The calculation of geometrical constants for irregular cross-sections of rods and beams
Menken, C.M.; Pasch, van de, J.A.M.

Published: 01/01/1986

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
THE CALCULATION OF GEOMETRICAL CONSTANTS FOR IRREGULAR CROSS-SECTIONS OF RODS AND BEAMS.

C.M. Menken and J.A.M. van de Pasch

Eindhoven University of Technology, The Netherlands.

ABSTRACT

Extruded rods and beams may have cross-sections with very complicated shapes; as a consequence, cross-sectional properties relating to stiffness, strength and stability cannot be calculated analytically, nor can they be obtained from a handbook or the manufacturer. For research purposes a special program was written in order to calculate geometric properties such as the torsion constant, the warping constant and the shear center location; it made use of an advanced mesh generator and was run on a large computer.

The need of a medium sized company, that only possessed an 8-bit microcomputer led to a simplification of the program. An investigation of the mechanical (torsional) behaviour of the members led to the choice of an element and a strategy for mesh-division such that calculation of the geometrical constants could be done simply on the microcomputer. The calculations could be confirmed by comparison with more elaborate calculations.

INTRODUCTION

Extruded beams and rods made of aluminium, for example, may have cross-sections with very complicated shapes (Fig. 1.a), due

Fig. 1. Example of a complicated cross-section and finite element mesh generated by TRIQUAMESH.
to the fact that a designer may take the liberty of adapting their shapes to suit his requirements. He may adapt the shape to suit the specific functions of a member and he is not constrained by the limited number of shapes for hot-rolled steel members. Cross-sections made by extrusion may comprise thin-walled parts, or solid like parts such as stiffeners shaped like a bulb, lip or bead; as well as, parts with intermediate ratios of their dimensions. As a consequence, cross-sectional properties relating to stiffness, strength and stability cannot be calculated analytically, nor can they be obtained from a handbook, and as far as we know, are not provided by the manufacturer. Especially, torsion-related properties such as the torsion constant, the warping constant and the location of the shear center may pose problems. For their determination, the so-called torsion function, $\psi$, must be known which is itself determined by the plane Laplace’s equation

$$\Delta \psi = 0$$  \hspace{1cm} (1)

The numerical solution of this equation is a beloved topic of textbooks on finite element methods, e.g. 1 and 2. Once the distribution of the torsion function, $\psi(y,z)$, is known, the aforementioned geometric constants can be determined in a straightforward manner. Little information is available however, in literature about accuracy and the proper use of elements, notwithstanding the widespread use of extruded members with complicated cross-sections. Finite element calculation of the torsion constant and shear-stress distribution was described by Herrmann already in 1965 3, followed by a paper by Mason and Herrmann on the determination of the shear center and shear deformation coefficient related to bending. In these papers, three-node linear triangular elements were used. As a consequence, the examples presented by Mason and Herrmann to demonstrate the versatility of their method, contained a large number of elements.

Surana presented a higher order isoparametric finite element formulation, thus combining a better simulation of curved parts, with reducing the number of elements. His derivations were based on torsion and flexure due to end shears, and the warping constant and mono-symmetry parameter needed for lateral-torsional buckling problems were absent.

As we were primarily interested in calculating lateral-torsional buckling loads of extruded members accurately, we developed some computer programs that can calculate all the relevant geometrical properties. The interest of a medium-sized industry, that possessed only a microcomputer, led us to make simplifications. Especially, to give an insight into the underlying mechanics; this enabled us to calculate all the relevant geometrical properties with a small number of elements. Notwithstanding the fact that calculating these properties poses no fundamental pro-
blems once the torsion function is known, we felt that it was appropriate to publish our results, since we believe that neither designers nor codes take sufficient advantage of the possibilities of modern computing techniques, which are now accessible to owners of microcomputers.

RELEVANT DEFINITIONS AND EXPRESSIONS.

Consider a homogeneous slender prismatic beam of length $l$. Choose a Cartesian coordinate system such that the $x$-axis coincides with the centroids of the cross-sections, whilst the $y$ and $z$ axes coincide with the principal axes of the cross-section. If this beam is loaded in such a way that at lower loads bending only occurs in the $x, z$-plane, which means that the workline of the load must pass through the so-called shear-center. Then, when increasing the load, so-called lateral-torsional buckling may occur at some critical load. We confined ourselves to the case where no distortion of the cross-section occurred; thus, we excluded local or interactive buckling. The quadratic expression for the additional potential energy when the beam goes from the unbuckled to the buckled state, by means of a lateral displacement $v_0(x)$ and a rotation $\alpha(x)$ is:

$$
P_2[u] = \int_0^1 \left[ \frac{1}{2} EI \alpha''^2 + \frac{1}{2} GI_t \alpha'^2 + \frac{1}{2} EI \alpha''^2 \right] - (Ma)'v_0' + I_c(Ma)'\alpha' \, dx.
$$

The critical load can be obtained by making the first variation of this expression zero. From this expression (2), and from the preliminaries, we know which geometrical properties are needed to determine the critical load:
- the centroid of the cross-section
- the orientation of the principal axis
- the principal moments of inertia:

$$
I_y = \int z^2 \, dA \quad \text{and} \quad I_z = \int y^2 \, dA \quad (3a,b)
$$

- Saint Venants torsional constant:

$$
I_t = \int_{A} \left( (\psi, y - z)^2 + (\psi, z + y)^2 \right) \, dA \quad (4)
$$
- the coordinates $y_0$ and $z_0$ of the center of shear with respect to the principal axis:

$$y_0 = -\frac{1}{I_y} \int y \psi \, dA$$  \hspace{1cm} (5a)$$

$$z_0 = \frac{1}{I_z} \int z \psi \, dA$$  \hspace{1cm} (5b)$$

- the warping constant:

$$\gamma = -y_0^2 I_y - z_0^2 I_z + \int_A \psi^2 \, dA$$  \hspace{1cm} (6)$$

This constant may be particularly important for open thin walled parts, even if warping (i.e. a distribution of axial displacements) of the ends is free to occur.

- the mono-symmetry integral:

$$I_c = z_0 - \frac{1}{2I_y} \int z(y^2+z^2) \, dA$$  \hspace{1cm} (7)$$

In many practical situations, its influence is omitted; however, for accurate calculations, it may be important.

The torsion function $\psi(y,z)$, occurring in these expressions can be obtained by solving the following Laplace's equation:

$$\psi_{yy} + \psi_{zz} = 0 \quad \text{in } A$$  \hspace{1cm} (8)$$

with the Neumann type boundary condition:

$$\psi_{y y} n_y + \psi_{z z} n_z = z n_y - y n_z \quad \text{on } S,$$  \hspace{1cm} (9)$$

where: $S$ is the boundary of $A$. For a unique solution, it is necessary that:

$$\int \psi \, dA = 0$$  \hspace{1cm} (10)$$
This implies that we must make $\psi=0$ on a line of symmetry of the cross-section. The solution of this problem when the cross-section has a complicated shape, can only be done numerically. The numerical calculation is based on the stationary requirement of functional (4) (divided by two), in discretised form. If we require this functional to be stationary with respect to all admissible $\psi$-fields, equation (8) and boundary condition (9) will be obtained. If we use the functional for finite element calculations, the stiffness matrix will be obtained from the discretisation of expression

$$
\frac{1}{2} \int_{A} (\psi_y^2 + \psi_z^2) \, dA, \quad (11)
$$

whereas, the righthand members are determined by discretising the expression

$$
\int_{A} (z\psi_y - y\psi_z) \, dA. \quad (12)
$$

STRATEGIES FOR REDUCING THE NUMBER OF ELEMENTS, AND NUMERICAL ILLUSTRATIONS.

Two finite element programs have been written consecutively. The first one used numerically integrated four-node isoparametric elements for solving the discrete values of the torsion function $\psi(y,z)$ and calculating the desired geometrical properties. We had the opportunity to use this program with the mesh generator TRIQUAMESH$^7$ (6000 sentences ALGOL, 200 $\&$ 300 kbytes) and it was run on a large computer (BURROUGHS B7700). Fig. 1.b shows a mesh division made by TRIQUAMESH. Amongst the calculated geometric properties of such cross-sections, the torsion constant proved to be rather sensitive to the number of elements and the mesh division. As a numerical example of the use of this program, we show in the first column of Table I some results for a thin-walled T-profile with stiffeners along the edge of the flange (Fig. 2).

The second example concerns a thin walled open beam with circular cross-section. The dimensions are given in Fig. 3. The results are presented in Table II. In the second example, making use of the symmetry of the structure can even halve the number of elements and degrees of freedom.

The second program was a modification of the aforementioned one, and was intended for use on a micro-computer. In this case, the mesh-generator cannot be used. In order to make calculations with a reduced number of elements we utilized the available
knowledge about the torsional behaviour of thin-walled parts. Classical literature, e.g., however, considers the stress-function approach of Prandtl when dealing with thin-walled sections. Less attention has been given to the warping-(or torsion) function approach of Saint Venant, although the latter approach is directly applicable to the finite element displacement method. An advantage of the approach of Prandtl is that it contains the so-called stress-function which is analogous with the obvious behaviour of a deflected membrane.

Fig. 2. Element meshes for test problem.

Table I.

<table>
<thead>
<tr>
<th></th>
<th>120 four-node elements</th>
<th>9 eight-node elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-coordinate of shear center</td>
<td>31,53 mm</td>
<td>31,54 mm</td>
</tr>
<tr>
<td>Torsion constant $I_t$</td>
<td>957 mm$^4$</td>
<td>963 mm$^4$</td>
</tr>
<tr>
<td>Warping constant $\Gamma$</td>
<td>162900 mm$^6$</td>
<td>162770 mm$^6$</td>
</tr>
</tbody>
</table>

This analogy is very helpful when constructing approximate solutions for the stress-function for thin-walled sections. However, we can utilize this powerful tool in the displacement approach too and supplement it with numerical calculations.

If we consider a homogeneous membrane supported at its edges, with the same outline as that of the cross-section, and subjected to a uniform tension at the edges and a uniform lateral pressure; the shape of this deflected membrane will be analogous to the stress function, whereas the slope of the membrane will be proportional to the shearing stress. With this analogy, we can see that, in the case of a narrow cross-section, the shape of the membrane is for the greater part independent of the longitudinal direction, although it has local deviations at the
shorter ends. We made some analogous finite element
calculations for the torsion function and plotted the
contribution of the torsion function to the torsion constant as
a function of place. Fig. 4 shows some examples, confirming the
qualitative picture of the membrane: a distribution, homogeneous
for the greater part of the longitudinal direction, but with
local deviations near the shorter ends, at corners and at
junctions. These disturbances of the homogeneous picture are concentrated
in an area of about half or the same as the local thickness of
the material.

\[ \begin{align*}
\text{69 four-node elements} & \quad 158 \text{ degrees of freedom} \\
R_l = 100 \text{ mm.} \\
\text{12 eight-node elements} & \quad 68 \text{ degrees of freedom} \\
R_u = 110 \text{ mm.}
\end{align*} \]

Fig. 3. Element meshes for test problems.

Table II.

<table>
<thead>
<tr>
<th></th>
<th>69 four-node elements</th>
<th>12 eight-node elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)-coordinate of the shear center</td>
<td>-209.4 mm</td>
<td>-210 mm</td>
</tr>
<tr>
<td>Torsion constant (I_t)</td>
<td>(0.218 \times 10^6 \text{ mm}^4)</td>
<td>(0.220 \times 10^6 \text{ mm}^6)</td>
</tr>
<tr>
<td>Warping constant (\Gamma)</td>
<td>(0.108 \times 10^{13} \text{ mm}^6)</td>
<td>(0.103 \times 10^{13} \text{ mm}^6)</td>
</tr>
</tbody>
</table>

Another factor is that, for thin-walled sections, the warping
is dominated by the warping of the center line of the material.
A slight, linear variation across the thickness of the wall
forms a good approximation for warping of material points out-
side the center line.

This led us to the decision to use eight-node isoparametric
elements, because these complied with the aforementioned warping
description for thin-walled parts and, at the same time, simu-
late complicated solid parts. Now, thin-walled parts can be si-
mulated with a small number of elements: Since a bi-linear form
is a good approximation for the warping of a straight part, and
the bilinear form is included in the eight-node element, only
one element is needed for the greater part of a straight component. At junctions, some smaller elements are needed for continuity, whereas at free ends, only one small element can be used to meet the boundary condition. For these smaller elements, a dimension of the order of the wall-thickness proved satisfactory. Figure 2b shows such a simple element mesh, whilst, the relevant results are given in table I.

![Element Meshes](image)

**Fig. 4.** Contours of the local contribution to the torsion constant.

Whereas the greater part of a thin-walled straight component can be simulated with just one element, we found that for curved parts, more elements were needed. Numerical and analytical calculations showed that this could not be attributed to the following sources:
- inadequate description of the shape of the cross-section by means of the isoparametric element.
- inaccurate description of the thickness of the curved thin-walled part (the torsional constant is proportional to the cube of the thickness).
- inaccurate numerical integration of the stiffness matrix and/or righthand member, the integrants being complicated by the isoparametric mapping.

The main reason, however, proved to be that, for a curved part, the description of the torsion function, $\psi(y,z)$, is more approximate, especially, in the circumferential direction. Moreover, curvature induces a coupling between the approximate torsion function distribution in the radial direction and in the circumferential direction (Appendix A).

With circular elements, including an angle of 30 degrees satisfactory results were obtained. Fig. 3b shows the mesh for the thin-walled open beam with a circular cross-section; whereas, Table II presents the pertinent results.

![Circular Element Mesh](image)

**Fig. 5.**
Making use of the aforementioned programs and strategies, geometrical constants for irregularly shaped cross-sections, as shown in Fig. 1, have been calculated successfully. Fig. 5 gives an example of a simple element subdivision for a complicated cross-section.

ACKNOWLEDGEMENT

The author gratefully acknowledges the contribution of Mr. W.J. Groot to this paper.

APPENDIX A

Curved versus straight thin-walled cross-sections.

In order to explain why a curved thin-walled component requires more elements than a straight one, we will consider a straight cross-section and a curved cross-section with a constant radius of curvature $R$:

For a simple comparison, we will give both cross-sections the same dimensions. We have already seen that the straight cross-section has two axes of symmetry, whereas, the curved cross-section has only one. We will introduce the dimensionless (curvilinear) coordinates $\xi$ and $\eta$: $\xi = 2y/t$ resp. $2(\varphi-R)/t$, and $\eta = z/\alpha R$ resp. $\varphi/\alpha$.

Now, we consider a polynomial for the torsion function, $\psi(\xi, \eta)$, in such a way that it contains the quadratic terms, used when formulating an 8-node element. Moreover, we will add one additional higher-degree term. Since it was observed that more than one element was needed in the circumferential direction and for
thin-walled open cross-sections, the amount of warping away from the center line is negligible with respect to the warping along the center line, we choose $\eta^3$ as an additional term. The polynomial then becomes:

$$\psi(\xi, \eta) = a_{00} + a_{10} \xi + a_{01} \eta + a_{20} \xi^2 + a_{11} \xi \eta + a_{02} \eta^2 + a_{21} \xi^2 \eta + a_{12} \xi \eta^2 + a_{03} \eta^3$$

Now if we utilize the antimetric behaviour of the torsion function, this means for the narrow rectangle that:

$$\psi(-\xi, \eta) = -\psi(\xi, \eta) \quad \text{and} \quad \psi(\xi, -\eta) = -\psi(\xi, \eta)$$

The remaining alternative for the torsion function becomes:

$$\psi(\xi, \eta) = a_{11} \xi \eta$$

This behaviour can be described exactly by one quadratic element.

The curved cross-section, however, has only one axis of anti-

metry:

$$\psi(\xi, -\eta) = -\psi(\xi, \eta)$$

Now the allowable polynomial is:

$$\psi(\xi, \eta) = a_{01} \eta + a_{11} \xi \eta + a_{21} \xi^2 \eta + a_{03} \eta^3 + \ldots$$

After introducing this polynomial into the functional (4) and making the first variation zero, all constants $a_{01}$, $a_{11}$, $a_{21}$ and $a_{03}$, in general will appear to be nonzero.

Thus, the warping distribution will be more complicated and can, due to the presence of the cubic term, no longer be described exactly by means of only one quadratic element, but can only be approximated by means of a number of such elements.

This invokes a typical problem for torsion: in the case of thin-walled open cross-sections, the torsion constant, $I_t$, be-

comes very sensitive to the number of elements. This is due to the fact that one and the same functional is used for calculating the torsion constant for both solid-, thin-walled closed and thin-walled open cross-sections:
\[ I_t = \int_A \left( \psi^2 + \psi^2 + 2z \psi + 2y \psi + y^2 + z^2 \right) dA \]

<table>
<thead>
<tr>
<th>Cross-section:</th>
<th>( I_t )</th>
<th>( \int (y^2 + z^2) dA )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid</td>
<td>( R^4 )</td>
<td>( R^4 )</td>
</tr>
<tr>
<td>Thin-walled closed</td>
<td>( R^3_t )</td>
<td>( R^3_t )</td>
</tr>
<tr>
<td>Thin-walled open</td>
<td>( R^3 )</td>
<td>( R^3_t )</td>
</tr>
</tbody>
</table>

Whereas, in the case of a thin-walled open cross-section, the \( \psi \)-independent component of the functional is of the order \( R^3_t \); the value of the functional, i.e. the torsional constant, is of the order \( R^3 \) and is thus an order \((t/R)^2\) smaller. This is only permissible if the contribution of the approximate torsion function to the value of the functional almost cancels out the \( \psi \)-independent part, whereas, the remaining relatively small difference is responsible for the small torsion-constant. Approximation errors for \( \psi \) will thus have a marked influence on \( I_t \).

**NOMENCLATURE**

- **A**: Cross-sectional area
- **E**: Modulus of elasticity
- **G**: Shear modulus
- **I_c**: Mono-symmetry parameter
- **I_y**: Moment of inertia about the centroidal y-axis
- **I_z**: Moment of inertia about the centroidal z-axis
- **I_t**: Torsion constant
- **l**: Length of beam
- **M**: Local cross-sectional moment
- **n_y**: Directional cosine between the outer normal and the y-direction
- **n_z**: Directional cosine between the outer normal and the z-direction
- **P_2**: Increment of the total potential energy quadratic in displacements
- **R**: Radius of curvature of the center line
- **S**: Boundary of \( A \)
- **t**: Wall-thickness
Displacement vector

Lateral displacement of the shear center

Cartesian coordinate in the longitudinal direction of the beam

Cartesian coordinates originating at the centroid and in the plane of the cross-section

Coordinates of shear center

Rotation about the x-axis

Warping constant

Laplace's operator

Torsion function

Dimensionless coordinates

Polar coordinate

Superscript

Differentiation with respect to x

Subscript

Differentiation with respect to the coordinate that follows

REFERENCES