B = C_1 A_1^i B_1 \]  

for \( i = 0, 1, 2, 3, \ldots \)  

(140)
With the knowledge of Markov-parameters it is possible to construct the finite Hankel matrix $H_r = (q_0 \times p_r)$, which is a submatrix of the double infinite Hankel matrix form relation (8).

\[
H_r = \begin{bmatrix}
M_0 & M_1 & \ldots & M_{r-1} \\
M_1 & M_2 & \ldots & M_r \\
\vdots & \vdots & \ddots & \vdots \\
M_{r-1} & M_r & \ldots & M_{2r-2}
\end{bmatrix}
\]  

(141)

It is very easy to notice that $H_r$ can be expressed as a product of the observability and controllability matrices (see Ho-Kalman (1966)).

\[
W = \begin{bmatrix}
B & AB & A^2B & \ldots & A^{r-1}B
\end{bmatrix}
\]  

(142)

\[
V = \begin{bmatrix}
C^T & A^T & A^2C^T & \ldots & (A^T)^{r-1}C^T
\end{bmatrix}
\]  

observability Matrix

(143)

\[
H_r = V^T W = \begin{bmatrix}
C & B & AB & A^2B & A^{r-1}B \\
C & A & AB & A^2B & A^{r-1}B \\
C & A^2 & & & \\
\vdots & & & & \\
C & A^{r-1} & & & \\
\end{bmatrix}
\]  

(144)

From (144) it is obvious that a row (column) of $H_r$ is dependent if and only if it corresponds to a dependent row (column) of the observability (controllability) matrix (see Niederlinski, Hajdasiński (1979)). This means that there exists equivalence between canonically observable and canonically controllable forms (described in 2.2.1. and 2.2.2.) and forms which can be generated from the Hankel matrix. This is stated in the following theorem by Candy, Warren and Bullock (1978).
Theorem 7 1. If the rows of the $H_r$ matrix are examined for predecessor independence, then $\ell$th (dependent) row, where $\ell = i + qv_i$, $i = 1, 2, \ldots, q$ is given by

$$
\Psi_{i}^{T} = \sum_{j=1}^{\ell-1} \sum_{k=0}^{q} \alpha_{ij,k} \Psi_{j+k}^{T} + \sum_{j=1}^{q} \sum_{k=0}^{q} \alpha_{ij,k} \Psi_{j+k}^{T}
$$

where $\alpha_{ij,k}$ and $v_i$ are invariant parameters for canonically observable couple $(A, C)$.

2. If the columns of the $H_r$ matrix are examined for predecessor independence, then the $\ell$th (dependent) column, where $\ell = i + pn_i$, $i = 1, 2, \ldots, p$ is given by

$$
\Psi_{i} = \sum_{j=1}^{\ell-1} \sum_{k=0}^{p} \tilde{\alpha}_{ij,k} \Psi_{j+k} + \sum_{j=1}^{p} \sum_{k=0}^{p} \tilde{\alpha}_{ij,k} \Psi_{j+k}
$$

where $\tilde{\alpha}_{ij,k}$ and $n_i$ are invariants for the canonically controllable couple $(A, B)$.

Candy, Warren and Bullock (1978) present an algorithm for the transformation of the $H_r$ matrix to a form enabling the simultaneous determination of the $v_i$ and $n_i$ invariants by inspection. This algorithm may be treated as the determination of the canonical structure of the conceptual, nondisturbed system, having the advantage that both sets of Kronecker indices are formed in parallel.

An infinite number of canonical forms can be derived from the Hankel matrix $H_r$, depending on the decomposition of the latter. No completely satisfactory research has been made in order to reflect the hidden possibilities of the Hankel canonical forms. As an illustrative example the minimal realization algorithm by Ho-Kalman will be presented as a method using a "general" decomposition of the Hankel matrix and the minimal realization algorithm.
with use of the singular value decomposition as an illustration of a particularly interesting decomposition of the Hankel matrix, giving possibilities for interpretation of the noisy experiments in view of least squares fit to a given data set.

3.4.1. The Ho-Kalman minimal realization algorithm

The minimal realization problem is referred to as finding the triplet of matrices \( \{A, B, C\} \) using the external data like input-output signals or the transfer function matrix. This problem is very well known in the system theory, and quite extensively worked out at present. There is a huge number of minimal realization algorithms, but still the original work by Ho-Kalman (1966) shows the majority over another method. Let us present the Ho-Kalman algorithm in the form of the following theorem:

**Theorem 8** For an arbitrary, finite dimensional, linear, dynamical system, given the input-output map, the canonical realization exists in the following form:

Let \( q \) - be the number of outputs

\( p \) - the number of inputs

\( n \) - dimension of the realization

1. There is defined the following matrix:

\[
E_k^\ell = \begin{cases} 
  k \times \ell \text{ matrix} & \begin{bmatrix} I_k & O_{\ell-k} \end{bmatrix} \quad \text{if } k < \ell \\
  k \times \ell \text{ matrix} & \begin{bmatrix} I_{\ell} \end{bmatrix} \quad \text{if } k = \ell \\
  k \times k \text{ matrix} & \begin{bmatrix} O_{k-\ell} \end{bmatrix} \quad \text{if } k > \ell
\end{cases}
\]

(145)
2. Choose \( r \) such that the relation

\[
M_{r+j} = \sum_{i=1}^{r} a_i M_{r+j-i} \quad \text{for all} \; j \geq 0
\]  

(146)

holds.

3. Find a nonsingular matrix \( P(q_r \times q_r) \) and a nonsingular matrix \( Q(p_r \times p_r) \) such

\[
P H_r Q = \begin{bmatrix}
I_k & \frac{Q^{pr-n}}{n} \\
\frac{Q^{pr-n}}{qr-n} & \frac{Q^{pr-n}}{qr-n}
\end{bmatrix} = \begin{bmatrix}
E^{n} & \frac{p^{pr}}{q^{r-n}} \\
\frac{q^{r-n}}{p^{r-n}} & \frac{q^{r-n}}{p^{r-n}}
\end{bmatrix}
\]

(147)

where

\[
H_r = \begin{bmatrix}
M_0 & \cdots & M_{r-1} \\
\vdots & \ddots & \vdots \\
M_{r-1} & \cdots & M_{2r-2}
\end{bmatrix}; \quad \sigma H_r = \begin{bmatrix}
M_1 & \cdots & M_r \\
\vdots & \ddots & \vdots \\
M_r & \cdots & M_{2r-1}
\end{bmatrix}\quad \text{shifted (148)}
\]

Hankel Matrix

4. A canonical realization of the considered system is given by:

\[
A = E_n P(\sigma H_r)Q \frac{E^{n}}{p^{pr}}
\]

\[
B = E^{q r} P \frac{H E^p}{r^{pr}}
\]

\[
C = E^{q r} \frac{H E^n}{p^{pr}}
\]

(149)
The proof of this theorem can be found in Ho B.L., Kalman R.E. (1966), Kalman R.E., Arbib, Falb (1967), Schwarz H. (1971), Hajdasinski (1976).

In the Ho-Kalman algorithm the Hankel matrix decomposition is non-unique and depends on the choice of the transformation matrices $P$ and $Q$. Kalman (1967) suggested the upper and lower triangular structure for the $(P, Q)$ or $(Q, P)$ respectively. For this sake the Andree (1951) algorithm was perfectly suited. But it can be shown (Hajdasinski (1976) that by a different choice $P$ and $Q$ the phase-canonical form of the state equations can be achieved. However, in general, when state variables are not well defined, it is not possible.

3.4.2. The minimal realization algorithm with the use of Singular Value Decomposition of the Hankel matrix

The following decomposition of the Hankel matrix will be called the Singular Value Decomposition (s.v.d.) (see Hajdasinski - Damen (1979)):

$$ H_r = U D V^T $$

where

$D$ is the $n \times n$ diagonal matrix

$$ D = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_n) \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n $$

$\sigma_i$, for $i = 1, 2, \ldots, n$ are called the singular values

$U$ - $(r q \times n)$ matrix consisting of $n$ orthogonal columns $u_j$,

$$ U^T U = I_n $$

$V$ - $(r p \times n)$ matrix consisting of $n$ orthogonal columns $v_j$,

$$ V^T V = I_n $$

It can be easily demonstrated that the Moore-Penrose inverse of $H_r$ given the
s.v.d. (150) is:

\[ H^*_r = V D^{-1} U^T \]  \hspace{1cm} (152)

as the other hand from (147) we have.

\[ P H Q = E^n_{qr} E^{pr} = J \]  \hspace{1cm} (153)

by definition \[ P H_r H_r^T Q = J \]

\[ P H_r Q \xi P H_r Q = J \hspace{1cm} \xi = J^T \]  \hspace{1cm} (154)

thus

\[ H^*_r = Q J^T P = Q E^n_{pr} E^{qr}_{pq} \]  \hspace{1cm} (155)

Comparing (152) and (155)

\[ E^{qr}_{pq} = D^{-1} U^T \]  \hspace{1cm} (156)

\[ Q E^n_{pr} = V \]  \hspace{1cm} (157)

A canonical realization of the considered system is given by

\[ A = D^{-1} U^T (CH_r) V \]

\[ B = D^{-1} U^T U D V E^p_{pr} = V^T E^p_{pk} \]  \hspace{1cm} (158)

\[ C = E^{qr}_{pq} U D V^T V = E^{qr}_{pq} U D \]

Once the (s,v,d.) of the \( H_r \) is given, the realization (158) is unique. This canonical form plays a dominant role in indentification of both the order and parameters of the noise corrupted dynamical multivariable systems, and will be more broadly explained in chapters 3 and 4.

Example 5 Let us consider a two input, two output dynamical system, whose realization \( \{ F, G, H \} \) is:

\[
F = \begin{bmatrix}
0 & 1 & -1 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
-2 & -5 & -3 & -1 & -2 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 1 & 0 & -2 \\
\end{bmatrix};
\quad
G = \begin{bmatrix}
1 & -1 \\
0 & 1 \\
-1 & -1 \\
1 & 0 \\
1 & 1 \\
\end{bmatrix};
\quad
H = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
constructing

\[
H = \begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix}
\]

where

\[
\begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} h_2^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

and performing the test for linear independence according to relation (52)

we get

\[
v_1 = 3, \quad v_2 = 2, \quad v_{12} = v_{21} = 3
\]

so the transformation matrix \( T^T \) will be (see (54))

\[
T^T = \begin{bmatrix} h_1^T ; \left(F^T h_1 \right) ; \left(F^T \right)^2 h_1 ; h_2^T ; F^T h_2 \end{bmatrix}
\]

\[
F^T = \begin{bmatrix} 0 & 0 & -2 & 0 & -1 \\ 1 & 0 & -5 & 0 & 1 \\ -1 & 1 & -3 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 \end{bmatrix}, \quad \left(F^T \right)^2 = \begin{bmatrix} 2 & -2 & 8 & -1 & 0 \\ 5 & -5 & 11 & 1 & -8 \\ 4 & -3 & 4 & 0 & -3 \\ 2 & -1 & 0 & 0 & 1 \\ 1 & -1 & 9 & -2 & 2 \end{bmatrix}
\]

thus

\[
T^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
Now the canonical form may be completed:

\[
A = TFT^{-1} =  \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-2 & -3 & -3 & 2 & -1 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 2 & 1 & -1 & -2
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\] (see (58),(60))

\[
B = TG =  \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\]

\[
C = HT^{-1} =  \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\] (see (61))

The realization \( \{A, B, C, \} \) is a canonically observable realization of the considered system, and is equivalent (in the sense of Definition 15) to the realization \( \{F, G, H, \} \).
Example 6  Let us consider a two input, two output dynamical system, characterized by the following series of Markov-parameters:

\[
\{M_i\}_i = 0,1,2 \ldots \quad = \begin{cases} 
\begin{bmatrix} 1.0 & 0.0 \\ 0.8 & 0.2 \\ 0.64 & 0.28 \\ 0.512 & 0.296 \end{bmatrix}, \\
\begin{bmatrix} 0.0 & 1.0 \\ 0.0 & 0.6 \\ 0.0 & 0.36 \\ 0.0 & 0.216 \end{bmatrix}, \\
\begin{bmatrix} 0.4096 & 0.28 \\ 0.0 & 0.1296 \end{bmatrix}, \\
\begin{bmatrix} 0.32608 & 0.24992 \\ 0.0 & 0.07776 \end{bmatrix}, \\
\begin{bmatrix} 0.260864 & 0.215488 \\ 0.0 & 0.466264 \end{bmatrix} \ldots \ldots 
\end{cases}
\]

I. Applying the Ho-Kalman algorithm we will derive the realization \( \{A,B,C\} \) of this system:

It is sufficient to consider \( H_1 \) for

\[
\begin{align*}
\det H_1 &= 1 \quad \Rightarrow \quad \text{rank } H_1 = 2 \\
\det H_2 &= 0 \quad \Rightarrow \quad \text{rank } H_2 = 2
\end{align*}
\]

thus the system is of the first order and has the dimension \( n = 2 \). In such a case it is trivial to find \( P \) and \( Q \) matrices, which are:

\[
P = \alpha I_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}
\]

\[
Q = \alpha I_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}
\]

and \( \alpha = \frac{1}{\alpha_q} \)
The Ho-Kalman algorithm (149) provides us with:

\[
A = \alpha_p (\sigma_H) \alpha_q = \sigma_H M = \begin{bmatrix} 0.8 & 0.2 \\ 0.0 & 0.6 \end{bmatrix}
\]

\[
B = \alpha_p H = \alpha_I p^{-1}
\]

\[
C = H \alpha_q = \alpha_I q^{-2}
\]

Eigenvalues: \( \lambda_1 = 0.8 \)
\( \lambda_2 = 0.6 \)

Transfer function matrix of this system is:

\[
K(z) = C(Iz - A)^{-1}B = \begin{bmatrix} 1 & 0.2 \\ (z - 0.8) & (z - 0.8)(z - 0.6) \\ 0.0 & (z - 0.6) \end{bmatrix}
\]

II. Applying the s.v.d. realization algorithm for example for \( H_n \), we get a following set of singular values:

\[
\sigma_1 = 2.5701194
\]

\[
\sigma_2 = 1.4734138
\]

\[
\sigma_3 = 8.6947604 \cdot 10^{-12} \approx 0
\]

\[
\sigma_4 \approx 0
\]

... thus if it is clear that the system dimension \( n = 2 \), \( r \) - order = 1.

Set of equations (158) provides us with:

\[
\tilde{A} = \begin{bmatrix} 0.834575 & -0.068194 \\ -0.011896 & 0.565417 \end{bmatrix}
\]

Eigenvalues: \( \lambda_1 = 0.8 \)
\( \lambda_2 = 0.6 \)

\[
\tilde{B} = \begin{bmatrix} -0.568066 & -0.259609 \\ -0.288067 & 0.772964 \end{bmatrix}
\]
Realizations \{A, B, C\} and \{\bar{A}, \bar{B}, \bar{C}\} are equivalent in the sense of Definition 15.

Realization\{\bar{A}, \bar{B}, \bar{C}\} delivers exactly the same \(K(z)\) as the realization \{A, B, C\}.

The great advantage of the s.v.d. realization is its uniqueness, and an easy test for the system order.
3. **Identification of the structure of the multivariable dynamical systems.**

The whole chapter will be based on the definitions 13 and 14, and on structural properties of dynamical models for MIMO systems as described in chapter 2. The task of the structural identification it is to determine the suitable "complexity" of the chosen model. Thus in view of definitions 13 and 14 it will be determination of the order \(-r\), or (and) the minimal dimension \(-n\) of the considered system (MIMO).

Such a posing of the problem is possible only for some strictly conceptual systems having both a finite order and a finite dimension. In the real, noisy systems identification, however, one cannot search for any exact \(r\) and \(n\), because due to the noise these are systems of infinite order and dimension. The only goal which we can aim at is to find a reasonably simple, and resulting in good outputs of the model, a finite dimensional approximation of the real system — i.e. to find finite estimates of the \(r\) and \(n\).

3.1. **Estimation of structural invariants - Guidorzi's method**

The Guidorzi's method is based on the canonically observable form of the state equations — see 2.2.1, relations (56),(57),(58),(59),(60),(61),(62). According to Guidorzi, the structural identification is defined as:

**Definition 16** The structural identification of a multivariable system it is the determination of the set of integers \(\nu_1 \ldots \nu_q\), (Kronecker invariants) defining the structure of the couple \((A, C)\) from input-output relations, without the intermediate construction of a parametric model.
Thus the Guidorzi's method requires the input-output description of the dynamical system, which would use the Kronecker invariants.

\[ p(z)\psi(k) = q(z)u(k) \]  \hspace{1cm} (159)

where

\[ y^T(k) = [y_1(k), y_2(k), \ldots, y_q(k)] \]
\[ u^T(k) = [u_1(k), u_2(k), \ldots, u_q(k)] \]
\[ P(z) = \begin{bmatrix} p_{11}(z) & \cdots & p_{1q}(z) \\ \vdots & \ddots & \vdots \\ p_{q1}(z) & \cdots & p_{qq}(z) \end{bmatrix} \quad ; \quad Q(z) = \begin{bmatrix} q_{11}(z) & \cdots & q_{1p}(z) \\ \vdots & \ddots & \vdots \\ q_{q1}(z) & \cdots & q_{qp}(z) \end{bmatrix} \]

From the \( j \)-th component of the canonically observable form it can be written:

\[
\hat{x}(k)_{i} \overset{\text{def}}{=} \hat{x}_i(k)
\]

\[
\hat{x}(k)_{(v_1+ v_2 + \ldots + v_{j-1}+1)} = y_j(k)
\]
\[
\hat{x}(k)_{(v_1+ v_2 + \ldots + v_{j-1}+2)} = zy_j(k) - b^T_{(v_1+ \ldots + v_{j-1}+1)}u(k)
\]
\[
\hat{x}(k)_{(v_1+ v_2 + \ldots + v_{j-1}+3)} = z^2y_j(k) - b^T_{(v_1+ \ldots + v_{j-1}+2)}u(k) - \]
\[
- z^2b^T_{(v_1+ \ldots + v_{j-1}+1)}u(k) \quad (160)
\]

\[
\hat{x}(k)_{(v_1+ v_2 + \ldots + v_j)} = z^jy_j(k) - b^T_{(v_1+ \ldots + v_{j-1})}u(k) - \]
\[
- z^{j-2}b^T_{(v_1+ \ldots + v_{j-1}+1)}u(k)
\]

Thus the canonically observable state equation can be rewritten as:

\[
\hat{p}(k) = v(z)\hat{p}(k) - w(z)u(k) \quad (161)
\]
where

\[
\mathbf{V}(z) = \begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
0 & \cdots & z^{v_1-1}
\end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & z^{v_q-1}
\end{bmatrix}
\]

\[v_M = \max_i (v_i) \quad ; \quad \mathbf{Z}(z) = \begin{bmatrix}
1 \\
z \\
\vdots \\
z^{v_{\max} - 1}
\end{bmatrix}
\]

Finally the relation between \( \mathbf{y}(k) \) and \( \mathbf{u}(k) \) is found to be:

\[
((zI - A)\mathbf{V}(z))\mathbf{y}(k) = ((zI - A)\mathbf{WZ}(z) + \mathbf{B})\mathbf{u}(k)
\] (162)

Comparing coefficients of (159) and (162) we obtain:

\[
p_{ii}(z) = z^{v_i - a_{ii}}, v_i^{v_i - 1} - \cdots - a_{ii}, z - a_{ii}, 1
\] (163)

\[
p_{ij}(z) = -a_{ij}, v_i^{v_i - 1} - \cdots - a_{ij}, z - a_{ij}, 1
\] (164)

and for \( q_{ij}(z) \) it is necessary to construct a matrix \( \mathbf{\hat{M}} = \mathbf{M} \mathbf{B} \)
where

\[
\mathbf{M} = \begin{bmatrix}
-a_{11}, 2 & -a_{11}, 3 & \cdots & -a_{11}, v_1 & 1 & \cdots & -a_{1q}, 2 & \cdots & -a_{1q}, v_1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
-a_{11}, v_1 & 1 & \cdots & -a_{1q}, v_1 & 0 & \cdots & -a_{q1}, 2 & \cdots & -a_{q1}, v_1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
\end{bmatrix}
\]

(165)

and comparing coefficients we get:

\[
q_{ij}(z) = \beta_{i1} z^{v_1-1} + \cdots + \beta_{i-2} z^{v_{i-2}+2} + \beta_{i-1} z^{v_{i-1}+1}, j
\]

From relations (162), (163), (159) it is seen that the set of indices

\[v_1, v_2, \ldots, v_q\]

can be deduced by inspection or from the knowledge of \(\mathbf{A}\) or of \(P(z)\); also from the parametric standpoint \(\mathbf{A}\) and \(P(z)\) are equivalent.

Matrix \(\mathbf{C}\) can be directly written if the Kronecker invariants are known. See also Guidorzi (1973), Bonivento C, Guidorzi R (1971), Bonivento C, Guidorzi R (1972).

Guidorzi considers the matrix of input-output data given by:
From relations (162), (161) and (166) the following relation between \( y_i(k) \) and \( u_i(k) \) can be derived:

\[
y_s(k+v_s) = \sum_{i=1}^{q} \sum_{j=1}^{s_i} a_{s_i,j} y_i(k+j-1) + \sum_{i=1}^{p} \sum_{j=1}^{s} \beta_{s,1+j} u_i(k+j-1)
\]

(168)

and \( v_{ss} = v_s \)

Relation (168) shows clearly the linear dependence between present and former samples of outputs and between present outputs and former inputs.

If we look closely to the relation (167), this property may very easy be forecasted. Each subsystem in (167) is the Hankel matrix and relation (168) has the form of the realizability criterion. It first is due to Guidorzi (1971), that this property has been discovered letting determination of \( v_1 \ldots \ldots v_q \) by selecting of nonsingular matrices being products of these output-input vectors:

\[
L_i(y_j) = \begin{bmatrix} y_j(k) & y_j(k+1) & \ldots & y_j(k+i-1) \end{bmatrix} - i \text{ elements}
\]

(169)

\[
L_i(u_j) = \begin{bmatrix} u_j(k) & u_j(k+1) & \ldots & u_j(k+i-1) \end{bmatrix} - i \text{ elements}
\]

(170)

Constructing following \( R \) and \( S \) matrices:
To complete the structural identification we need to build up a sequence of increasing dimension matrices:

\[ S(2,1,\ldots,1); S(2,2,1,\ldots,1); \ldots; S(2,2,\ldots,2); \ldots \]

and selecting nonsingular ones, find indices \( v_i \) while a singular matrix is found. Suppose \( S(1,1,\ldots,1) \) is a singular matrix and let \( l_i \) is the index increased by one, with respect to previous nonsingular matrix in the sequence. Then \( v_i = l_i - 1 \) and \( v_{ij} = 1 \) for \( j = 1,2,\ldots,q \) for \( i \neq j \). This ends the structural identification.

The Guidorzi’s method is the first coherent approach to the structural and parametric identification. For "really" multivariable systems with a high number of inputs and outputs, however, the selecting procedure for \( S \) matrices increases complexity of already complicated algorithm. In noisy cases the structural identification produces higher dimensions than the realizations based for example on the Hankel model or on partial minimal realizations - see Tether(1970), Anderson(1977), Roman, Bullock(1975).

Also for noisy cases the Guidorzi’s method requires a priori knowledge of a covariance matrix of the noise. In noisy cases the structural (and parametric) identification, according to Guidorzi, must be performed on the base of following relations:

\[ y_j^*(k) = y_j(k) + d(y_j(k)) \]  \hspace{1cm} (174)
\[ u_j^*(k) = u_j(k) + d(u_j(k)) \]  \hspace{1cm} (175)

where \( u^* \) are for the noisy signals.

Assuming that the covariance matrix of the noise vector

\[ \bar{u}(d) = \left[ d(y(k))^T \right], \bar{N}(d) \text{ is known:} \]

\[ \bar{N}(d) = \text{prob lim}_{N \to \infty} \frac{1}{N} \left( \bar{u} \bar{u}^T \right) \]  \hspace{1cm} (176)

it is possible to find

\[ \bar{R}^* = \bar{R} + \bar{n}^T \]  \hspace{1cm} (177)

and

\[ \bar{S}^* = (R^*)^T (R^*) \]  \hspace{1cm} (178)

Thus for the structural identification we need the "compensated estimates" of \( \bar{R} \) and \( \bar{S} \). Assuming that input and output noises are "zero mean" white noises and that \( \hat{\bar{N}}(d) \) - the estimate of \( \bar{N}(d) \) can be found, the compensated estimate of \( \bar{S} \) is

\[ \hat{\bar{S}} = \bar{S}^* - \bar{N} \hat{\bar{N}}(d) \]  \hspace{1cm} (179)

**Discussion.** The main difficulty for application of this method lies in the fact, that practically \( \hat{\bar{S}} \) will never be a singular matrix and the structural identification will become very unreliable and tedious, unless the additional constraints are imposed on the problem. It also excludes the existence of another assumption about the noise nature which were previously made.
3.2. Order tests based on the "innovation approach" - Tse-Weinert's order test.

Tse and Weinert (1975) started from the following model:

\begin{align*}
\tilde{x}(k+1) &= A \tilde{x}(k) + \tilde{w}(k) \\
\tilde{y}(k) &= C \tilde{x}(k) + \tilde{v}(k)
\end{align*}

where:
\( \tilde{x}(k) \in \mathbb{R}^n \); \( \tilde{y}(k) \in \mathbb{R}^q \); \( \tilde{w}(k) \) and \( \tilde{v}(k) \) are zero-mean Gaussian noises with covariances:

\[
E(\tilde{w}(k)\tilde{w}(j)^T) = \tilde{W} \delta_{kj} \quad \text{and} \quad E(\tilde{v}(k)\tilde{v}(j)^T) = \tilde{V} \delta_{kj} \quad \text{and} \quad E(\tilde{w}(k)\tilde{v}(k)^T) = \tilde{D} \delta_{kj}
\]

It means that there is considered a multivariable dynamical system for which the input signal is stabilized, and we observe deviations from the steady state caused by the state noise \( \tilde{w}(k) \). Unknown parameters are \( \{x_0, n, A, C, W, V, D\} = \{\gamma\} \).

The objective is to estimate \( \{\gamma\} \), using output data \( \tilde{y}_N = \{\tilde{y}(1), \tilde{y}(2), \ldots, \tilde{y}(N)\} \).

As it is seen, a part of the objective is to estimate the system dimension \( n \), (called by Tse and Weinert - the order of the MIMO system) and only this part of the algorithm will be discussed here. The set of parameters \( \{\gamma\} \) is quite extended and requires a large amount of data. As it is pointed out by Tse and Weinert, \( \{\gamma\} \) may not be identifiable and therefore the innovation representation is proposed:

supposing that

\[
\hat{x}(k+1|k) - \text{is the predicted state vector (the conditional mean at time } k+1, \text{ given the estimate } \hat{x}(k) \text{ at time } k \)
\]

\[
\hat{x}(k+1) - \text{corrected state vector at time } k+1
\]

\[
K - \text{steady state Kalman filter gain}
\]

we get:
\[ \hat{x}(k+1|k) = A \hat{x}(k) \quad \text{one step prediction equation} \] (181)

\[ \hat{x}(k+1) = \hat{x}(k+1|k) + K(\hat{y}(k+1) - C \hat{x}(k+1|k)) \quad \text{the correction eq.} \] (182)

\[ y(k) = C \hat{x}(k|k-1) + v(k) \quad \text{output equation} \] (183)

\[ v(k) = y(k) - C \hat{x}(k|k-1) \quad \text{innovation equation (or "zero mean innovation process")} \] (184)

From equations (181) \(\cdots\) (184) we get:

\[ \hat{x}(k+1|k) = A \hat{x}(k|k-1) + A K y(k) \] (185)

\[ y(k) = C \hat{x}(k|k-1) + v(k) \] (186)

The set of equations (185) and (186) is called the innovation model of the dynamical multivariable system. (see references to chapter 2.3.3. of this report)

Additional assumptions are:

- \( A \) is stable
- \( (A, C) \) is observable
- \( (A, B) \) is controllable
- \( Q \) is the unknown covariance matrix of \( \hat{y}(k) \)
- \( B = A K \) is the optimum gain
- \( \dim \{ \hat{x}(k|k-1) \} \) is finite but unknown

The canonically observable form of (185) and (186) is applied. If \( v_i \) again denote the Kronecker indices such that

\[ \sum_{i=1}^{q} v_i = n \] (187)
from the observability matrix and from the definition of the Kronecker indices there follows already described the observable canonical form (56),(57),(58)-(62), and this implies an existence of a unique set of \( \{a_{ij,k}\} \) such that for \( i = 1,2,\ldots,q \)

\[
\mathbf{c}_i^T \mathbf{A}^{k} \mathbf{v}_i = \sum_{j=1}^{i} \sum_{k=0}^{v_i-1} a_{ij,k} \mathbf{c}_j^T \mathbf{A}^k \quad \text{if} \quad v_i > 0 \quad (188)
\]

\[
\mathbf{c}_i^T \mathbf{A}^{-1} = \sum_{j=1}^{i} \sum_{k=0}^{v_i-1} a_{ij,k} \mathbf{c}_j^T \mathbf{A}^k \quad \text{if} \quad v_i = 0 \quad (189)
\]

\( \mathbf{c}_i^T \) is the \( i \)th row of \( \mathbf{C} \) matrix.

Introducing the covariance matrix of states \( \mathbf{S}_x \),

\[
\mathbf{S}_x = \mathbf{E} \{ \mathbf{\hat{x}}(k+1|k) \mathbf{\hat{x}}^T(k+1|k) \} \quad (190)
\]

\[
\{ \mathbf{\hat{x}}(k+1|k) \mathbf{\hat{x}}^T(k+1|k) \} = \mathbf{A} \{ \mathbf{\hat{x}}(k|k-1) \mathbf{\hat{x}}^T(k|k-1) \} \mathbf{A}^T + \mathbf{B} \mathbf{v}(k) \mathbf{v}^T(k) +
\]

\[
+ \mathbf{A} \{ \mathbf{\hat{x}}(k|k-1) \mathbf{v}^T(k) \} \mathbf{B}^T + \mathbf{B} \mathbf{v}(k) \mathbf{\hat{x}}^T(k|k-1) \mathbf{A}^T \quad (191)
\]

Taking expectations of both sides of (191) we get

\[
\mathbf{S}_x = \mathbf{A} \mathbf{S}_x \mathbf{A}^T + \mathbf{B} \mathbf{Q} \mathbf{B}^T \quad (192)
\]

for \( \mathbf{v}(k) \) was assumed to be a zero-mean noise

Defining

\[
\mathbf{R}(\sigma) \overset{\text{def}}{=} \mathbf{E} \{ \mathbf{\gamma}(k+\sigma) \mathbf{\gamma}^T(k) \} \quad (193)
\]

and considering

\[
\mathbf{\gamma}(k) = \mathbf{C} \mathbf{\hat{x}}(k|k-1) ; \quad (194)
\]
finally we get:

\[ R(O) = C S C^T + Q \] (195)

\[ R(\sigma) = C A^{\sigma-1} S \] (196)

\[ S = A S C^T + B Q \] (197)

denoting as \( r_{ij}(\sigma) \) the \( ij \)th element in \( R(\sigma) \), and as \( s_j \) the \( j \)th column in \( S \), using (188) from (195), (196) we get:

\[
r_{ij}(v_i + \tau) = \sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} c_{l}^{T} A^{k} A^{\tau-1} s_j \quad v_i > 0
\]

\[
= \begin{cases} 
\sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} c_{l}^{T} A^{k} A^{\tau-1} s_j & v_i > 0 \\
\sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} c_{l}^{T} A^{k} A^{\tau-1} s_j & v_i = 0
\end{cases}
\] (198)

where \( \tau = 1, 2, \ldots \)

Basing on (196) and (198) we get:

\[
r_{ij}(v_i + \tau) = \sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} r_{lj}(k+\tau) \quad v_i > 0
\]

\[
= \begin{cases} 
\sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} r_{lj}(k+\tau) & v_i > 0 \\
\sum_{l=1}^{i-l} \sum_{k=0}^{v_i-1} a_{il,k} r_{lj}(k+\tau) & v_i = 0
\end{cases}
\] (199)

Thus for \( i=1 \)

\[
r_{ij}(v_i + \tau) = \sum_{k=0}^{v_i-1} a_{il,k} r_{lj}(k+\tau) \quad \tau = 1, 2, 3, \ldots .
\]

(200)

defining

\[ \bar{r}_1^T = [r_{1j}(v_{1} + 1), \ldots , r_{1j}(2v_{1})] ; \quad \bar{b}_1^T = [a_{11,0}, \ldots , a_{11,(v_{1} - 1)}] \]
Remark:

This is worth a notice, that \( r_{1j}(\sigma) \) are the Markov parameters of the 1st subsystem in the canonical representation, treated as the multi-input/single-output subsystem. Thus (200) is merely the realizability criterion for the first subsystem, and (201) is the Hankel matrix of the first subsystem. In such a case the Hankel model can be applied:

\[
\Phi_1(k) = \begin{bmatrix}
  r_{1j}(1) & r_{1j}(2) & \ldots & r_{1j}(k) \\
  r_{1j}(2) & r_{1j}(3) & \ldots & r_{1j}(k+1) \\
  \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{1j}(k) & \ldots & \ldots & r_{1j}(2k-1)
\end{bmatrix}
\]  

Remark:

The Tse-Weinert's order test is nothing but a Hankel matrix determinant test which will be described for a more general case later in this chapter.

If \( d_1(k) \overset{\text{def}}{=} \left| \det \Phi_1(k) \right| \)

\[
\begin{cases}
  d_1(k) > 0 & \text{for } k = 1, 2, 3, \ldots v_1 \\
  d_1(k) = 0 & \text{for } k > v_1
\end{cases}
\]  

thus if \( R(\sigma) \) were exactly known, \( v_1 \) could be found by testing \( d_1(k) \) for \( k=1, 2, 3, \ldots \) until \( d_1(i) = 0 \) and then \( v_1 = i-1 \), which completes the order test.

Since only \( \left\{ Y \right\}_N \) is available, \( R(\sigma) \) must be estimated the following way:

\[
\widehat{R}(\sigma) = \frac{1}{N} \sum_{i=1}^{N-\sigma} Y(k+\sigma) Y^T(k)
\]
\( \hat{R}(\sigma) \) is a strongly consistent estimate of \( R(\sigma) \) for a stable \( A \) and \( N^{\infty} \). Replacing \( \hat{\Phi}_1(k) \) by \( \hat{\Phi}_1(k) \), we can perform the order test. If the first sharp decrease in \( \hat{d}_1(k) \) occurs at time \( k = k^* \), then \( \hat{\nu}_1 \) is chosen as \( \hat{\nu}_1 = k^* - 1 \) (in the original paper by Tse - Weinert (1975) it is \( \hat{\nu}_1 = k^* \), but there is a different meaning of what is the "point where the sharpest decrease occurs". In Tse and Weinert's work it is always one point before the decreased value of \( \hat{d}_1(k) \) occurred).

\[ k^* = 4 \Rightarrow \hat{\nu}_1 = 3 \]

(according to Tse-Weinert :
\[ k^* = \hat{\nu}_1 = 3 \])

Behaviour of the \( \hat{d}_1(k) \)

The estimate of \( \hat{\beta}_1 \) is found from:

\[ \hat{\xi}_1 = \hat{\Phi}_1(\hat{\nu}_1) \hat{\beta}_1 \]

(205)

According to the order test criterion \( \hat{\Phi}_1(\hat{\nu}_1) \) is always nonsingular,

\[ \hat{\beta}_1 = \hat{\Phi}_1^{-1}(\hat{\nu}_1) \hat{\xi}_1 \]

(206)

\( \hat{\beta}_1 \) is a strongly consistent estimate of \( \beta_1 \), provided \( \hat{\nu}_1 = \nu_1 \). For \( i=2,3,\ldots,q \), \( \hat{\nu}_1 \) and \( \hat{\beta}_1 \) are computed in analogous manner. For \( i=2, \tau = 1,2,\ldots (\nu_1 + \nu_2) \)
\[ r_2 = \Phi_2(v_2) \beta_2 \]  

(207)  

\[ \Phi_2(k) = \begin{bmatrix} \Phi_1(v_1) & r_{2j}(1) & \ldots & r_{2j}(k) \\ r_{1j}(v_1+1) & \ldots & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ r_{1j}(v_1+k) & \ldots & \ldots & r_{2j}(v_1+2k-1) \end{bmatrix} \begin{bmatrix} \beta_{21,0} & \ldots & \beta_{21,(v_1-1)} & \beta_{22,0} & \ldots & \beta_{22,(v_2-1)} \end{bmatrix} \]

and again for the estimate \( \hat{\Phi}_2(k) \) of \( \Phi_2(k) \) we get:

\[ \hat{r}_2 = \hat{\Phi}_2(v_2) \hat{\beta}_2 \]  

(208)  

and \( v_2 \) is found testing \( d_2(k) \).

Remarks:

The method by Tse and Weinert is a combination of the innovation approach to the state space with the canonical representation of the innovation model, which decomposes the original model into q interconnected subsystems having a structure guaranteeing the complete observability of the j th subsystem from the j th output component. Each of such subsystems is driven by the white noise. The stochastic realization theory is applied to determin orders of subsystems ( orders understood as dimensions)

The method is more convincing then the Guiderzi's method and delivers very good results of the structural test.

The concept of the stochastic minimal realization of the sequence of the output autocovariance coefficients was proposed by Rissanen(1974).
3.3. Order tests proposed for the transfer matrix model of the MIMO system

Furuta's approach.

The main idea of the Furuta's approach is to identify coefficients appearing in the transfer matrix of the considered system, assuming a certain degree of the Common Denominator of the transfer matrix. This identification is performed employing the criterion function \( J \).

\[
J = \frac{1}{N-\ell + 1} \sum_{k=\ell}^{N} \| \varepsilon(k) \|^2
\]

where

\[
\varepsilon(k) = \gamma(k) + \sum_{i=1}^{\ell} a_i \gamma(k-i) - \sum_{i=1}^{\ell} \delta_i u(k-i)
\]

\[
\gamma(k) = u(k) + m(k) - (p \times 1) \text{ dimensional input measurement vector.}
\]

\[
u(k) - \text{ deterministic input (} p \times 1\text{)}
\]

\[
m(k) - \text{ stochastic input (} p \times 1\text{)}
\]

\[
E(m(k)) = 0 ; \quad E(m(k)m^T(t)) = Q I \delta_{kt}
\]

\[
\gamma(k) + \eta(k) - (q \times 1) \text{ dimensional output measurement vector}
\]

\[
\gamma(k) - \text{ deterministic output (} q \times 1\text{)}
\]

\[
\eta(k) - \text{ stochastic output (} q \times 1\text{)}
\]

\[
E(\eta(k)) = 0 ; \quad E(\eta(k)\eta^T(t)) = R I \delta_{kt}
\]

\[
E(m(k)\eta^T(t)) = 0 ;
\]

Furuta assumes further the following model of the transfer function matrix:
\[
K(z,\lambda) = \frac{1}{Q(z,\lambda)} \left[ \frac{1}{\lambda} z^{-1} + \frac{1}{\lambda^2} z^{-2} + \ldots + \frac{1}{\lambda^L} z^{-L} \right] \tag{215}
\]

\[
Q(z,\lambda) = 1 + a_1 z^{-1} + \ldots + a_L z^{-L}
\]

\[
\Gamma_i = \begin{bmatrix} T_{\mathbf{G}_{i1}} \\ \vdots \\ T_{\mathbf{G}_{ip}} \end{bmatrix} \quad \text{(q x p) matrix} ; \quad \mathbf{G}_{ij} = \text{(q x 1) vector}
\]

which corresponds with (209). Using (215) and (209) the vector of parameters

\[
\Phi = \begin{bmatrix} a_1, a_2, \ldots, T_{\mathbf{G}_{11}}, T_{\mathbf{G}_{12}}, \ldots, T_{\mathbf{G}_{ip}} \end{bmatrix}^T
\]

is identified minimizing \( J \) with respect to \( \Phi \).

The parameter estimation procedure is described in Furuta(1973). Assuming the transfer function matrix has already been identified, the nonminimal realization of the transfer function matrix can be found as:

\[
x(k+1) = F x(k) + G u(k) \tag{216}
\]

\[
y(k) = H x(k) \quad \text{where} \quad x(k) \in \mathbb{R}^n
\]

\[
F = \begin{bmatrix}
0 & \cdots & 0 & -a_L I \\
1 & & & \\
0 & & & \\
\vdots & & & \\
0 & \cdots & 0 & -a_1 I
\end{bmatrix} ; \quad G = b^{-1} \begin{bmatrix}
\mathbf{f}_L \\
\vdots \\
\mathbf{f}_1
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
0, \ldots, 0, I
\end{bmatrix}
\]

This is a very well known Frobenius canonical form (see for ex. Hajdasinski(1976)), and it is seen \( Q(z,\lambda) \) - is the annihilating polynomial of \( F \). From this form Ho and Kalman(1966) extracted a controllable and observable realization,
called the \textit{minimal realization}. It is based on the Hankel matrix, which can be derived as the product of the controllability and observability matrices.

\begin{align}
\mathbf{w} & = [\mathbf{c}, \mathbf{pc}, \ldots, \mathbf{p}^{n-1}\mathbf{c}] \\
\mathbf{v} & = [\mathbf{H}^T, \mathbf{F}^T\mathbf{H}^T, \ldots, (\mathbf{F}^T)^{n-1}\mathbf{H}^T] \\
\mathbf{H}_n & = [\mathbf{v}^T \mathbf{w}] 
\end{align}

(217) \quad (218)

Furuta introduces the concept of \(\varepsilon\) - practical controllability and observability, which gives a solution to the order test and further to the \(\varepsilon\) - minimal realization:

\textbf{Definition 17}  The state \(\mathbf{x}\) of the system (216) is said to be \(\varepsilon\) - practically controllable (observable) if \(\mathbf{x}\) is element in the space spanned by the eigenvectors of \(\mathbf{w}\mathbf{w}^T(\mathbf{v}\mathbf{v}^T)\) corresponding to the eigenvalues larger than \(\varepsilon\), \(\mathbf{w}\) and \(\mathbf{v}\) denote controllability and observability matrices.

\textbf{Remark} : It is easily seen that seeking for eigenvalues of the \(\mathbf{w}\mathbf{w}^T(\mathbf{v}\mathbf{v}^T)\) matrix is equivalent to the diagonal decomposition of the \(\mathbf{H}_n^T\) matrix, which is only slightly different from the singular value decomposition of the Hankel matrix.

\textbf{Definition 18}  The system represented by the states which are \(\varepsilon\) - practically controllable and observable is said to be \(\varepsilon\) - practically minimal realization.

The order test is made by comparison of eigenvalues of the \(\mathbf{w}\mathbf{w}^T\), and truncation of the selection matrix consisting of normal eigenvectors of \(\mathbf{w}\mathbf{w}^T\), such that consists only of these eigenvectors which correspond to eigenvalues larger than \(\varepsilon\).
If $\varepsilon$ is chosen as

$$v_{n_0} > \varepsilon > v_{n_0+1}; \quad v_i (i = 1, \ldots, n) \text{ - eigenvalues of } W W^T$$

(220)

then the order test is considered as completed, and $\varepsilon$ - minimal realization of the system is found as:

$$\begin{align*}
\{ F_0 &= s X^T s ; \quad G_0 = s T G ; \quad H_0 = H s \} \\
(222)
\end{align*}$$

Remark: This refers to the notion of the numerical rank of the Hankel matrix which will be discussed while considering the singular value decomposition of the Hankel matrix.

The Furuta's method is rather inefficient and incorporates an unnecessary step of the transfer matrix identification with arbitrarily chosen $\lambda$. This method, however, pointed out the way to the more advanced methods of the approximate minimal realization and to the eigenvalue problem in determination of the system order.
3.4. Miscellaneous order tests. The pattern recognition method by Thiga and Gough.

The approach presented by Thiga and Gough is direct in the sense the model parameters are not estimated at each step. The test is based on a measure of the linear dependence (or independence) of features displayed by each model order. This method is totally empirical and assumes identification of the order (understood as a degree of the transfer function denominator) of individual subsystems, such that the final system is described in form of the transfer function matrix.

\[ G = \begin{bmatrix} g_{ij} \end{bmatrix} ; i = 1,2,\ldots,q ; j = 1,2,\ldots,p \]  

(223)

where \( g_{ij} \) correspond to the linear differential or difference equations:

\[ \sum_{k=0}^{n_i} d_{ik} y_i(t) = \sum_{\ell=0}^{m} c_{\ell} u_j(t) \quad n_i > m \quad i=1,2,\ldots,q \]

(224)

\[ d_{ik} \neq 0 \quad j=1,2,\ldots,p \]

and \( d_{ik} \) and \( c_{\ell} \) are differential operators.

or

\[ y_i(k) = \sum_{\ell=0}^{m} b_{i\ell} u_j(k-\ell) - \sum_{\ell=0}^{n_i} a_{i\ell} y_i(k-\ell) \]

(225)

\[ m < n_i ; \quad a_{i\ell} \neq 0 \]

\[ k = 0,1,2,\ldots,N \]

The order test is completed when all subsequent \( n_i \) are found. Authors proposed to solve the order recognition problem by testing a single-valued function of pattern, which will reflect a decision surface \( f(x) = \epsilon \),
where \( c \) is the threshold of pattern recognition. For the sake of the order test for a noisy system, such a function must serve two following tasks:

1. help to spread the cluster of the patterns in the two classes further apart in the pattern space.
2. reduce the dimensionality of the pattern space by combining dimensions.

The input-output cross-correlation function is proposed as a one for the training of the system:

\[
R_{uy}(k) = \frac{T}{M} \sum_{j=1}^{M} u(j) y(j + k)
\]

Authors are reporting quite a success in the order discrimination during an experiment carried on for a set of "representative systems" chosen from amongst the vast possible combinations of characteristic roots of low-pass filters, up to seventh order. The machine was learned to extract from such patterns purported characteristics of a given system order and type of roots. This type of approach was tested with different types of processes proving its usefulness.

3.5. Akaike's FPE (final prediction error) and AIC (Akaike's maximum Information Criterion) as order tests for MIMO systems.

Akaike has proposed two new approaches to the order determination and consequently parameter estimation. These two methods - FPE and AIC are asymptotically equivalent in final results, as the distribution of the equation error (for the AR model - considered as the prediction error) converges to the Gaussian one.

These two methods employ much more statistical properties of given measurements, than some "a priori" assumptions about its nature. However one general
assumption, which in practical cases almost always holds, is the asymptotic Gausseness of the time series samples. To get some more insight into these two methods, let us study some important passages in their derivations, however no claim for completeness is being made.

3.5.1. Statistical predictor identification - Final Prediction Error Approach.

Let us consider first a single input-single output system and assume that the output of this system is a stationary and ergodic process $Y(n)$. In practice $Y(n)$ is given as a function of the recent values of $Y(n)$ and the structure or the parameter of the function is determined. There is considered the situation where the structure is identified using an observation of a process $X(n)$ and, using the structure, the prediction is made with another process $Y(n)$ which is independent of $X(n)$, but with one and the same statistical property as $X(n)$.

The FPE is defined as the mean square prediction error

$$FPE \text{ of } \hat{X}(n) = E \{(X(n) - \hat{X}(n))^2\}$$ (227)

for $\hat{X}(n)$ being the predictor of $X(n)$

When the process $X(n)$ is stationary and the predictor $\hat{Y}(n)$ of $Y(n)$ is linear and given by:

$$\hat{Y}(n) = \sum_{m=1}^{M} \hat{a}_M(m) Y(n-m) + \hat{a}_M(0)$$ (228)

where $\hat{a}_M(m)$ is a function of $X(n)$

Using (227) and (228) we have:

$$FPE \text{ of } \hat{Y}(n) = \sigma^2(M) + \sum_{\ell=0}^{M} \sum_{m=0}^{M} E\{\Delta a_M(m)\Delta a_M(\ell)\} \nu_{M+1}(\ell, m)$$ (229)
where:

$$\sigma^2(M) = \sum_{m=1}^{M} a_m(Y(n) - a_m) \sum_{m=1}^{M} a_m(Y(n-m) - a_m)^2$$

(230)

where \( a_m \) denotes the member of the set of parameters \( \{a(m)\} \) giving, in the sense of mean squares, the best linear predictor i.e.

$$\sigma^2(M) = \min_{\{a(m)\}} \sum_{m=1}^{M} a(m) Y(n-m) - a(0)^2$$

(231)

$$\Delta a_m(m) = \hat{a}_m(m) - a_m(m) \quad m = 0, 1, 2, \ldots, M$$

(232)

and

$$V_{M+1}(l, m) = E(Y(n-l)Y(n-m)) \quad l, m = 1, 2, \ldots, M$$

(233)

$$V_{M+1}(0, m) = V_{M+1}(m, 0) = E(Y(n)) \quad m = 1, 2, \ldots, M$$

(234)

$$V_{M+1}(0, 0) = 1$$

In the relation (229) all components containing expectations of products of noncorrelated quantities are neglected.

It is seen that FPE is composed of two components: prediction for a given \( M \)-th order, and the second due to statistical deviation of \( \hat{a}_m(m) \) from \( a_m(m) \).

Behaviour of the FPE, namely decrease of \( \sigma^2(M) \) for increasing \( M \) and increase of the second term for increasing \( M \), suggests that there exists an optimum for a certain \( M \), which is nothing but the order of the autoregressive model (228).
This idea is worked out further in Akaike (1970). Following this idea we get:

\[ X(n) = \sum_{m=1}^{M} a(m) X(n-m) + a(0) + \varepsilon(n) \]  

(235)

where \( \varepsilon(n) \) are the samples of the "white noise", uniformly distributed and

\[ E(\varepsilon(n)) = 0; \quad E(\varepsilon^2(n)) = \sigma^2 \]  

(236)

If there is a collection of data available, \{ \( X(n) ; n = -M+1, -M+2, \ldots, N \) \}, the parameter \( \hat{a}_M(m) \) is defined as the least squares estimate of \( a(m) \) (\( \hat{a}_M(m) \) is the parameter of the predictor (228) and we are going to find it basing on observations of \( X(n) \)).

Defining

\[ C_{XX}(m, \ell) = \frac{1}{N} \sum_{n=1}^{N} (X(n-m) - \bar{x}_m)(X(n-\ell) - \bar{x}_\ell) \]  

(237)

where

\[ \bar{x}_m = \frac{1}{N} \sum_{n=1}^{N} X(n-m) \quad m = 0, 1, 2, \ldots, M \]  

(238)

\[ C_{XX}(m, \ell) \hat{a}_M(m) = \frac{1}{N} \sum_{n=1}^{N} (X(n-m)\hat{a}_M(m) - \bar{x}_m \hat{a}_M(m))(X(n-\ell) - \bar{x}_\ell) \Rightarrow \]

\[ \sum_{m=1}^{M} C_{XX}(m, \ell) \hat{a}_M(m) = \frac{1}{N} \sum_{n=1}^{N} \left( \sum_{m=1}^{M} \hat{a}_M(m)X(n-m) - \hat{a}_M(m) \bar{x}_m \right) (X(n-\ell) - \bar{x}_\ell) \]

(239)
where
\[
\hat{X}_0 = \sum_{m=1}^{M} \hat{A}_N(m) \bar{X}_m = \frac{1}{N} \sum_{m=1}^{M} \sum_{n=1}^{N} \hat{A}_N(m) X(n-m) \tag{240}
\]

Thus for \( N \) large enough we can write:
\[
\sum_{m=1}^{M} C_{xx}(m,\ell) \hat{A}_N(m) = C_{xx}(0,\ell) \tag{241}
\]

for \( \ell = 1, 2, 3, \ldots, M \) and \( m = 0, 1, 2, \ldots, M \) according to (237) and (238). Relation (241) can be rewritten in the form:
\[
\begin{bmatrix}
C_{xx}(1,1) & C_{xx}(2,1) & \cdots & C_{xx}(M,1) \\
C_{xx}(1,2) & C_{xx}(2,2) & \cdots & C_{xx}(M,2) \\
\vdots & \vdots & \ddots & \vdots \\
C_{xx}(1,M) & C_{xx}(2,M) & \cdots & C_{xx}(M,M)
\end{bmatrix}
\begin{bmatrix}
\hat{A}_N(1) \\
\hat{A}_N(2) \\
\vdots \\
\hat{A}_N(M)
\end{bmatrix}
= \begin{bmatrix}
C_{xx}(0,1) \\
C_{xx}(0,2) \\
\vdots \\
C_{xx}(0,M)
\end{bmatrix} \tag{242}
\]

or
\[
C_{xx} \hat{A}_N = C_M \tag{243}
\]

if \( C_{xx} \) is nonsingular
\[
\hat{A}_N = C_{xx}^{-1} C_M \tag{244}
\]

if \( C_{xx} \) is singular or ill conditioned
\[
\hat{A}_N = C_{xx}^+ C_M \tag{245}
\]

where \( C_{xx}^+ \) is the Moore-Penrose pseudoinverse of \( C_{xx} \).
The "zero" coefficient $a_M(0)$ is estimated basing on (235)

$$a_M(0) = \bar{x}_0 - \sum_{m=1}^{M} \hat{a}_M(m) \bar{x}_m$$  \hspace{1cm} (246)

Following the definition given previously - (228) using (246), we get:

$$\hat{Y}(n) = \sum_{m=1}^{M} \hat{a}_M(m) (Y(n-m) - \bar{x}_m) - \bar{x}_0$$  \hspace{1cm} (247)

the relation for the predictor $\hat{Y}(n)$ of $Y(n)$.

Assuming that $Y(n)$ is generated by relation

$$Y(n) = \sum_{m=1}^{M} a_m Y(n-m) + a_0 + \delta(n)$$  \hspace{1cm} (248)

where

$\delta(n)$ has exactly the same statistical properties as $\varepsilon(n)$, we get:

$$Y(n) - \hat{Y}_{\text{opt}}(n) = \delta(n) - \sum_{m=1}^{M} \Delta a_M(m) y(n-m) - (\Delta \bar{x}_0 + \sum_{m=1}^{M} \hat{a}_M(m) \Delta \bar{x}_m)$$  \hspace{1cm} (249)

for $a_m = \hat{a}_M(m)$ are coefficients of the optimal predictor $\hat{Y}_{\text{opt}}(n)$.

Relation (249) is somewhat abstract and refers to statistical properties of the optimal predictor. It is assumed here, the process $Y(n)$ is stationary and ergodic, thus defining the following:

$$\Delta a_M(m) = \hat{a}_M(m)$$

$$y(n) = Y(n) - E(Y(n))$$

$$\Delta \bar{x}_\ell = \bar{x}_\ell - E(X(n))$$  \hspace{1cm} (250)
basing on (247) and (248), (249) we get two relations:

\[
Y(n) - \tilde{Y}_{\text{opt}}(n) = \delta(n) - \left( \sum_{m=1}^{M} \hat{a}_M(m)Y(n-m) - \sum_{m=1}^{M} a_M(m)Y(n-m) - \sum_{m=1}^{M} \hat{a}_M(m)E(Y(n-m)) + \sum_{m=1}^{M} a_M(m)E(Y(n-m)) \right) - \left( \bar{X}_o - E(X(n)) \right) + \sum_{m=1}^{M} \hat{a}_M(m) \left( \bar{X}_m - E(X(n)) \right)
\]

(251)

from (249)

\[
Y(n) - \tilde{Y}(n) = \delta(n) - \sum_{m=1}^{M} \hat{a}_M(m) (Y(n-m) - \bar{X}_m) - \bar{X}_o + \sum_{m=1}^{M} a_m Y(n-m) + a_o
\]

(252)

\[
a_o = \bar{X}_o - \sum_{m=1}^{M} \hat{a}_M(m) \bar{X}_m
\]

(253)

thus

\[
Y(n) - \tilde{Y}(n) = \delta(n) - \sum_{m=1}^{M} (\hat{a}_M(m) - a_m) Y(n-m) = \\
= \delta(n) - \sum_{m=1}^{M} ((\hat{a}_M(m) - a_m) + (a_M(m) - a_m)) Y(n-m) = \\
= \delta(n) - \sum_{m=1}^{M} \Delta a_M(m) Y(n-m) + \sum_{m=1}^{M} (a_M(m) - a_m) Y(n-m)
\]

(254)

From (251) and (254) we see that \((Y(n) - \tilde{Y}(n))\) will statistically be equal \((Y(n) - \tilde{Y}_{\text{opt}}(n))\) if:
From (255) it is seen that statistically left and right sides are equivalent if \( a_m \rightarrow a_M(m) \), thus (249) is proven. Taking into account independence of \( y(n) \) of \( \Delta a_M(m) \) and \( \Delta\bar{x} \), we get:

\[
\text{FPE of } \hat{Y}(n) = E( Y(n) - \hat{Y}(n) )^2 = \sigma^2 + \sum_{m=1}^{M} \sum_{\ell=1}^{M} E(\Delta a_M(m)\Delta a_M(\ell)R_{XX}(\ell-m) + \\
+ E(\Delta\bar{x} - \sum_{m=1}^{M} \hat{a}_M(m)\Delta\bar{x}_m)^2)
\]

(256)

where

\[
R_{XX}(\ell-m) = E( X(n-\ell)X(n-m) ) - (E( X(n) )^2)
\]

(257)

For the sake of numerical solution we have to be interested in assymptotic properties of the FPE, which hopefully can be a lot simpler to handle than statistical evaluation of all subsequent quantities of the relation (256). For this sake quantities \( \Delta\bar{x} \) and \( \Delta a_M(m) \) have been introduced.

For the asymptotic evaluation of the FPE we will need the following theorem:

**Theorem 9:** Under the assumption of the stationarity and ergodicity of \( X(n) \), the limit distribution \( \sqrt{N} \Delta\bar{x} = \sqrt{N}( \bar{X}_n - E(X(n)) ) \) and

\[
\sqrt{N} \Delta a_M(m) = \sqrt{N}( \hat{a}_M(m) - a_M(m) ) \text{ for } m = 1,2,\ldots, M,
\]

when \( N \) tends to infinity, is \( (M+1) \)-dimensional Gaussian with zero mean and the variance matrix:

\[
\sigma^2 \begin{bmatrix}
\delta^{-2} & \Theta^T_M \\
\Theta_M & R_M^{-1}
\end{bmatrix}
\]
where
\[ \delta = 1 + \sum_{m=1}^{M} a_{M}(m) \]

\[ R_{M} = \begin{pmatrix} M \times M \end{pmatrix} \text{ matrix of } R(\ell, m) = R_{xx}(\ell - m) \]

and
\[ \mathbf{0} \] denotes a zero column vector

( this theorem is a special case of the limit theorem as presented in the book of Anderson T.W. (1971) ).

From the ergodicity of the process it is clear that \( C_{xx}(\ell, m) \) converges to \( R_{xx}(\ell - m) \) as \( N \to \infty \), with probability one. Thus \( \tilde{a}_{M} \) is a consistent estimate of \( a_{M} \) ( a vector of parameters of the optimal predictor) with convergence with probability one.

From (244) we have:

\[ \tilde{a}_{M}(\ell) = \left( C_{xx}(m, \ell) \right)^{-1} \left( C_{xx}(0, \ell) \right) \] (258)

\[ \tilde{a}_{M}(\ell) = a_{M}(\ell) + \left( C_{xx}(m, \ell) \right)^{-1} \left( C_{\varepsilon x}(\ell) \right) \] (259)

where
\[ C_{\varepsilon x}(\ell) = N^{-1} \sum_{n=1}^{N} \varepsilon(n)(X(n-\ell) - \tilde{X}_{\ell}) \] (260)

\[ \Delta a_{M}(\ell) = \left( C_{xx}(m, \ell) \right)^{-1} \left( C_{\varepsilon x}(\ell) \right) \] (261)

From the consistency of \( C_{xx}(m, \ell) \) it is concluded that that the limit distribution of \( \sqrt{N} \Delta \tilde{x}_{o} \) and \( \sqrt{N} \Delta a_{M} \) is identical to that \( \sqrt{N} \Delta \tilde{x}_{o} \) and \( \sqrt{N} R_{M}^{-1} C_{\varepsilon x} \).

Now, instead of taking the expectation of \( (Y(n) - \hat{Y}(n) )^{2} \) as it was done in (256), the conditional expectation of \( (Y(n) - \hat{Y}(n) )^{2} \) for a given \( X(n) \) is considered. The last is denoted by

\[ E\{ (Y(n) - \hat{Y}(n) )^{2} | X(n) \} \]
From the independency of \( Y(n) \) and \( X(n) \) it comes

\[
E\{(Y(n) - \hat{Y}(n))^2 | X(n) \} = \sigma^2 + \sum_{m=1}^{M} \sum_{\ell=1}^{M} \Delta a_M(m) \Delta a_M(\ell) R_{xx}(\ell-m) + \\
+ (\Delta \overline{X}_o - \sum_{m=1}^{M} \hat{a}_M(m) \Delta \overline{X}_m)^2
\]  

(262)

Again in the limit the difference between \( \sqrt{N} \Delta \overline{X}_o \) and \( \sqrt{N} \Delta \overline{X}_m \) are stochastically vanishing \( (m = 1, 2, \ldots, M) \) and Akaike (1970) demonstrates that:

\[
N \left\{ E\{ (Y(n) - \hat{Y}(n))^2 | X(n) \} - \sigma^2 \right\}
\]  

(263)

has a limit distribution with expectation equal to \( (M+1)\sigma^2 \), thus

\[
N \left\{ (\text{FPE})_M - \sigma^2 \right\} = (M+1)\sigma^2
\]  

(264)

\[
(\text{FPE})_M = \left( 1 + \frac{M+1}{N} \right) \sigma^2
\]  

(265)

where \( (\text{FPE})_M \) stands for an asymptotic evaluation of FPE.

Now remains to find an appropriate estimate of \( \sigma^2 \). From the ergodicity of \( X(n) \) it is concluded that

\[
S(M) = C_{xx}(0,0) - \sum_{\ell=1}^{M} \hat{a}_M(\ell) C_{xx}(0,\ell)
\]  

(266)

is a consistent estimate of \( \sigma^2 \). It is shown further in Akaike (1970) that the expression

\[
(1 - \frac{M+1}{N})^{-1} S(M)
\]  

(267)

is the best estimate of \( \sigma^2 \) while \( N \to \infty \). Thus the final Akaike’s FPE criterion is:
The order $M$ is chosen which minimizes $(FPE)_M$, thus completing the order test.

For the sake of MIMO models identification, the order test can be performed in the very similar way. The only assumption to be made is that the multivariable system is described by the following autoregressive model:

$$
Y(j) = \sum_{m=1}^{M} A_m Y(j-m) + A_o + E(j)
$$

(269)

where

$$
Y(j) = \begin{bmatrix} Y(j) \\ u(j) \end{bmatrix}; \quad Y(j) = \begin{bmatrix} y_1(j) \\ y_2(j) \\ \vdots \\ y_q(j) \end{bmatrix}; \quad u(j) = \begin{bmatrix} u_1(j) \\ u_2(j) \\ \vdots \\ u_p(j) \end{bmatrix}
$$

$A_m$ - $(p+q) \times (p+q)$ matrix of parameters

$A_o$ - $(p+q) \times 1$ vector of initial conditions

$E(j)$ - $(p+q) \times 1$ random vector satisfying the relations

$$
E\{E(j)\} = 0
$$

$$
E\{E(j)Y^T(j-m)\} = 0 \quad \text{for } m \geq 1
$$

$$
E\{E(j)E^T(m)\} = \delta_{jm} S
$$

(270)

Given a set of observed data $(Y(j); j = 1,2,\ldots,N)$, where $y_i(j)$ $(i = 1,2,\ldots,q+p)$ will in general denote the $i$th component of $Y(j)$.

I. Define $\bar{y}_i(j)$ $(i = 1,2,\ldots,k, k=p+q)(j = 1,2,\ldots,N)$ by

$$
\bar{y}_i(j) = y_i(j) - \overline{\bar{y}_i}
$$

(271)
where
\[ \bar{y}_i = \frac{1}{N} \sum_{j=1}^{N} y_i(j) \]  
(272)

II. For \( m = 0, 1, 2, \ldots, L \), where \( L \) is maximally allowable order of the model and should be generally kept below \( N/5k \), define the \((i, k)\) th element of \((k \times k)\) matrix \( C_m \) as:
\[ C_m(i, k) = \frac{1}{N} \sum_{j=1}^{N-m} \bar{y}_i(j+m) \bar{y}_k(j) \]  
(273)

III. Basing on (270)(271)(272) and definition of \( \mathbf{S} \) the sample covariance matrix of residual error can be computed by the following recursive formulae:
\[ \mathbf{S}(M) = \mathbf{C}_0 - \sum_{m=1}^{M} A_m(M) C_m^T \]  
(274)

where \( A_m(M) \) can be evaluated using a recursive algorithm.

\[ \mathbf{R}(M) = \mathbf{C}_{M+1} - \sum_{m=1}^{M} A_m(M) C_{M+1-m} \]  
(275)

\[ \mathbf{Q}(M) = \mathbf{C}_0 - \sum_{m=1}^{M} B_m(M) C_m \]  
(276)

\[ \mathbf{D}(M) = \mathbf{R}(M)\mathbf{Q}(M)^{-1} \]  
(277)

\[ \mathbf{A}_m(M+1) = A_m(M) - \mathbf{D}(M) B_{M+1-m}(M) \]  
(279)

\[ m = 1, 2, \ldots, M \]
\( A_m(M+1) = D(M) \) for \( m = M+1 \) \hspace{1cm} (280)

\( B_m(M+1) = B_m(M) - \lambda(M) A_{M+1-m}(M) \) for \( m = 1,2,...,M \) \hspace{1cm} (281)

\( B_m(M+1) = \lambda(M) \) for \( m = M+1 \) \hspace{1cm} (282)

\( S(0) = Q(0) = C_0 \) ; \( R(0) = C_1 \) \hspace{1cm} (283)

IV. Subsequently the order test must be performed according to the \((FPE)_M\) criterion, which in this case takes the form

\[
FPC(M) = \parallel S_{q \times q}(M) \parallel \left( 1 + \frac{Mk+1}{N} \right)^q \left( 1 - \frac{Mk+1}{N} \right)^{-q} \hspace{1cm} (284)
\]

where \( S_{q \times q}(M) \) denotes the \( q \times q \) submatrix in the upper left corner of \( S(M) \).

Adopting the value of \( M \), which gives the minimum of \( FPE(M) \) for \( (M = 1,2,...L) \), as the order of the model (269), the necessary matrices of coefficients can be found by relations (274) - (283). \( \parallel \cdot \parallel \) denotes a determinant.

This completes the order identification in terms of Akaike's \( FPE(M) \) for MIMO systems.
3.5.2 Akaike's maximum Information Criterion approach

Akaike has proposed a new approach to the order determination and parameter estimation, based on the FPE statistics which was a simplified concept of the broader concept of AIC (Akaike's Information Criterion-(1972),(1975)). The AIC criterion is built on the extension of the maximum likelihood principle, very well known and broadly applied everywhere, where the computing time factor does not play a main role. A brief and elegant explanation of the classical maximum likelihood method, the reader can find in Eykhoff(1974).

Considering $y$ being the output vector and the components of $y$ random variables $\{y(1), ..., y(k)\}$ (for SISO systems for the time being), the joint probability function for $y$ will be:

$$p(y(1), ..., y(k); \theta) = p(\chi^k; \theta)$$

where

$\chi^k$ depends on $\theta$, the vector of parameters.

This is our "a priori" knowledge; a posteriori knowledge encompasses values of the random variable as measured. From this, an estimate $\hat{\theta}$ of the $\theta$ can be determined.

To distinguish it from relation(285), the joint probability function of $\chi^k$ and $\hat{\theta}$ is called the likelihood function and denoted as $L$

$$L(y(1), ..., y(k); \hat{\theta}) = L(\chi^k; \hat{\theta})$$

For the sake of convenience, $\ln L$ is usually considered and the logarithmic function, as the monotonic function, has the maximum at the same value of $\hat{\theta}$ as $L$. (Eykhoff(1974)). This value of $\hat{\theta}$ can be obtained by solving:
\[ \Delta \beta \ln L(y^k; \beta) = \frac{\partial}{\partial \beta} \ln L(y^k; \beta) \bigg|_{\beta = \hat{\beta}} = 0 \]  

(287)

\( \hat{\beta} \) is called a maximum likelihood estimate of \( \beta \). In the classical theory of estimation it is usually assumed that observations in \( y^k \) are independent so that

\[ p(y^k; \beta) = \prod_{i=1}^{k} p(y(i); \beta) \]  

(288)

and

\[ L(y^k; \beta) = \prod_{i=1}^{k} L(y(i); \beta) \]  

(289)

\[ \ln L(y^k; \beta) = \sum_{i=1}^{k} \ln L(y(i); \beta) \]  

(290)

To discuss properties of the Akaike's method it is necessary to discuss some properties of the maximum likelihood method.

Let us define a bias \( \Delta(b) \) in the estimator \( \hat{b} \), given the observations \( y^k \):

\[ \Delta(b) \triangleq b - E\left\{ \hat{b}(y^k) \mid y^k \right\} = b - \int \hat{b}(y^k) p(y^k; b) \, dy^k \]  

(291)

and the error covariance matrix \( \mathbf{R}(b) \) of the estimator \( \hat{b} \):

\[ \mathbf{R}(b) = E\left\{ (b - \hat{b}(y^k) - \Delta(b))(b - \hat{b}(y^k) - \Delta(b))^\text{T} \mid y^k \right\} = \int (b - \hat{b}(y^k) - \Delta(b))(b - \hat{b}(y^k) - \Delta(b))^\text{T} p(y^k; b) \, dy^k \]  

(292)

In general, it is not possible to compute either \( \Delta(b) \) or \( \mathbf{R}(b) \). However, for any unbiased estimator, we have the following Cramer-Rao lower bound.

\[ \mathbf{R}(b) \geq \frac{\mathbf{I}^{-1}}{d_y^k} \]  

(293)

where

\[ d_y^k \]
\[
\mathbf{H}_2 y_k (b) = -E \left\{ \frac{\partial^2}{\partial b^2} \ln p(y_k^*, b) | b \right\} = E \left\{ \left[ \frac{\partial}{\partial b} \ln p(y_k^*, b) \right] \left[ \frac{\partial}{\partial b} \ln p(y_k^*, b) \right]^T \right\}
\]

is the Fisher information matrix.

Relation (293) is equivalent to

\[
\mathbf{R}(b) - \mathbf{H}_2 y_k (b) \succ 0
\]

which also means that every diagonal element of \( \mathbf{R}(b) \) must be no smaller than the corresponding element of \( \mathbf{H}_2 y_k (b) \).

Thus the Cramer-Rao lower bound provides a lower bound on the accuracy to which any component of \( b \) can be estimated. The technical assumptions required in the derivation of the lower bound do not include any assumptions of linearity or Gaussianness. Thus the bound is well-suited for nonlinear problems such as the parameter identification problem for dynamical systems. It is also worth noticing that \( \Delta (b) \) and \( \mathbf{R}(b) \) and \( \mathbf{H}_2 y_k (b) \) depend on \( b \).

Some asymptotic properties of the maximum likelihood estimate will also be very interesting. The asymptotic properties are concerned with the limiting behaviour as the number of observations becomes infinite. The first assumption to be made is the identifiability condition.

\[
p(y_k^*; b_1) \neq p(y_k^*; b_2) \text{ for all } b_1 \neq b_2
\]

This assumption means that no two parameters lead to observations with identical probabilistic behaviour. It is obvious that if the identifiability condition is violated for some pair \( b_1, b_2 \) of parameters, then \( b_1 \) and \( b_2 \) cannot be distinguished no matter how many observations are made.

Assuming independent observations, identifiability and additional technical assumptions, the asymptotic results are as follows:
1. **Consistency**

\[ \{ \hat{\theta} \to b \} \text{ with probability 1 as } \{ k \to \infty \} \]

2. **Asymptotic unbiasedness**

\[ E \{ \hat{\theta} \mid b \} \to b \text{ as } \{ k \to \infty \} \]

3. **Asymptotic normality**

\( \hat{\theta} \) tends towards a Gaussian random variable as \( \{ k \to \infty \} \)

4. **Asymptotic efficiency**

\[ E \{ (\hat{\theta} - \theta)(\hat{\theta} - \theta)^T \} \to \frac{1}{k} \mathcal{N}^T(b) \text{ as } \{ k \to \infty \} \]

In other words, as the number of processed observations becomes infinite, the maximum likelihood estimate converges to the true value of \( b \), and the parameter estimate error is asymptotically normally distributed \( (\hat{\theta} - b) \), with covariance matrix \( \frac{1}{k} \mathcal{N}^T(b) \), so that the Cramer-Rao lower bound is asymptotically tight.

The independence assumption implies an additive form for the information matrix

\[
\mathcal{N}^T(b) = -E \left\{ \frac{\partial^2}{\partial b^2} \ln p(y \mid b) \right\} = -E \left\{ \frac{\partial^2}{\partial b^2} \ln \prod_{i=1}^k p(y_i \mid b) \right\} = \\
= \sum_{i=1}^k -E \left\{ \frac{\partial^2}{\partial b^2} \ln p(y_i \mid b) \right\} = k \frac{1}{\mathcal{N}^T(b)}
\]

(298)

where \( \frac{1}{\mathcal{N}^T(b)} \) is the information matrix for a single observation.

In terms of the asymptotic covariance matrix, we see that:

\[
E \{ (\hat{\theta} - b)(\hat{\theta} - b)^T \} \approx \frac{1}{k} \mathcal{N}^T(b) \text{ for large } k
\]

(299)
This was introductory information. Let us return to the AIC-criterion. Given a set of estimates \( \hat{\beta} \) of the vector of parameters \( \beta \) of a probability distribution with density function \( p(y^k;\beta) \), we will choose the one which will give the maximum of the expectation from the \( \ln \) (logarithm) of the likelihood function, which will be by the definition:

\[
E \left\{ \ln p(y;\hat{\beta}) \right\} = \int p(y;\beta) \ln p(y;\hat{\beta}) \, dy
\]  

(300)

where \( y \) is the random variable following the distribution with the density function \( p(y;\beta) \) and is independent of \( \hat{\beta} \).

For the purpose of discrimination between the two probability distributions with density functions \( p_i(y) \) (\( i = 0,1 \)), all the necessary information is contained in the likelihood ratio \( T(y) = p_i(y)/p_0(y) \). Thus Akaike suggests, instead of the classical maximum likelihood principle, maximization of information theoretic quantity, which is given by definition:

\[
E \ln \left( \frac{p(y;\hat{\beta})}{p(y;\beta)} \right) = \int p(y;\beta) \ln \left( \frac{p(y;\hat{\beta})}{p(y;\beta)} \right) \, dy
\]  

(301)

Relation (301) is very well known as the Kullback-Leibler’s mean value of information for discrimination between \( p(y^k;\beta) \) and \( p(y^k;\beta) \) and can be interpreted as the distance between the two distributions.

\[
E \ln \left( \frac{p(y;\hat{\beta})}{p(y;\beta)} \right) = \int p(y;\beta) \ln p(y;\hat{\beta}) \, dy - \int p(y;\beta) \ln p(y;\beta) \, dy = I(y;\beta;\hat{\beta}) - I(y;\beta)
\]  

(302)
Following the concept of the measure of information, Akaike defines the "mean amount of information to be discriminated per observation" by

\[ I(p_1, p_0, \Phi) = \int \Phi \left( \frac{p_1(y)}{p_0(y)} \right) p_0(y) \, dy \]  (303)

where \( \Phi \) must be "properly" chosen, and by \( dy \) demonstrates the measure with respect to which \( p_1(y) \) are defined. Considering a parametric situation, where probability densities are specified by a set of parameters \( b \) in the form

\[ p(y) = p(y \mid b) \quad \text{theoretical distribution} \]  (304)

\[ b^m = (b_1, b_2, \ldots, b_m) \]

\[ p(y \mid \hat{b}) \quad \text{the true distribution under observation} \]  (305)

\[ \hat{b}^m = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_m) \]

The quantity defined by (303) will be denoted \( I(b, \hat{b}, \Phi) \) with \( p(y) = p(y \mid b) \) and \( p_0(y) = p(y \mid \hat{b}) \). To find the most "proper" (suitable) form of the function, Akaike analysed the sensitivity of \( I(b, \hat{b}, \Phi) \) to the derivation from \( \hat{b} \) to \( b \).

It is assumed that either \( p(y \mid b) \) and \( \Phi(r) \) are the regular functions.

\[
\frac{\partial}{\partial b_i} I(b, \hat{b}, \Phi) = \int \left[ \frac{d}{dy} \Phi(r) \frac{\partial r}{\partial b_i} \right] \cdot p(y \mid \hat{b}) \, dy + \int \Phi(r) \frac{\partial p(y \mid \hat{b})}{\partial b_i} \Big|_{b=\hat{b}} \, dy =
\]

\[ = \Phi(r) \left[ \frac{\partial p(y, \hat{b})}{\partial b_i} \right]_{b=\hat{b}} \, dy \]  (306)
and assuming validity differentiation under integral sign as well as that

\[ \int p(y \mid b) dy = 1 \quad \text{(by def. of } p(y \mid b) \text{)}, \]

which implies that

\[ \Phi(u) \int \frac{\partial p(y, b)}{\partial b_c} \left|_{b = \hat{b}} \right. \, dy = \Phi(u) \frac{\partial^2}{\partial b_c \partial b_m} \left[ \int p(y \mid b) dy \right]_{b = \hat{b}} = 0 \quad (307) \]

In the same way it can be shown that

\[ \int \frac{\partial^2 p(y, b)}{\partial b_c \partial b_m} \, dy = 0 \quad (308) \]

which gives

\[ \frac{\partial^2 I(b, \hat{b}, \Phi)}{\partial b_c \partial b_m} \bigg|_{b = \hat{b}} = \Phi(u) \int \left[ \frac{\partial p(y, b)}{\partial b_c} \left( \frac{1}{p(y, \hat{b})} \right) \frac{\partial p(y, b)}{\partial b_m} \left( \frac{1}{p(y, \hat{b})} \right) \right] dy \quad (309) \]

Comparing (309) with (298) we see that integral

\[ \int \left[ \frac{\partial p(y, b)}{\partial b_c} \left( \frac{1}{p(y, \hat{b})} \right) \frac{\partial p(y, b)}{\partial b_m} \left( \frac{1}{p(y, \hat{b})} \right) \right] dy = \frac{1}{\lambda} \delta_{y, \lambda, m}(b) \quad (310) \]

is the \((l,m)\)th element of Fisher's information matrix.
Summarizing we get:

\[
\begin{align*}
I(\hat{c}, \hat{\theta}, \Phi) &= \Phi(\hat{\xi}) \\
\frac{\partial I(\hat{c}, \hat{\theta}, \Phi)}{\partial b_0} \bigg|_{b_0 = \hat{b}} &= 0 \\
\frac{\partial^2 I(\hat{c}, \hat{\theta}, \Phi)}{\partial b_0 \partial b_m} &= \Phi(b_0) I_{Y_{1m}}(b_0)
\end{align*}
\]

Relation (311) shows that \(\Phi(\hat{\xi})\) must be different from zero if \(I(\hat{b}, \Phi)\) ought to be sensitive to small variations of \(b\). Also the relative sensitivity of \(I(b, \hat{\theta}, \Phi)\) is high when \(\left| \frac{\Phi(\hat{\xi})}{\Phi(\hat{\xi})} \right|\) is large. The possible forms of \(\Phi(r)\) are e.g. \(\log e, (r-1)\) and \(r\).

To restrict further the form of \(\Phi(r)\), the increase of information is considered, by increase of \(N\) independent observations of \(Y\).

For this case

\[
I_N(b_0, \hat{\theta}, \Phi) = \Phi \left( \frac{\prod_{i=1}^{N} p(y_i|\hat{\theta})}{\prod_{i=1}^{N} p(y_i|\hat{\theta})} \right) \prod_{i=1}^{N} p(y_i|\hat{\theta}) dy_i \ldots dy_N
\]

thus

\[
\begin{align*}
I_N(\hat{\theta}, \hat{\theta}, \Phi) &= I(\hat{c}, \hat{\theta}, \Phi) \\
\frac{\partial I_N(b_0, \hat{\theta}, \Phi)}{\partial b_0} \bigg|_{b_0 = \hat{b}} &= 0 \\
\frac{\partial^2 I_N(b_0, \hat{\theta}, \Phi)}{\partial b_0 \partial b_m} \bigg|_{b_0 = \hat{b}} &= N \frac{\partial^2 I(b, \hat{e}, \Phi)}{\partial b_e \partial b_m} \bigg|_{b_e = \hat{b}}
\end{align*}
\]
From (313) it is seen that $I_N(b, \hat{\theta}, \Phi)$ is not very much influenced by the increase of information. It can be also seen from evaluation of (313) from (312) that the only quantity concerned with a final result is

$$\frac{\partial}{\partial b_c} \left[ \frac{1}{\prod_{i=1}^{N} p(y(i)|b)} \right] \bigg|_{b=\hat{b}} = \sum_{i=1}^{N} \frac{\partial p(y(i)|b)}{\partial b_c} \left[ \frac{1}{p(y(i)|\hat{b})} \right] \bigg|_{b=\hat{b}}$$

(314)

This last relation shows very clearly that taking into account the $\ln$ from the density function $p(y(i)|b)$, (314) will be fulfilled in a natural way:

$$\frac{\partial p(y|b)}{\partial b_c} \cdot \frac{1}{p(y|b)} = \frac{\partial \ln p(y|b)}{\partial b_c}$$

(315)

and thus this observation suggests the choice of $\Phi(r) = \ln r$ for the definition of the amount of information, - this simply leads to the Kullback-Leibler definition of information.

Remark: Any other definition of $\Phi(r)$ will be useful only if $\Phi$ is not vanishing.

For the purpose of the estimation Akaike proposes the following loss

$$W(b, \hat{\theta}) = (-2) \int p(y|b) \ln \left( \frac{p(y|b)}{p(y|\hat{b})} \right) dy$$

(316)

and the risk

$$R(b, \hat{\theta}) = E \{ W(b, \hat{\theta}) \}$$

(317)

functions, which are based on the Kullback-Leibler formula. Akaike pestilates that when $N$ independent realizations $y(i)$ ($i = 1, 2, \ldots, N$) of $Y$ are available, $(-2)$ times the sample mean of the $\ln$-likelihood, will be a consistent estimate of $W(b, \hat{\theta})$,

$$(-2) \sum_{i=1}^{N} \ln \left( \frac{p(y(i)|\hat{b})}{p(y(i)|b)} \right)$$

(318)
thus, at least for large values $N$, the value $\hat{\beta}$ which will give the maximum of

$$
\sum_{i=1}^{N} \ln \left( \frac{p(y_{i} | \hat{\beta})}{p(y_{i} | \beta)} \right)
$$

will nearly minimize $W(b, \hat{b})$.

The whole idea of Akaike is to, instead of considering a single estimate of $b$, to consider estimates corresponding to various possible restrictions of the distribution. This whole idea can be simply realized by comparing $R(b, \hat{b})$ or $W(b, \hat{b})$ if possible, for various $\hat{\beta}$'s and taking the one with the minimum of $R(b, \hat{b})$ or $W(b, \hat{b})$ as a final choice.

This approach may be viewed as a natural extension of the classical maximum likelihood principle. The only problem in applying this extended principle in practical situations is how to get a reliable estimate of $R(b, \hat{b})$ and $W(b, \hat{b})$.

Akaike gives a procedure which enables finding estimates of $R(b, \hat{b})$ and $W(b, \hat{b})$ which are used later for estimation of the order of the model and the vector of parameters $\hat{\beta}$.

Assuming $b_k$ ($k = 0, 1, 2, \ldots, L$) - components of the vector $b$, Akaike is looking for $\hat{b}_k$ ($k = 0, 1, 2, \ldots, L$) being maximum likelihood estimates of $b_k$. Considering the situation where the results $y(i)$ ($i = 1, 2, \ldots, N$) are obtained as independent $N$ observations, $\hat{b}_k$ will be the value of $b_k$ which gives the maximum of the likelihood function

$$
\prod_{i=1}^{N} p(y(i) | b_k).
$$

Thus Akaike suggests that

$$
W_{k,L} = \left( -\frac{2}{N} \right) \sum_{i=1}^{N} \ln \left( \frac{p(y_{i} | \hat{\beta}_k)}{p(y_{i} | \hat{\beta}_L)} \right)
$$

(320)
as an estimate of $\hat{W}(\hat{\beta}, \hat{\beta}_k)$. In Akaike's paper (1972) it is precisely proven that the
\[
\nu(\hat{\beta}, \hat{\beta}_k) = \frac{1}{N} \left[ -2 \sum_{i=1}^{N} \ln \left( \frac{p(y_i|\hat{\beta})}{p(y_i|\hat{\beta}_k)} \right) + 2k - L \right]
\]  
(324)
is the good estimate of $E\{W(\hat{\beta}, \hat{\beta}_k)\}$ (where $\hat{\beta}$ is the estimate of $\beta_k$ according to Euclidean norm), at least in cases where $N$ is sufficiently large and $L$ and $k$ are relatively large integers. Since we are only concerned with finding out the $\hat{\beta}_k$ which will give the minimum of $r(\hat{\beta}, \hat{\beta}_k)$, we have only to compute:
\[
\nu_{k,L} = -2 \sum_{i=1}^{N} \ln p(y_i|\hat{\beta}_k) + 2k
\]  
(322)
\[
\lambda_{k,L} = -2 \sum_{i=1}^{N} \ln \left( \frac{p(y_i|\hat{\beta})}{p(y_i|\hat{\beta}_k)} \right) + 2k
\]  
(323)
Relation (322) is the essence of the Akaike's method.

3.5.3. Concluding remarks
As it was proven by Söderström (1977) Akaike's FPC criterion and Akaike's AIC criterion are asymptotically equivalent. However, for the MIMO systems the evidence of equivalence of $\text{FPC}(M)$ and $\text{AIC}(M)$ is not proven. From a practical point of view it would seem that $\text{FPC}(M)$ should be singular and less time consuming. Söderström (1977) once again points out that it is not the case and in view of his work $\text{FPE}(M)$ and $\text{AIC}(M)$ are equivalent for sufficiently large observation samples.
3.6. Structural identification based on the Hankel model

3.6.1. Behaviour of the error function with respect to the number of Markov-parameters in the Hankel model

The error function for the Hankel model of the multivariable dynamical system is defined as:

\[ V_w = \| (\hat{Y} - \overline{Y}) W (\hat{Y} - \overline{Y}) \| \]  \hspace{1cm} (324)

where \( W \) is the gain scaling matrix which is positive definite. Quantities \( \hat{Y} \) and \( \overline{Y} \) are calculated according to the Hankel model (see chapter I).

It can be easily shown that the error function \( V_w(s) \), where "s" is the number of different Markov-parameters encountered in \( \hat{Y}^T = N S_m \), rapidly decreases in value when the model reaches the proper order i.e.

\[ \hat{r} = \text{Entier} \left( \frac{s}{2} + 1 \right) \]  \hspace{1cm} (325)

where \( \hat{r} \) is the estimate of the order \( r \). Then "s" is also the number of different Markov-parameters contained in the \( H_r \) matrix. Behaviour of the error function is not always clear. Sometimes it is difficult, at first sight, to decide for an optimal \( s \). Then the difference ratio

\[ K = \frac{V_w(i+1) - V_w(i + 2)}{V_w(i) - V_w(i + 1)} \]  \hspace{1cm} (326)

for \( i = (s-2),(s-1),s,... \)

gives the clearer picture of changes.

Difficulties will occur in cases with a relatively high noise power and in the case when eigenvalues of the state matrix of the identified system are close to each other. Another possibility, in cases where the order of the system is smaller than 3, a direct decrease rate of the error function can be used.

\[ D = \frac{\Delta V_w(k)}{\Delta k} = V_w(k) - V_w(k + 1) \]  \hspace{1cm} (327)
This order test may be compared with the error function order test for SISO systems as presented by A. J. W. van den Boom, A.W.M. van den Enden (1973), and more details can be found in A. Hajdasinski (1979).

3.6.2. Behaviour of the determinant of the $H_r H_r^T$ matrix

This order test is based on the property that the rank of the Hankel matrix $H_r$ is equal to $n$ if

$$r \geq \frac{n}{\min (p,q)} \Rightarrow \text{rank} \{H_r\} = n.$$  \hspace{1cm} (328)

(which means that when constructing the sequence of matrices $\{H_r\}$ for $r = 1, 2, 3, \ldots$ and checking rank $\{H_r\}$ for $r = 1, 2, \ldots$, the proper $r$ is already found while for

$$r \geq \frac{n}{\min (p,q)} \Rightarrow \text{rank} \{H_r\} = n.$$  \hspace{1cm} (328)

For purely deterministic systems having finite realization, we may find this way a real $n$ and $r$. Increasing the Hankel matrix dimension as long as singularity is not detected, we find the rank $n$, the $r$ can be deduced from structural properties of the $H_r$.

In cases where $q \neq p$, it is necessary either to check all minors in subsequent Hankel matrices, or (which was found to be easier) to check the singularity of the $H_r H_r^T$ matrix.

Talking about the rank of the matrix, we understand this as the "numerical rank" defined as follows:

**Definition 19** Let $N(n)$ be a set of ordinal numbers $\{1, 2, 3, \ldots\}$ and $R(\delta)$ a set of reals. The $(k \times k)$ matrix $H_k$ of the rank $\min (k \times k)$, has a "numerical rank" $(\epsilon, \delta, n)$ with respect to the spectral norm if:

$$n = \inf \{\text{rank} \hat{H}_k : \|H_k - \hat{H}_k\|_2 \leq \epsilon\}, \text{ rank} \hat{H}_k \in N(n)$$

$$\delta = \sup \{\delta : \|H_k - \hat{H}_k\|_2 \leq \delta \Rightarrow \text{rank} \hat{H}_k \geq n\} \delta \in R(\delta)$$  \hspace{1cm} (329)
where the spectral norm of the $H_k$ matrix is:

$$
\| H_k \|_2 = \sup \frac{\| H_k x \| e}{\| x \| e};
$$

$x$ - any $(k \times 1)$ vector $\mathbb{R}^{kp}$

$$
\| \cdot \|_e - \text{Euclidean norm}
$$

The method again incorporates observation of changes in the behaviour of

$\det \{ H_r \}$ or $\{ H_r H_r^T \}$. The rapid decrease in the value of the $\det \{ H_r \}$ or $\det \{ H_r H_r^T \}$ indicates a proper $r$. Then for the chosen $r$, a test for the dimensionality of the state space must be performed.

Usually this order test is a very convincing one, and when applied together with the test for the error function behaviour, it appears to be a very efficient one. In cases where this order test would be confusing, it is again desirable to look for the ratio test:

$$
K = \left| \frac{\det \{ H_{r-1} \}}{\det \{ H_{r+1} \}} \right|; \quad \text{or} \quad K = \left| \frac{\det \{ H_k H_k^T \}}{\det \{ H_{k+1} H_{k+1}^T \}} \right|
$$

For more details see: Hajdasinski (1980), Isidori (1972).

3.6.3 Singular value decomposition of the Hankel matrix

The singular value decomposition of the Hankel matrix is a very efficient order test which can be combined with the new realization algorithm (A. Hajdasinski, A.A.H. Damen (1979)). The whole procedure is based on the following:

Having exact Markov-parameters, it is always possible to find $k > r$ - the system order for which (see relation (150))

$$
H_k = U D V^T \quad \text{D} - \text{is the } n \times n \text{ diagonal matrix.}
$$

(332)
$D = \text{diag. } (\sigma_1, \sigma_2, \ldots, \sigma_n)$

$\sigma_i$ for $i = 1, 2, \ldots, n$ are called singular values.

$U$ - is the $(k \times n)$ matrix consisting of $n$ eigenvectors of the

$H_k^T H_k$ i.e. $H_k^T H_k = U D U^T$

$U^T U = I_n$

$V$ - is the $(k \times n)$ matrix consisting of $n$ eigenvectors of the

$H_k^T H_k$ i.e. $H_k^T H_k = V D V^T$

$V^T V = I_n; \quad \sigma_i = \sqrt{\lambda_i}$, where $\lambda_i$ for $i = 1, 2, \ldots, n$

are eigenvalues of $H_k^T H_k$ or $H_k H_k^T$. In the ideal case, the singular value
decomposition performed on the $H_k$, delivers the "n" - the system dimension equal
to the number of nonzero singular values.

By a simple deduction the $r$ can be found from structural properties of the Hankel
matrix.

In the noisy cases an easy test can be performed to discriminate between which
singular values are substantial and which can be neglected by comparing their
rate of decrease. The decision concerning which singular values can be
neglected will depend on the accuracy we will impose on our model.

Two types of approach can be proposed:

I. The relative error of the least squares fit on the Hankel matrix.

Defining the Euclidean norm of the matrix as:

$$\| H_k \|_e^2 = \text{tr}(H_k^T H_k) = \text{tr}(H_k H_k^T)$$

we can prove the following theorem: (Kam J.J. van der, Damen A.A.H. (1978),
Golub G.H., Reinsch C. (1970)).
Theorem 10: Given a s.v.d. for a \((k_q \times k_p)\) matrix \(H_k\):

\[ H_k = UDVT, \quad D = \text{diag.} (\sigma_1, \sigma_2, \ldots, \sigma_{\min(k_q \times k_p)}) \]

the \((k_q \times k_p)\) matrix \(\hat{H_k}\) of the rank \(q \leq \min(k_q \times k_p)\) and such that \(\| \hat{H_k} - H_k \|_e\), is given the following:

\[
\hat{H_k} = U \begin{bmatrix}
D_{q \times q} & 0 \\
0 & 0
\end{bmatrix} V^T = U \frac{D_{q \times q} V}{\sqrt{q}} T
\]

where

- \(U_{q \times q}\) contains the first \(q\) columns of \(U\)
- \(V_{p \times p}\) contains the first \(p\) columns of \(V\)
- \(D_{q \times q}\) = diag. \((\sigma_1, \sigma_2, \ldots, \sigma_q)\)

Remark: Thus setting the smallest \((\min(k_q, k_p) - q)\) singular values to zero in the s.v.d., through the relation (17), we obtain the best, in the least squares sense, approximation of the \(H_k\) matrix, being of a smaller rank than \(H_k\).

To decide which singular values can be neglected the absolute error criterion

\[
\| \hat{H_k} - H_k \|_e = \sqrt{\sum_{j=q+1}^{\min(k_q, k_p)} \sigma_j^2} < \varepsilon;
\]

\[ 0 < \varepsilon \ll 1 \]

or the relative error criterion

\[
\frac{\| \hat{H_k} - H_k \|_e}{\| H_k \|_e} = \sqrt{\frac{\sum_{i=q+1}^{\min(k_q, k_p)} \sigma_i^2}{\sum_{i=1}^{\min(k_q, k_p)} \sigma_i^2}} \leq \delta
\]

\[ 0 < \delta < 1 \]
may be applied.

II. The numerical rank approach

Let us start with the following Lemma.

**Lemma 1:** Let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(k_q, k_p)} \) be the nonzero singular value of the matrix \( H_k \), then \( H_k \) has a numerical rank \((\varepsilon, \delta, n)_2\) iff:

\[
\sigma_n \geq \delta > \varepsilon \geq \sigma_{n+1}
\]

(337)

The definition of the numerical rank and relations (329), (330) together with Lemma 1 provide another criterion of the absolute error of the approximate realization based on \( H_k \). Assuming the upper bound \(-\delta\) and lower bound \(-\varepsilon\) of the absolute error, through relations (337) and (334), we obtain the least square optimal approximation of the \( H_k \) matrix \( \hat{H}_k \), where

\[
\text{rank } \hat{H}_k = \text{numerical rank } \{H_k\} = (\varepsilon, \delta, n)_2
\]

(338)

Assuming that 2k Markov-parameters are estimated, it is possible to construct the following Hankel and shifted Hankel matrices \( H_k \) and \( \phi H_k \). Performing the s.v.d., there also is found a vector of singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_s \), where \( s = \min(k_q, k_p) \). The relative error of the least squares fit or the numerical rank absolute error (337) decides which singular values may be neglected. In such a way the estimate \( \hat{n} \) of the system dimension is determined.

**Example 7.** Let us compare the last three order tests for the system having the following properties:

\[
\begin{align*}
\chi(k+1) &= \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.8 \end{bmatrix} \chi(k) + \begin{bmatrix} 0.0 & -1.0 \\ 1.0 & 1.0 \\ 1.0 & -1.0 \end{bmatrix} u(k) \\
\chi(k) &= \begin{bmatrix} 1.0 & 0.0 & -1.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \chi(k)
\end{align*}
\]
for this dynamical system \( r = 2, n = 3 \). The output of the dynamical system is affected by the noise which is generated from the white gaussian noise by the filter:

\[
\begin{bmatrix}
0.1 & 0.0 \\
0.0 & 0.7
\end{bmatrix}
\begin{bmatrix}
\xi(k) \\
\xi(k)
\end{bmatrix}
\begin{bmatrix}
1.0 \\
0.0
\end{bmatrix}
\begin{bmatrix}
.0 \\
.0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.0 \\
0.0
\end{bmatrix}
\begin{bmatrix}
\xi(k) \\
\xi(k)
\end{bmatrix}
\begin{bmatrix}
1.0 \\
0.0
\end{bmatrix}
\]

Thus the measured output \( y_m(k) = y(k) + n(k) \).

The intensity of the simulated noise - \( n(k) \) is 10% of the output signal amplitude - \( y(k) \).

Ideal and estimated Markov-parameters are the following:

<table>
<thead>
<tr>
<th></th>
<th>( M_0 )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( M_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal Markov parameters</td>
<td>-1.0 0.0</td>
<td>-0.8 0.6</td>
<td>-0.64 0.6</td>
<td>-0.512 0.5</td>
<td>-0.41 0.4</td>
<td>-0.33 0.32</td>
</tr>
<tr>
<td>Estimated Markov parameters</td>
<td>1.0 1.0</td>
<td>0.4 0.4</td>
<td>0.16 0.16</td>
<td>0.064 0.064</td>
<td>0.026 0.026</td>
<td>0.01 0.01</td>
</tr>
<tr>
<td></td>
<td>0.99 0.0</td>
<td>-0.79 0.59</td>
<td>-0.61 0.57</td>
<td>-0.47 0.47</td>
<td>-0.35 0.38</td>
<td>-0.25 0.28</td>
</tr>
<tr>
<td></td>
<td>1.0 1.0</td>
<td>0.403 0.4</td>
<td>0.162 0.170</td>
<td>0.07 0.07</td>
<td>0.028 0.030</td>
<td>0.015 0.01</td>
</tr>
</tbody>
</table>

and values of the \( V_w \) as the function of Markov param. number - \( s \), are:

<table>
<thead>
<tr>
<th></th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Estim. Mark. par.</td>
<td>0.50606</td>
</tr>
<tr>
<td>Ideal Mark. par.</td>
<td>0.3855</td>
</tr>
</tbody>
</table>
At first sight there seems to be some confusion as to which $s$ should be chosen.

One may choose $s = 2$ or $s = 3$. For example for:

\[ s = 2; \hat{r} = \text{Entier} \left( \frac{s}{2} + 1 \right) = 2 \]

\[ s = 3; \hat{r} = \text{Entier} \left( \frac{s}{2} + 1 \right) = 2 \]

for $\hat{r} = 2$, $\hat{n} = 3$ is sufficient to provide a good approximation of the original system.

Performing the determinant test on the $H_r$ matrix we get:
we have exactly the same answer: $\hat{r} = 2$, $\hat{n} = 3$.

For the s.v.d. order test we choose the $H_n$ matrix and we get the following set of singular values:

- $\sigma_1 = 2.7707$
- $\sigma_2 = 1.7185$
- $\sigma_3 = 3.5826 \times 10^{-1}$
- $\sigma_4 = 3.709 \times 10^{-2}$
- $\sigma_5 = 8.1052 \times 10^{-3}$
- $\sigma_6 = 3.8196 \times 10^{-3}$
- $\sigma_7 = 2.8425 \times 10^{-3}$
- $\sigma_8 = 1.0788 \times 10^{-3}$

It is seen that:

$$\sigma_3 > 0.1 > 0.05 > \sigma_n$$

thus the choice $\hat{n} = 3$ assures the numerical rank of the $H_n$ matrix equal to:

$$(\varepsilon, \delta, \hat{n})_2 = (0.05; 0.1; 3)_2$$

and provides the relative error with respect to the Euclidean norm

$$\frac{\| H_n - \hat{H}_n \|_e}{\| H_n \|_e} = 0.03829$$

The system order $\hat{r} > \hat{n} \min(p,q)$, thus it is the smallest integer fulfilling $\hat{r} > 1.5$. And again, we have as the result of the order test $\hat{r} = 2$, which is sufficient to provide $\hat{n} = 3$.

It is demonstrated then, that all three order tests provide us with the same result $\hat{r} = 2$ and $\hat{n} = 3$.

Conclusions

There exists already a quite extended class of order tests. Certainly this choice of order tests mentioned here is one of many possible choices. However, it shows quite well the mainstream in the methodology of model building. The Guidorzi's order test has been applied for many practical problems providing quite satisfactory results.
However, in many cases it was possible to find lower order approximations of investigated systems. Such a comparison between the Guidorzi's method, Ho-Kalman's algorithm and the partial minimal realization by Tether (Tether A. (1970)) was made by A. Krause (1976), showing that it was always possible, for noisy data, to find a lower order model for Tether's and Ho-Kalman's methods, whereas for Guidorzi's methods a higher order model was found.

Tse-Weinert's methods, being a special case of the determinant test for the Hankel matrix, gives very good results. Examples can be found in an original publication by Tse and Weinert, (1975).

Furuta's approach is rather unnecessarily complicated, but leading to quite good results for low dimensional model.

Akaike's methods have already been discussed. They are very interesting from a theoretical point of view. However, in practical applications demand a large amount of data.

Order tests proposed for the Hankel model are rather simple and efficient. Especially good results are with use of the s.v.d. method. This is, however, the beginning of research in this field and many unexpected events may occur during application of these methods in practice.

4. **Multivariable system identification**

There has already been much said about identification and parameter estimation. However, the general definition of the identification has not yet been given. According to Zadeh (1962), identification is the determination, on the basis of input and output, of a system within a specified class of systems, to which the system under test is equivalent.
This short definition covers a huge range of knowledge about the modelling and model building basing an input/output data. A good survey of problems concerning modelling and identification of single input – single output systems has been written by Fasol K.H., and Jörgel H.P. as the theoretical lecture for the 5th IFAC Symposium on Identification and Parameter Estimation - Darmstadt 1979. For there is no substantial difference in general classification of MIMO and SISO systems it is fair to quote this tutorial lecture as a good reference.

Of our particular interest, there is an experimental analysis of a process, which in general is considered just as the "identification". If the structure of the model is known in advance, or at least can be estimated (see the previous chapter), parametric methods can be used for identification of the mathematical model. In cases where the structure cannot be estimated non-parametric techniques have to be applied.

Among the various goals of identification the below mentioned are most frequently quoted:

- deeper insight into the nature of the process
- prediction of the dynamical behaviour of the process
- verification of theoretical models
- computation (or better estimation) of non-measurable variables
- design of the controller
- optimization of the process.

Most of all, our interest will be attached to discrete time models which will be split into stochastic models and deterministic models. However, the deterministic models, as purely conceptual ones, usually serve as the preliminary information about the process.

A survey of methods for identification of multivariable systems has recently been written by Niederliński A. and Hajdasiński A. (1979). This paper discusses
multivariable systems structures, parametric and non-parametric identification
techniques for multivariable systems.

In this report we will present only the outline of a few of the most important
techniques, which are related to already discussed order tests. Also more
attention will be given to the Markov-parameter approach.

It should also be noted that the parametric identification of canonical models
by Guidorzi and Tse-Weinert has already been presented in chapters 3.1 and 3.2,
(however briefly), the Akaike's information criterion approach has also been
presented in quite some detail. Thus the following algorithms, as in the
illustrative examples, will be discussed further:

- Tether's minimal partial realization algorithm
- Gerth's algorithm
- The Approximate Gauss-Markov Scheme with the s.v.d. Minimal Realization
  algorithm

4.1. The Tether's minimal partial realization algorithm

The problem of minimal partial realization is formulated as follows:

1. Given an infinite sequence of Markov-parameters \( \{M_i\} \), which does not
   have a finite dimensional realization, it is desired to find such a finite
dimensional realization so that its first \( N_0 \) Markov-parameters are
correspondingly equal to the first \( N_0 \) Markov parameters of the first
sequence, or

2. Given a finite sequence of Markov-parameters \( M_0, M_1, \ldots, M_{N_0-1} \), find a
   finite-dimensional realization whose first \( N_0 \) Markov-parameters are
correspondingly equal to the Markov-parameters of the given finite sequence.
The second alternative is much more suited to the purpose of identification and gives a powerful tool for handling the noisy case. It does not, however, incorporate any stochastic considerations, assuming that only some extraction of the Markov-parameters from the noise corrupted systems is prior to the minimal partial realization procedure. The following two definitions are useful:

Definition 20: \( \{A, B, C\} \) is said to be a partial realization of order \( N_0 \) of the sequence \( \{M_i\} \), if and only if \( M_i = C A_i B \) holds for \( i = 0, 1, 2, \ldots, N_0 - 1 \).

Definition 21: \( \{A, B, C\} \) is said to be a partial minimal realization of order \( N_0 \) if and only if the dimension of \( A \) is minimal among all other \( \{A_i, B_i, C_i\} \) satisfying definition 20.

Tether (1970) proved that for every finite series of Markov-parameters \( \{M_i\} \), \( i = 0, 1, 2, \ldots, N_0 - 1 \) there exists an extension sequence \( \{M_{N_0 + 1}, \ldots\} \), for which a completely controllable and observable partial realization exists via the Ho-Kalman's algorithm.

It can also be demonstrated that among all possible partial realizations must exist at least one minimal partial realization which is unique if and only if the extension sequence is unique. Defining

\[
H_{N_0} = \begin{bmatrix}
M_0 & M_1 & \cdots & M_{N_1 - 1} \\
M_1 & M_2 & \cdots & M_{N_1} \\
\vdots & \vdots & \ddots & \vdots \\
M_{N-2} & \cdots & \cdots & M_{N_1 + N - 2}
\end{bmatrix}
\]

(339)
it is possible to prove that given a finite sequence of $r \times m$ matrices $\{M_0, M_1, \ldots, M_{N_0-1}\}$ satisfying

$$\text{rank } H_{N_1, N} = \text{rank } H_{N_1+1, N} = \text{rank } H_{N_1, N+1}$$

(340)

for some $N, N_1$ such that $N_1 + N = N_0$ the extension of the sequence $\{M_0, M_1, \ldots, M_{N_0-1}\}$ to $\{M_0, \ldots, M_{N_0-1}, \ldots, M_{N_0+k-1}\}$ with $0 \leq k < \infty$ for which

$$\text{rank } H(m_1, m) = \text{rank } H_{N_1, N}$$

(341)

where $m_1 + m = N_0 + k$, is unique.

This result proven by Tether (1970) leads to the following realizability criterion:

Let $\{M_0, M_1, \ldots, M_{N_0-1}\}$ be an arbitrary finite sequence of $q \times p$ real matrices.

Let $H_{i,j}$ for $i + j < N_0$ be a corresponding Hankel matrix. Then a minimal partial realization $\{A, B, C\}$ given by the Ho-Kalman algorithm is unique and realizes the sequence up to and including the $N_0$-th term if and only if there exists positive integers $N_1$ and $N$ such that:

1. $N_1 + N = N_0$

(342)

2. $\text{rank } H_{N_1, N} = \text{rank } H_{N_1+1, N} = \text{rank } H_{N_1, N+1}$

(343)

If the realizability criterion is satisfied then the Ho-Kalman algorithm can be applied with

$$n = \text{rank } H_{N_1, N}$$

(344)

using for the Ho-Kalman algorithm $H_{N_1, N}$ and $C H_{N_1, N}^{-1}$.

The resulting minimal partial realization is unique because the extension sequence $M_k = C A^k B$ for $k = N_0, N_0 + 1, \ldots$ generated by the realization satisfies

$$\text{rank } H_{N_1+1, j, N+i} = \text{rank } H_{N_1, N}$$

(345)

for all $i, j \geq 0$. 
If 1 and 2 are not satisfied, then a minimal partial realization - if it exists - may not be unique, for in order to use the Ho-Kalman algorithm new matrices \( \{ \tilde{M}_0, \ldots, \tilde{M}_{P-1} \} \) must be found until

\[
\text{rank} \tilde{H}_{M+1,M} = \text{rank} \tilde{H}_{M+1,M+1} = \text{rank} \tilde{H}_{M+1,M} \tag{346}
\]

where \( M + M = P_0 \). These matrices can be partially or completely arbitrary. Since \((\tilde{A}, \tilde{B}, \tilde{C})\) are functions of the sequence \( \{ \tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_{P-1} \} \) they may also not be unique.

One of the most important results of Tether (1970) is the determination of a lower bound for the minimal partial realization dimension. This is formulated as the following Lemma.

**Lemma 2** Let \((\tilde{A}, \tilde{B}, \tilde{C})\) be a partial realization where first \( N_0 \) Markov-parameters are equal to the given sequence \( \{ \tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_{N_0-1} \} \). Then the dimension of the minimal partial realization satisfies the following inequality:

\[
n \geq n(N_0) = \sum_{j=1}^{N_0} \text{rank} \tilde{H}_{j,N_0+1-j} - \sum_{j=1}^{N_0} \text{rank} \tilde{H}_{j,N_0-j} \tag{347}
\]

This lower bound can be achieved for a "suitably chosen extension". This "suitable choice" is the subject of the fundamental theorem of minimal partial realization given by Tether (1970). In order to state this theorem, it is necessary to formulate one more definition and one more lemma.

**Definition 22**: Let \( N^1(N_0) \) equal the first integer such that every row of the block row \( \left[ \tilde{M}^1_{N_0} \ldots \tilde{M}^1_{N_0-1} \right] \) is linearly dependent on the rows of the Hankel matrix \( \tilde{H}^1_{N_0}, N_0 - N^1(N_0) \), let \( N(N_0) \) equal the first integer such that every column of the block column \( \left[ \tilde{M}^1_{N_0} \ldots \tilde{M}^1_{N_0-1} \right]^T \) is linearly dependent on the columns of the Hankel matrix \( \tilde{H}^1_{N_0-N(N_0), N(N_0)} \).

**Lemma 3** Let \( n(N_0), N^1(N_0) \) and \( N(N_0) \) be integers defined previously. Then any extension \( \{ \tilde{M}_{N_0}, \tilde{M}_{N_0+1} \ldots \} \) of \( \{ \tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_{N_0-1} \} \) whose realization
achieves the minimal lower bound \( n(\text{No}) \) for its dimension also satisfies

\[
\text{rank } H_{N(\text{No}), N(\text{No})} = \text{rank } H_{N(\text{No})+1, N(\text{No})} = \text{rank } H_{N(\text{No})+1, N(\text{No})+1}
\]  
(348)

for that extension.

From this Lemma follows the Minimal Partial Realization Theorem, which is central in Tether's algorithm:

**Theorem II**  
Let \( \{M_0, M_1, \ldots, M_{N_{\text{No}}-1}\} \) be a fixed partial sequence of \( q \times p \) matrices with real coefficients and let \( n(\text{No}), N(\text{No}) \) and \( N^1(\text{No}) \) be integers defined previously. Then:

1. \( n(\text{No}) \) is the dimension of the minimal partial realization
2. \( N(\text{No}) \) and \( N^1(\text{No}) \) are the smallest integers such that (348) holds simultaneously for all minimal extensions
3. There is a minimal extension of order \( P(\text{No}) = N(\text{No}) + N^1(\text{No}) \) for which \( n(\text{No}) \) is the dimension of the realization computed by the Ho-Kalman algorithm, but which in general is not unique.
4. Every extension which is fixed up to \( P(\text{No}) \) is uniquely determined thereafter.

It must be stressed that Tether's algorithm does not explicitly recognize the stochastic nature of experimentally determined Markov-parameters. Nevertheless it proves useful for real-life noisy and incomplete data because it can generate an approximate model which agrees with the data. A number of successful applications of this algorithm have been described (see Rossen-Lapidus (1972), Hajdu (1969), Roman (1975)).

4.2. **Gerth's algorithm**

Tether's algorithm was assuming the availability of Markov-parameters. However, they must be estimated. Therefore, an asymptotically unbiased and efficient estimation of Markov-parameters is an essential step for the successful identification of multivariable systems via the realization theory.
Gerth (1971, 1972) proposed a multistage procedure for the initial estimation of Markov-parameters and further refinement of the finite series \( \{ \mathbf{M}_i \} \) in a way that assures the linear dependence of the extension series. It works on a noise-corrupted data set of input-output pairs, assumes a preselected degree \( r \) of the minimal polynomial and applies the Ho-Kalman algorithm to determine a realization.

Gerth's algorithm consists of two separate steps:

1. Estimation of the finite series of Markov-parameters \( \{ \mathbf{N}_i \} \) for \( i = 0, 1, 2, \ldots, k \)
2. Determination of the minimal polynomial coefficients \( a_i \) and matrices \( \mathbf{M}_i \) for \( i = 0, 1, \ldots, r - 1 \), which are as "close as possible" to \( \mathbf{N}_i \), for the same \( i \) and specify an infinite realizable sequence via the relation

\[
\mathbf{N}_j = - \sum_{i=1}^{r} a_i(j) \mathbf{M}_{j-i}
\]  

(349)

with \( a_i(j) = -a_{r-i} \) being the coefficients of the minimal polynomial with \( a_r = 1 \).

The Gerth's algorithm assumes \( r \) - the order equal \( n \) - the dimension of the realization. Assuming further that:

1. initial conditions are zero
2. noises corrupting the signals are white and zero mean
3. number of samples is sufficiently large

and using the Hankel model and the least-squares method, the algorithm results in the estimate of \( (k + 1) \) Markov-parameters:

\[
\mathbf{N} = \left( \mathbf{S} \mathbf{S}^T \right)^{-1} \mathbf{S} \mathbf{Y}
\]

(350)
where $N^T = [N_0, N_1, \ldots, N_k]$ and matrices $S_m$ and $Y$ are defined by

$$Y^T = \begin{bmatrix} y(\ell), y(\ell + 1), \ldots, y(\ell + m) \end{bmatrix}$$

$$Y^T(i) = \begin{bmatrix} y_1(i), y_1(i) \ldots, y_q(i) \end{bmatrix}$$

$$S_m = \begin{bmatrix} u(\ell - 1) \ldots u(\ell + m - 1) \\ \vdots \\ \vdots \\ u(\ell - k - 1) \ldots u(\ell + m - k - 1) \end{bmatrix}$$

$$U^T(i) = \begin{bmatrix} u_1(i), u_2(i) \ldots, u_p(i) \end{bmatrix}$$

Using an input excitation $\{u_1\} = \{1,0,0 \ldots\}$ for each of the $p \times q$ partial systems and using the Hankel model (8), the excitation matrices for these partial systems will be

$$T_{uij\ell} = w_{ji} \begin{bmatrix} u(-1)_{j\ell} & u(0)_{j\ell} & \ldots & u(k-1)_{j\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & \ddots & u(-1)_{j\ell} \end{bmatrix}$$

for $j = 1, 2, \ldots, q$, $\ell = 1, \ldots, p$, where $w_{j\ell}$ is the individual weighting factor for every partial system.

For every partial system the following Hankel matrix is constructed from the elements of $\{N_i\}$ for $i = 0, 1, \ldots, k$.

$$H_{ij} = \begin{bmatrix} N_{0,j\ell} & N_{v_{j\ell}} & \ldots & N_{k-r,j\ell} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ N_{r-1,j\ell} & \ldots & \ldots & N_{k-1,j\ell} \end{bmatrix}$$
where \( N_{i,j} \) is the \( j \)-th element of \( N_i \). The vector containing elements of \( \{N_i\} \) for \( i = r, r+1, \ldots k \) is defined as:

\[
\begin{align*}
\nu_{j,k} &= N_r, j \cdot N_{r+1, j} \cdot \ldots \cdot N_{k, j} \\
\end{align*}
\]

(355)

The estimation of the minimal polynomial coefficients is also performed using the least-squares method and the following loss function.}

\[
\mathcal{J} = (h_r^T h - \nu^T) T_T T (h_r^T h - \nu^T)^T
\]

obtained from (349), with the coefficients vector

\[
h_r^T = [-a_0, -a_1, \ldots, -a_{r-1}]
\]

(357)

and

\[
H = \begin{bmatrix}
H_{11}, H_{12}, \ldots, H_{1p}, H_{21}, H_{22}, \ldots, H_{q1}, \ldots, H_{qT}
\end{bmatrix}
\]

(358)

\[
\nu^T = \begin{bmatrix}
\nu_{11}^T, \nu_{12}^T, \ldots, \nu_{qT}^T
\end{bmatrix}
\]

(359)

\[
T_u = \text{diag}[T_{u11}, T_{u12}, \ldots, T_{uqp}]
\]

(360)

The minimization of \( \mathcal{J} \) gives:

\[
h_r = (H T_u T_u^T h_r^T)^{-1} H T_u T_u^T \nu
\]

(361)

It can be proven that the least-squares estimation of the minimal polynomial coefficients for a given degree \( r \) is unique if and only if rank \( H = r \). Next the finite sequence \( \{M_i\} \), for \( i = 0, 1, \ldots, r-1 \), must be determined using again least-squares estimation for computing.

\[
\min_{\nu_{j,k}} \sum_{i=0}^{k} \left[ y(i)_{j,k}(h_r, \nu) - y(i)_{j,k}(\nu) \right]^2
\]

(362)
for \( j = 1, 2, \ldots, p, \ell = 1, 2, \ldots, q \), where \( y(i)_{j\ell}(h, M) \) is the output sequence generated for the \( j\ell \)-th partial system by \( T_{uj\ell}^{-1} \) for the already determined \( h\ell \) but unknown \( \{M_i\} \), and \( y(i)_{j\ell}(N) \) is the output sequence generated for the same partial system and the same excitation, but for known \( \{N_i\} \). The solution to (362) is given by:

\[
m_{j\ell} = (G T_{u}^{-1} T_{u}^{-1} G^T)^{-1} G T_{u}^{-1} T_{u}^{-1} n_{j\ell}
\]

where

\[
m_{j\ell} = \begin{bmatrix} M_{0,j\ell}, M_{1,j\ell}, \ldots, M_{r-1,j\ell} \end{bmatrix}
\]

\[
n_{j\ell} = \begin{bmatrix} N_{0,j\ell}, N_{1,j\ell}, \ldots, N_{k,j\ell} \end{bmatrix}
\]

\[
G = \begin{bmatrix} I_r^T, R e_r, R^2 e_r, \ldots, R^{k-r+1} e_r \end{bmatrix}
\]

with

\[
E = \begin{bmatrix} \frac{R}{I_r-1} & R \end{bmatrix}
\]

\[
e^T_r = \begin{bmatrix} 0 \ldots 0 \ldots 1 \ldots 0 \end{bmatrix}
\]

\( r \)-th position

If all \( T_{uj\ell} \) are nonsingular, the sequence \( \{M_i\}, i = 0, 1, \ldots, r-1 \), with a given vector \( h\ell \) is always unique and can be used to determine the minimal realization using the Ho-Kalman's algorithm.

Extensions of Gerth's idea can be found in papers by Hajdasiński (1976, 1978).

4.3. The approximate Gauss-Markov scheme with the singular value decomposition minimal realization algorithm

Using the Hankel model (8) and relations (351) (352), defining the noise vector

\[
E^T = \begin{bmatrix} e(1), e(\ell + 1), \ldots, e(\ell + m) \end{bmatrix}
\]

\[
e^T(i) = \begin{bmatrix} e_1(i), e_2(i), \ldots, e_q(i) \end{bmatrix}
\]
and

\[ M^T = \begin{bmatrix} M_0, M_1, \ldots, M_k \end{bmatrix} \]  \hspace{1cm} (371)

the block vector containing estimates of the first \( k + 1 \) Markov-parameters

\[ M^T_\infty = \begin{bmatrix} M_{k+1}, M_{k+2}, \ldots, M_{k+m} \end{bmatrix} \]  \hspace{1cm} (372)

the block vector containing the remaining Markov-parameters and the matrix of initial conditions \( S_\infty \), the Hankel

\[ S_\infty = \begin{bmatrix} u(-k-2) & \ldots & u(-k+m-2) \\ u(-k-3) & \ldots & u(-k+m-3) \\ \vdots & & \vdots \\ u(-1) & & 0 \\ 0 & & \ddots \\ \vdots & & \vdots \\ 0 & & \ddots & \ddots & \ddots \\ & & & & u(-1) \end{bmatrix} \]  \hspace{1cm} (373)

model, for the noisy system will be following:

\[ Y = S_m^T N + S_\infty^T M_\infty + E \]  \hspace{1cm} (374)

Assuming that input samples \( \{u_i\} \) and the noise samples \( \{e_i\} \) are mutually uncorrelated stationary processes and that \( E\{e(i)\} = 0 \), an estimate of the first \( k + 1 \) Markov-parameters can be found minimizing the error function \( V_w \) (see Hajdasiński (1976)(1978)(1979), Niederliński-Hajdasiński (1978)).

\[ V_w = \| (Y - \hat{Y})^T W (Y - \hat{Y}) \| \]  \hspace{1cm} (375)

where

\[ \hat{Y} = S_m^T N \]  \hspace{1cm} (376)

is the estimate of the \( Y \). Thus applying the well known formalism, an expression for a minimal trace of \( V_w \) is found:

\[ \frac{\partial \text{tr} V_w}{\partial N} = 0; \]

\[ N = (S_m W S_m^T)^{-1} S_m W Y - (S_m W S_m^T)^{-1} S_m W S_\infty^T M_\infty \]  \hspace{1cm} (377)
Expressing $N$ in terms of $E$

$$N = M + (S_m W S_m^T)^{-1} S_m W E$$  \hspace{1cm} (378)$$

it is seen that the term $(S_m W S_m^T)^{-1} S_m W E$ is the bias of the estimate (377). This expression depends on properties of the noise $E$ and asymptotically vanishes when $E\{e(i)\} = 0$ and there is no correlation between samples of $E$ and $S_m$.

Also for the zero initial conditions the second term in the expression (377) vanishes. In case $E\{E\} \neq 0$, for this type of model i.e. (374), it is very easy to estimate the bias of the estimate - see Hajdasiński (1978).

Considering the following expression:

$$E\{(M - N)(M - N)^T\}_W$$  \hspace{1cm} (379)$$
as an accuracy criterion, it follows that:

$$E\{(M - N)(M - N)^T\}_W = E\{(S_m W S_m^T)^{-1} S_m W E E^T W S_m^T (S_m W S_m^T)^{-1}\}_W$$  \hspace{1cm} (380)$$

Substituting $E\{E E^T\} = R$ and $W = R^{-1}$ we get:

$$E\{(M - N)(M - N)^T\}_R = (S_m R^{-1} S_m^T)^{-1}$$  \hspace{1cm} (381)$$
see also Hajdasiński (1976, 1978, 1979)

Thus choosing $W = R^{-1}$, this results in an efficient estimate of the Markov-parameters, since it is easy to show that:

$$E\{(M - N)(M - N)^T\}_W \gg E\{(M - N)(M - N)^T\}_R$$  \hspace{1cm} (382)$$

This way of weighting resembles very much the classical Gauss-Markov estimation for SISO systems.
Defining
\[ n^T = \begin{bmatrix} n_1^T, n_2^T, \ldots, n_q^T \end{bmatrix} \] (383)
\[ n_i^T = \begin{bmatrix} e_i(k), e_i(k+1), \ldots, e_i(k+m) \end{bmatrix} \] (384)
the noise covariance matrix will take the form:
\[ Q = E(n n^T) \] (385)

It is seen that the trace of \( Q \) is the following sum of squares:
\[ \text{tr} Q = E\left\{ \sum_{i=1}^{q} \sum_{j=1}^{m+1} e_i(j) \right\} \] (386)

Writing the \( R \) matrix in an explicit form it can be seen that the trace of \( Q \) is equal to the trace of \( R \). (Hajdański (1978)).
\[ \text{tr} Q = \text{tr} R \] (387)

Treating the \( R \) matrix as the covariance matrix of a hypothetical single-input single-output noise filter it is possible to reconstruct this noise and its covariance matrix. This "composite" noise model is but a mathematical fiction, having no strictly physical interpretation. According to (386) and (387), minimization of the (379), which appears in the minimal \( \text{tr} R \), results in the minimal trace of the noise covariance matrix, attaining in this way the main goal of an efficient estimation.

If all noises corrupting a multivariable system are stationary, the \( R \) matrix has also a very simple structure:
\[ R = \begin{bmatrix} s^2 & s^2 & s^2 & \ldots & s_{m-1}^2 & s_m^2 \\ s_1^2 & s_2^2 & \ldots & s_{m-2}^2 & s_{m-1}^2 & \ldots \\ s_2^2 & \ldots & \ldots & s_{m-3}^2 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ldots \\ s_m^2 & s_{m-1}^2 & \ldots & \ldots & \ldots & s^2 \end{bmatrix} \] (388)
In practice the $\mathbf{R}$ matrix must be estimated using a finite number of samples, taking as $s^2$ and $s^2_m$:

$$s^2 \approx \frac{1}{m+1} \sum_{j=1}^{m+1} \sum_{i=1}^q \hat{e}_i^2(j), \quad (389)$$

$$s^2_m \approx \frac{1}{m-k+1} \sum_{j=1}^{m-k+1} \sum_{i=1}^q \hat{e}_i(j) \hat{e}_i(j+k), \quad (390)$$

where $\hat{e}_i(.)$ are standing for estimates of $e_i(.)$.

Thus only the first part of the $\hat{\mathbf{R}}$ matrix, being the estimate of $\mathbf{R}$, can be computed with a sufficient accuracy, for in an explicit method there are only $m+1$ input-output and residual error samples available. Higher ordered elements in $\hat{\mathbf{R}}$ will be increasingly less accurate. As it is demonstrated by Hajdasiński (1978), the quality of the $\hat{\mathbf{R}}$ matrix estimation plays a key role in the convergence of Markov-parameters estimates. The realization theory once again helps to get improved results, providing means to reconstruct the composite noise covariance matrix. For a detailed derivation of this result the reader is referred to Hajdasiński (1978).

Treating $\mathbf{E}$ as the set of specially arranged samples of a one-dimensional noise with the covariance matrix $\mathbf{R}$, it can be assumed that this noise is generated by a colouring filter from a hypothetical white noise. In such a case an estimate $\hat{\mathbf{R}}$ of $\mathbf{R}$ is given by the following expression:
where \( \hat{R} \) is the estimate of the multivariable noise, calculated using the residual error of the L.S. estimation of Markov-parameters.

Assuming that the noise \( e(k) \) is generated from \( \xi(k) \) (white noise) by a moving-average filter,

\[
e(k) = (C_0 + C(z^{-1}) \xi(k)
\]

a very interesting decomposition of the \( \hat{R} \) matrix can be found. From (391) it follows

\[
e = C \xi
\]

with

\[
e^T = \begin{bmatrix}
     e^T(k) & e^T(k + 1) & \cdots & e^T(k + n)
   \end{bmatrix}
\]

\[
\xi = \begin{bmatrix}
     \xi^T(k-v) & \xi^T(k-v+1) & \cdots & \xi^T(k + n)
   \end{bmatrix}
\]

\[
C = \begin{bmatrix}
     C_{v-1} & C_{v-2} & \cdots & C_0 & 0 & \cdots & 0 \\
     0 & C_{v-1} & \cdots & C_1 & C_0 & \cdots & 0 \\
     0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
   \end{bmatrix}
\]

Since \( E(e \ e^T) = E(\hat{E} \ e^T) = C \ C^T \sigma^2 \)

for \( E(\xi \ \xi^T) = \sigma^2 I \)

Then with accuracy to a constant factor the composite noise covariance matrix is

\[
\hat{R} = \alpha C \ C^T
\]
However, for the Gauss-Markov estimation it is only necessary that the weighting matrix $\mathbf{W}$ is similar to the covariance matrix $\mathbf{R}$.

$$\mathbf{W} = \mathbf{T}^{-1} \mathbf{R} \mathbf{T}$$  \hspace{1cm} (400)

Knowing a finite number of initial elements in the $\hat{\mathbf{R}}$ matrix and assuming they are sufficiently exact, it will be possible to find, via the realization theory, the remaining elements and reconstruct the full rank covariance matrix $\mathbf{R}$.


Having identified Markov-parameters $\{\mathbf{M}_i\}$, the minimal realization algorithm with the s.v.d. of the Hankel matrix was proposed by Hajdasiński and Damen (1979). Application of the last is straightforward using results derived in the chapter 3.4.2. - (150) - (158). For the numerical example see Example 6.
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