ORTHOGONALLY SCATTERED MEASURES ON NON-BOOLEAN SEMI-RINGS

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Summary

The definition of a countably additive orthogonally scattered measure on an orthogonal semi-ring is given. It is proved that there is no non-trivial c.a.o.s. measures mapping the lattice of orthogonal projections in a separable Hilbert space into a finite dimensional Hilbert space. Global properties of families of c.a.o.s. measures are investigated in connection with inductive limits of families of Hilbert spaces.

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INTRODUCTION

Orthogonally scattered measures appeared in our theory of inductive-projective limits of Hilbert spaces in a natural way ([EGK], [EK], [M]). We recall shortly the idea of our construction.

Let $\mathcal{R}$ be a generating family of operators i.e. a family of bounded selfadjoint operators in a Hilbert space $H$ with the following properties:

i) $\forall \ a \in \mathcal{R} \quad 0 \leq a \leq 1$

ii) $\forall \ a, b \in \mathcal{R} \quad ab = ba$

iii) $\forall \ a, b \in \mathcal{R} \exists c \in \mathcal{R} \quad a \leq c$ and $b \leq c$

iv) $\forall \ a \in \mathcal{R} \exists b \in \mathcal{R} \quad a^\perp \preceq b$.

Let $aH$ denote the Hilbert space consisting of vectors $ah$, where $h \in H$, with the scalar product $(ah \mid af)_a := (r(a)h \mid r(a)f)_H$, where $r(a)$ is the right (left) support of $a \in \mathcal{R}$.

$S_{\mathcal{R}}$ will denote the inductive limit of the family of Hilbert spaces $\{ aH \}_{a \in \mathcal{R}}$ i.e. $S = \bigcup_{a \in \mathcal{R}} aH$.

Let $E$ be the joint resolution of identity of the family $\mathcal{R}$, i.e. a projection valued countably additive measure defined on the $\sigma$-ring of Borel subsets of the spectrum $\Lambda$ of the $\mathcal{W}^*$-algebra $\mathcal{W}^*(\mathcal{R})$ generated by $\mathcal{R}$.

Let $\Sigma$ denote the semi-ring of Borel subsets of $\Lambda$ defined as follows:

$\Delta \in \Sigma$ if and only if $\Delta$ is a Borel subset of $\Lambda$ and there exists $a \in \mathcal{R}$ such that for some positive number $c$, $E(\Delta) \leq c \cdot a$.

$\Sigma$ is called a family of $\mathcal{R}$-bounded subsets of $\Lambda$. If $\lambda \notin \mathcal{R}$, then $\Sigma$ need not be a $\sigma$-ring.
We define a completely additive orthogonally scattered measure on $\Sigma$ (c.a.o.s.m.) as a function $\mu: \Sigma \to H$ with the properties:

i) if $\{ \Delta_\alpha \}_{\alpha \in I} \subseteq \Sigma$, $\Delta_\alpha \cap \Delta_\beta = \emptyset$ for $\alpha \neq \beta$ and $\bigcup_{\alpha \in I} \Delta_\alpha \subseteq \Sigma$ then

$$\Sigma \mu(\Delta_\alpha) = \mu(\bigcup_{\alpha \in I} \Delta_\alpha),$$

where the series converges in norm in $H$.

ii) if $\Delta_1, \Delta_2 \subseteq \Sigma$ and $\Delta_1 \cap \Delta_2 = \emptyset$ then $(\mu(\Delta_1) \mid \mu(\Delta_2))_H = 0$.

(cf. [M]).

A c.a.o.s. measure $\mu$ is said to be generated by a spectral measure $E$ on $\Sigma$ if $\forall \Delta_1, \Delta_2 \subseteq \Sigma$, $E(\Delta_1) \mu(\Delta_2) = \mu(\Delta_1 \cap \Delta_2)$.

For a given family $\mathfrak{A} \subseteq \mathcal{B}(H)$ a c.a.o.s. measure $\mu$ is called $\mathfrak{A}$-bounded if it is generated by $E$ and $\forall \ a \in \mathfrak{A}$ the mapping

$$\Sigma \ni \Delta \mapsto a \mu(\Delta) \in H$$

is a c.a.o.s. measure such that

$$\sup \| a \mu(\Delta) \| < \infty.$$ The set of $\mathfrak{A}$-bounded c.a.o.s. measures is denoted by $T_{\mathfrak{A} \Sigma}$ and we endow it with the locally convex topology given by the following family of seminorms:

$$T_{\mathfrak{A} \Sigma} \mu \mapsto \| \mu \|_a = \sup_{\Delta \subseteq \Sigma} \| a \mu(\Delta) \|.$$  

In [EK] we proved that the following duality holds:

$$S_{\mathfrak{A} \Sigma} = T_{\mathfrak{A} \Sigma}^*, \quad S_{\mathfrak{A} \Sigma}^* = T_{\mathfrak{A} \Sigma}.$$

Moreover we introduce a natural embedding $j: S_{\mathfrak{A} \Sigma} \subseteq T_{\mathfrak{A} \Sigma}$ putting:

$$j(s)(\Delta) = aE(\Delta)h,$$

where $s \in S$ is such that $s = ah \in aH$. Note that then $h \in r(a)H$ is unique.

In this way we can describe the inductive limit $S_{\mathfrak{A} \Sigma}$ and its strong dual $S_{\mathfrak{A} \Sigma}^*$ in terms of c.a.o.s. measures on the semi-ring $\Sigma$.

We can extend our definition of a c.a.o.s. measure $\mu$ defined on "characteristic functions" on $\Sigma$ onto an "integral" defined on the "family of functions" $\mathfrak{A}$ by:

$$\mathfrak{A} \ni a \to \mu(a) \in H$$

where $\mu(a)$ is the unique vector in $H$ such that

$$E(\Delta) \mu(a) = a \mu(\Delta)$$

for all $\Delta \subseteq \Sigma$ (cf. Lemma 2.13. [EK]).
The function $\mu : \mathcal{R} \to H$ has interesting properties: if $r(a) \perp r(b)$ for $a, b \in \mathcal{R}$ then $\mu(a) \perp \mu(b)$, and if $a \leq b$ then $\|\mu(a)\| \leq \|\mu(b)\|$.

In this way we have represented the order structure of the family $\mathcal{R}$ or, equivalently, the order type of the inductive limit $S_\mathcal{R}$ in terms of c.a.o.s. measures.

It is an easy observation that we can originally define c.a.o.s. measures on the family of projections $\mathcal{E} = \{ E(\Delta) | \Delta \in \Sigma \}$ instead of defining them on $\Sigma$, by:

$$\exists \exists e \to \mu(e) = \mu(\Delta), \text{ where } e = E(\Delta) \text{ for some } \Delta \in \Sigma.$$

Suppose now that a family of operators $\mathcal{R}$ fulfils conditions i), iii) and iv), i.e. it is not commutative. Then we can define an inductive limit $S_\mathcal{R}$ of the family of Hilbert spaces $\{ aH \}_{a \in \mathcal{R}}$, taking as before $S_\mathcal{R} = \bigcup_{a \in \mathcal{R}} aH$ with an adequate topology. This time however there is no joint resolution of identity for $\mathcal{R}$ and our previous idea of the representation of $S_\mathcal{R}$ as a space of c.a.o.s. measures is not applicable directly.

This leads to the following idea: define orthogonally scattered measures on non-distributive (non-commutative) order structures, for instance such as a lattice of projections of a non-commutative von Neumann algebra of operators in a Hilbert space.

The present paper is devoted to the study of such measures including existence problem.
1. EXISTENCE OF C.A.O.S. MEASURES ON GENERAL DIRECTED SETS

We consider here certain particular type of directed sets which is modelled after the order structure of sets of orthogonal projections onto subspaces of a Hilbert space.

1.1. Definition

A set $\mathcal{L}$ is called an orthogonal semi-ring if:

i) $\mathcal{L}$ is a partially ordered directed set with respect to a relation $\leq$ such that every finite family of elements of $\mathcal{L}$ has its least upper bound. $\mathcal{L}$ contains the minimal element $0$.

ii) There is given an orthogonality relation $\perp \subseteq \mathcal{L} \times \mathcal{L}$, such that:

1. $a \perp b \implies b \perp a$.
2. $a \perp b$ and $a \leq b \implies a = 0$.
3. if $a, c, d \in \mathcal{L}$ have the property that for each $b \in \mathcal{L}$ $a \perp b$ implies $c \perp b$, then $c \leq a$.
4. If $a \perp b$ and $c \leq a$ then $c \perp b$.
5. If $a \leq b$ then there exists (unique) $c \in \mathcal{L}$ such that $c \perp a$ and $a \vee c = b$.
6. (Weak modularity) If $a \perp b$, $a \perp c$, $a \vee b = a \vee c$ and $b \leq c$ then $b = c$.
7. If $a \leq b$, $a \perp c$, $b \perp d$ and $a \vee c = b \vee d$ then $d \leq c$.

Although the conditions 1 - 7 of ii) are not independent it seems useful to display all of them at once.
1.2. Definition

Let $H$ be a Hilbert space and $\mathcal{L}$ be an orthogonal semi-ring. Then a function $\mu: \mathcal{L} \rightarrow H$ is called a countably additive orthogonally scattered measure on $\mathcal{L}$ (c.a.o.s.m.) if:

i) For any $a_1, a_2 \in \mathcal{L}$, $a_1 \perp a_2$ implies $\mu(a_1) \perp \mu(a_2)$.

ii) For each countable family $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathcal{L}$, $a_i \perp a_j$ if $i \neq j$ and $\bigcup_{i \in \mathbb{N}} a_i \in \mathcal{L}$ implies $\mu\left(\bigcup_{i \in \mathbb{N}} a_i\right) = \sum_{i \in \mathbb{N}} \mu(a_i)$, where the series converges in the Hilbert space norm.

For our purposes it is enough to consider the orthogonal semi-ring $\mathcal{L}_p$ consisting of projections onto closed subspaces of a Hilbert space $H$. Then for $e_1, e_2 \in \mathcal{L}_p$ we put $e_1 \leq e_2$ if $e_1 H \subseteq e_2 H$, for $e \in \mathcal{L}_p$, $e_3 = e_1 \lor e_2$, we take the orthogonal projection onto the closed subspace $e_1 H \lor e_2 H$ and we put $e_1 \perp e_2$ if $e_1 e_2 = 0$.

An example of a c.a.o.s. measure on $\mathcal{L}_p$ is given by

$\mathcal{L}_p \ni e \mapsto \mu(e) := e x$, where $x \in H$.

Now let $\mathcal{L}_n$ denote the lattice of all orthogonal projections onto subspaces of the Euclidean space $\mathbb{R}^n$ with the above orthogonality relation.

1.3. Proposition

If $m, n \in \mathbb{N}$, $n > m$, $n \geq 3$ and $\mu: \mathcal{L}_n \rightarrow \mathbb{R}^m$ is a c.a.o.s. measure with values in the real Hilbert space $\mathbb{R}^m$ then $\mu = 0$.

Proof:

We will use the following result

1.4. Lemma ([G])

Let $\rho: \mathcal{L}_n \rightarrow [0,1]$ be a function with the properties:

$\rho(1) = 1$ and if $e \perp f$ then $\rho(e + f) = \rho(e) + \rho(f)$. Assume $n \geq 3$.

Then there exists a density matrix $W$ such that $\rho(p) = \text{Tr}(Wp)$ for all $p \in \mathcal{L}_n$. 
In particular it follows that the function \( \rho \) is continuous with respect to the natural parametrization of projections in \( \mathbb{R}^n \).

Observe that if \( \mu : \mathcal{L}_n \rightarrow \mathbb{R}^m \) is a c.a.o.s.m. then its components satisfy the assumptions of the above Lemma (up to normalization). It also can be applied to the scalar measure \( \mathcal{L}_n \mathcal{E} \rightarrow \|\mu(e)\|^2 \).

Thus the measure \( \mu \) is continuous with respect to the natural parametrization of projections in \( \mathbb{R}^n \).

We prove the proposition by induction.

Consider at first \( \mu : \mathcal{L}_n \rightarrow \mathbb{R}^1 \).

Suppose that there exists an one-dimensional projection \( p \in \mathcal{L}_n \), such that \( \mu(p) \neq 0 \). Then for any \( q \in \mathcal{L}_n \), such that \( q \perp p, \mu(q) = 0 \). Let \( e \in \mathcal{L}_n \) be any one-dimensional projection. Put \( f = p \perp (e \lor p) \neq 0 \). Then \( e \lesssim p + f \) and \( \mu(f) = 0 \). We have:

\[
\mu(e) + \mu(p + f - e) = \mu(p).
\]

Because \( \mu(e) \mu(p + f - e) = 0 \) so either \( \mu(e) = 0 \) or \( \mu(e) = \mu(p) \).

This is a contradiction to the continuity of \( \mu \). Hence \( \mu = 0 \).

Let \( n = 3, m = 2 \).

For any triple \( \{p_1, p_2, p_3\} \) of mutually orthogonal one-dimensional projections we can assume that \( \mu(p_3) = 0 \). We are going to show that if \( \mu(p_1) \neq 0 \) then \( \mu(p_2) = 0 \).

Assume the contrary, i.e. \( \mu(p_1) \) and \( \mu(p_2) \) non-zero. Let \( e \lesssim p_1 + p_3 \) be an one-dimensional projection. Then \( \mu(p_1 + p_3 - e) + \mu(e) = \mu(p_1) \).

Because \( \mu(p_1 + p_3 - e) \perp \mu(p_2) \) and \( \mu(e) \perp \mu(p_2) \) so

\[
\mu(p_1 + p_3 - e) \sim \mu(e) \sim \mu(p_1).
\]

But \( \mu(p_1 + p_3 - e) \perp \mu(e) \) hence either \( \mu(e) = 0 \) or \( \mu(e) = \mu(p_1) \) which contradicts the continuity of \( \mu \) restricted to the projections \( e \lesssim p_1 + p_3 \).
Thus only one of the values $\mu(p_1)$ or $\mu(p_2)$ can be non-zero. It follows that for any triple of mutually orthogonal one-dimensional projections in $\mathbb{R}^3$ at most one of them has non-zero measure.

Assume that $p$ and $q$ are one-dimensional projections in $\mathbb{R}^3$ such that $\mu(p) \neq 0$ and $\mu(q) \neq 0$ (hence $p \not\perp q$).

Let $p' = p^\perp \wedge (p \vee q)$, so $\mu(p') = 0$. We have:

$$\mu(p \vee q) = \mu(p + p') = \mu(p)$$

and similarly $\mu(p \vee q) = \mu(q) = \mu(p)$.

It means that $\mu : \mathcal{L}_3 \to \mathbb{R}^1$. Thus $\mu = 0$.

Let $m = n - 1$.

Let us assume now that the only c.a.o.s. measure $\mu : \mathcal{L}_{n-1} \to \mathbb{R}^{n-2}$ is $\mu = 0$. We will show that from this follows that if $\mu : \mathcal{L}_n \to \mathbb{R}^{n-1}$ then $\mu = 0$.

Let $\{p_1, p_2, \ldots, p_n\}$ be a collection of mutually orthogonal one-dimensional projections in $\mathcal{L}_n$ and let $\mu : \mathcal{L}_n \to \mathbb{R}^{n-1}$ be a c.a.o.s.m. Suppose that $\mu \neq 0$.

We can assume that $\mu(p) = 0$ and that there is such an index $i_0 \in \{1, 2, \ldots, n-1\}$ that $\mu(p_{i_0}) \neq 0$. The restriction of $\mu$ to the lattice of projections $\mathcal{L}_n = \{q \in \mathcal{L}_n \mid q \leq p_{i_0}^\perp\}$ has its values in $\mathbb{R}^{n-2}$. Indeed: for each $q \leq p_{i_0}^\perp$ we have $\mu(q) \perp \mu(p_{i_0})$; thus, by the assumption, $\mu|\mathcal{L}_{n-1} = 0$.

It means that for an arbitrary collection $\{p_1, p_2, \ldots, p_n\}$ there may by only one index $i_0 \in \{1, 2, \ldots, n\}$ for which $\mu(p_{i_0}) \neq 0$. It follows that for any one-dimensional projection $q \in \mathcal{L}_n$ such that $\mu(q) \neq 0$, we have $\mu(q^\perp) = 0$.

Let $p, q \in \mathcal{L}_n$ be a couple of one-dimensional projections. Then again

$$\mu(p \vee q) = \mu(p) = \mu(q)$$

and as before $\mu : \mathcal{L}_n \to \mathbb{R}^1$. Hence $\mu = 0$.

\[\square\]

1.5. Remark

The case $\mu : \mathcal{L}_2 \to \mathbb{R}^1$ obviously admits non-zero c.a.o.s. measures.

On the other hand the example $\mathcal{L}_n \ni e \to \vee(e) = e x \in \mathbb{R}^n$, for some $x \in \mathbb{R}^n$ shows that the assumption $n > m$ is essential.
1.6. Corollary

Let $H$ be a separable Hilbert space ($\dim H \geq 3$). Let $\mu$ be a c.a.o.s.m. on the lattice $\mathcal{L}_H$ of projections onto closed subspaces of $H$ with values in a Hilbert space $K$. Let $\dim H > \dim(\text{lin.span.}\{ \mu(E) | E \in \mathcal{L}_H \})$. Then $\mu = 0$.

1.7. Corollary

Let $\phi : \mathcal{L}_n \rightarrow \mathcal{L}_m$ be a mapping which preserves the orthogonality relation and which is additive on mutually orthogonal projections. If $n > m$ and $n \neq 2$ then $\phi = 0$. In particular, if $\pi : M_{n \times n} \rightarrow M_{m \times m}$ is a Jordan homomorphism then $\pi = 0$ for $n > m$.

Proof:

It is enough to notice that for any vector $x \in \mathbb{C}^m$ the map $\mathcal{L}_n \triangledown e \rightarrow \phi(e) x \in \mathbb{C}^m$ is a c.a.o.s. measure. Hence $\phi(\cdot) = 0$. 

$\Box$
2. REPRESENTATION OF A C.A.O.S. MEASURE

We extend here some of our results of [EK] onto the case of non-commutative domains of definition of c.a.o.s. measures. We exploit here the notion of the bi-orthogonality relation (i.e. compatibility) between two c.a.o.s. measures.

2.1. Definition

Let \( \mathcal{L} \) be an orthogonal semi-ring. Let \( \mu, \nu : \mathcal{L} \to \mathbb{H} \) be a couple of c.a.o.s. measures with values in a Hilbert space \( \mathbb{H} \). Then \( \{ \mu, \nu \} \) is called a bi-orthogonal couple ([M]) or a compatible couple of c.a.o.s. measures if for every \( a, b \in \mathcal{L} \) from \( a \perp b \) follows that \( \mu(a) \perp \nu(b) \).

Example

Let \( \mathcal{L} = \mathcal{L}_\mathbb{H} \), \( x, y \in \mathbb{H} \). Then \( \mu(E) = Ex, \nu(E) = Ey \) is a bi-orthogonal couple of c.a.o.s. measures.

We say that a family of c.a.o.s. measures is bi-orthogonal if every pair of members of it is a bi-orthogonal couple. Every bi-orthogonal family of c.a.o.s. measures can be extended to a maximal one with respect to the set-inclusion relation.

Let \( \Theta \) be a maximal bi-orthogonal family of c.a.o.s. measures on \( \mathcal{L} \).
Denote: \( \Theta(e) = \{ \mu(e) \mid \mu \in \Theta \} \) for \( e \in \mathcal{L} \).

2.2. Theorem

Let \( \Theta \) be a maximal bi-orthogonal family of c.a.o.s. measures on an orthogonal semi-ring \( \mathcal{L} \). Then:

i) For each \( e \in \mathcal{L} \) the set \( \Theta(e) \) is a closed linear subspace of the Hilbert space \( \mathbb{H} \).
ii) If $e, f \in \mathcal{E}$ and $e \succ f$ then $\Omega(e) \subseteq \Omega(f)$.

iii) If $e, f \in \mathcal{E}$ and $e \perp f$ then $\Omega(e) \perp \Omega(f)$.

iv) The set $S = \bigcup_{e \in \mathcal{E}} \Omega(e)$ is a linear subspace of $H$.

**Proof:**

At first we notice that iii) easily follows from the definition of $\Omega(e)$.

i) Let $\mu, \nu \in \Theta$, $\alpha, \beta \in C^1$. Then $\alpha \mu + \beta \nu$ is a c.a.o.s.m. compatible with each element of $\Theta$. Thus by the maximality of $\Theta$ it belongs to $\Theta$.

Let $\Phi(e)$ denote the orthogonal projection onto the closure of $\Omega(e)$ in $H$.

Then for $e \perp f$ in virtue of iii) we have $\overline{\Omega(e)} \perp \overline{\Omega(f)}$ and $\Phi(e) \perp \Phi(f)$.

Moreover we have then $\overline{\Omega(e \vee f)} = \overline{\Omega(e)} \oplus \overline{\Omega(f)}$, i.e. $\Phi(e \vee f) = \Phi(e) \oplus \Phi(f)$.

Because $\Phi(e) \leq 1$ so for any family $\{e_n\}_{n \in N}$ such that $e_n \perp e_m$ for $n \neq m$ and $\forall e_n \in \mathcal{E}$ there exists $\phi(\vee_{n \in N} e_n) = \sum_{n \in N} \phi(e_n)$, where the series converges strongly in $B(H)$. Thus for any $x \in H$ the mapping: $\mathcal{E} \ni e \mapsto \Phi(e)x \in H$ is a c.a.o.s.m. By the construction this measure is compatible with $\Theta$ hence it belongs to $\Theta$.

Now take $x \in \overline{\Omega(e)}$. Then $\Phi(e)x = x \in \Omega(e)$. Thus $\Omega(e) = \overline{\Psi(e)}$.

This proves i).

ii) It is easy to notice that for $e \preceq f$ we have $\Phi(e) \preceq \Phi(f)$. Thus $\Omega(e) \subseteq \Omega(f)$.

iv) The linearity of $S$ follows from ii).

\[ \Box \]

### 2.3. Remark

In distinction to the "commutative" case it may happen that the linear manifold $S$ is not dense in $H$. For instance let $\mathcal{E} = \mathcal{E}_3$ and $H = C^4$. Let $\Theta$ be a maximal bi-orthogonal family of c.a.o.s. measures containing all measures of the form: $\mathcal{E}_3 \ni e \mapsto e \cdot x \in C^4$ where $x \in C^3$ and we embed $C^3 \subseteq C^4$ in a fixed way.
Then we have $L_3 \simeq$ all $4 \times 4$ matrices of the form:

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

where $P$ is a $3 \times 3$ Hermitian idempotent matrix, i.e., a projection in $C^3$.

So $C^3 \subset S$. Let $\mu \in \Theta$. Then taking any one-dimensional projection $P_x$ onto a unit vector $x \in C^3$, we have: $P_x^\perp \mu(P_x) = 0$. Thus $\mu(P_x) = \gamma(P_x)x + \lambda(P_x)e_4$, where $e_4 \perp C^3$ in $C^4$ is a unit vector.

However, for any triple of mutually orthogonal one-dimensional projections in $C^3$ we have $\mu(P_i) \perp \mu(P_j), i \neq j$, in particular $\lambda(P_i) \perp \lambda(P_j) = \delta_{ij}\lambda(P_i)$. Thus there is only one index for which the number $\lambda(P_i)$ is different from 0.

Notice that the map $L_3 \ni P_i \rightarrow \lambda(P_i)e_4$ can be extended to a c.a.o.s. measure on $L_3$ with values in $C^1$, hence $\lambda = 0$, i.e., $S = C^3 \neq C^4$.

2.4. Corollary

For any maximal bi-orthogonal family of c.a.o.s. measures on $L$ there exists a map, which preserves the order and orthogonality relations $\phi: L \rightarrow B(H)^P$, with values in projections in $H$, such that:

If $e \perp f$ then $\phi(e \vee f) = \phi(e) + \phi(f)$ and for every $\mu \in \Theta$ and each $e \in L$

$\phi(e) \mu(e) = \mu(e)$.

Let us consider now a particular case in which $L = A^P$, i.e., $L$ is the set of all projections of a given $W^*$-algebra $A$ of operators acting in a separable Hilbert space $H$. 
2.5. Theorem

Let \( \mathcal{L} = \mathcal{A}^P \) and let \( \mu : \mathcal{L} \to H \).

Then there exists a linear positive map \( \tilde{\phi} : \mathcal{A} \to \mathcal{B}(H) \) and a vector \( x \in H \) such that for each \( e \in \mathcal{L} \) \( \mu(e) = \tilde{\phi}(e)x \).

Proof:

Let \( \Theta \) be a maximal biorthogonal family of c.a.o.s. measures containing \( \mu \). Let \( \phi(e) \) be the orthogonal projection onto the space \( \Theta(e) \). Then by Corollary 2.4. we obtain the complete additivity of \( \phi \) on families of mutually orthogonal projections in \( \mathcal{A}^P \). For \( e \perp f \) we have \( \phi(e) \perp \phi(f) \) and for \( e \leq f \) \( \phi(e) \leq \phi(f) \).

Moreover for \( e \leq f \) \( \phi(e) \mu(f) = \phi(e)(\mu(f-e) + \mu(e)) = \mu(e) \).

Now consider the net \( \{\mu(e)\} \}_{e \in \mathcal{L}} \). Because it is bounded in norm it has cluster points. Let \( x \in H \) be a cluster point of it. Then for every \( \varepsilon > 0, z \in H \) and \( e \in \mathcal{L} \) there exists \( f \in \mathcal{L} \) such that \( f \succ e \) and \( |(\phi(e)z - \mu(f) - x)| < \varepsilon \), i.e. \( \varepsilon > |(z| \phi(e) \mu(f) - \phi(e)x)| = |(z| \mu(e) - \phi(e)x)| \).

Since \( \varepsilon \) is arbitrary we have \( \phi(e)x = \mu(e) \).

Because \( \phi : \mathcal{L} \to \mathcal{B}(H)^P \) is a normal map it can be extended by the spectral theorem to a positive map \( \tilde{\phi} \) defined on the whole algebra \( \mathcal{A} \). Linearity follows from the generalization of Gleason theorem: there exists a linear positive (normal) functional \( \omega \) on \( \mathcal{A} \) such that \( \|\mu(e)\|^2 = \omega(e) = \|\phi(e)x\|^2 = (x| \phi(e)x) \).

By the definition \( \tilde{\phi} \) is positive.

\[ \square \]

2.6. Corollary

The map \( \mu : \mathcal{A}^P \to H \) can be extended to a linear map \( \mu : \mathcal{A} \to H \) with the property \( \mu(a) = \tilde{\phi}(a)x \). This map coincides with the "integral" \( \mu : \mathcal{A} \to H, \mu(a) \), defined in the Introduction for the commutative case.
CONCLUDING REMARKS

This is easy to observe that our description of c.a.o.s. measures can be closely related to the eigen-packet theory ([M]). Hence it would be desirable to describe the global properties of the dual of the inductive limit $S_{\mathcal{F}}$ for a non-commutative generating family $\mathcal{F}$ in terms of c.a.o.s. measures defined on the lattice of projections of $\mathcal{W}^*(\mathcal{F})$.

Also a factorization of unbounded c.a.o.s. measures in the sense of Theorem 2.5. seems interesting.

At last we pose two technical problems, strictly connected with the present paper:

Problems

1. Find conditions under which the map $\hat{\phi}$ described in Theorem 2.5. is a Jordan homomorphism.

2. Extend the technics used in Corollary 1.7. onto the case of von Neumann factors - i.e. an easy proof of the existence or non-existence of isomorphisms between factors.
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