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Note on a physical application of the main theorem of Chronogeometry

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1. Historical preface

There are three principally different methods that can be used in measuring geometrical quantities, such as lengths, angles and so on.

a) direct geometrical measurements

In this case one uses rods, triangles, theodolites and other devices for on-earth measuring of distances and angles.

In the pre-satellite era these were the main instruments of geodesy and still now vast and very expensive geodesic programs are based on such measurements.

b) chronogeometrical *) measurements.

These measurements do not take place in space only, but in space-time. We divide this kind of measurement in two parts, viz.

b 1) measurements, using light beams (and in the twentieth century, after the invention of radio, also radio beams).

This type of measurement also goes back to antiquity where directions (e.g. to determine the position of a ship) were determined by observing the sun, stars and so on.

b 2) measurements using clocks

This is a more recent type of measurement. Since middle ages this method was also used for geometrical measurements, especially in navigation. Up to the middle of the twentieth century, the most precise method was a).

*) Chronogeometry has the same relation to space-time as geometry to space.
One could, of course measure the position of a ship by observing stars and using clocks but, only with the preciseness of several miles; in order to do more precise measurements, rods and theodolites were used. The situation became quite different after world war II. The necessity of detecting enemy's airplanes as early as possible led to a quick progress in radiotechnique and this led to the rapid increase in preciseness of radio measurements (that is $b_1$).

A second reason for such increase was the invention of laser in 1964, that allowed first to increase the preciseness of light beam measurements (also $b_1$) and second to create very precise clocks (maser clocks) and that gave rise to much more preciseness of $b_2$. In course of that progress the preciseness of $b$ became greater than the preciseness of purely geometrical methods, as described in a). Hence application of methods like a) in the new situation led to the unwanted phenomenon that, although the results of type $b$ were very precise, in calculating geometrical quantities one had also to use relatively very coarse type a) results.

**Example 1**

Let us consider the measurement of a position of a very distant radio object (e.g. a quasar) by means of an interferometer. An interferometer consists of two antennas and it allows to measure the time delay $\tau$ between the times when the same signal arrives to different antennas (see fig. 1)
Denote the vector, uniting the antennas by $\mathbf{B}$. The source is very distant, therefore we can consider both radio beams, coming from it to two antennas, as parallel.

Assume that the unit vector in that common direction, is $\mathbf{s}$, then, as one easily sees, we have

$$\tau = \frac{\mathbf{B} \cdot \mathbf{s}}{c}.$$ 

We can measure $\tau$ very precisely, but if we measure $\mathbf{B}$ geometrically (by means of constructing a triangular net) the resulting preciseness of $\mathbf{s}$ will be very small.

**Example 2. (laser location)**

One can measure the distance to celestial bodies with cm-preciseness, but the preciseness of the corresponding earth measurements is much worser ($\sim$ km.)

That is the reason why for on-earth measurement new methods were necessary; methods that are unlike the classical ones and use only chronogeometrical information. But before analyzing such real methods it is necessary to solve a more principal question: Is it possible to determine all geometrical quantities by means of analyzing light beams and clocks?

In section 2 we shall show that the (affirmative) answer to that question follows from one of the cases of the so-called main theorem of chronogeometry (due to A.D. Aleksandrov, 1949).
2. Aleksandrov's theorem, applied to our situation

In 1949, A.D. Aleksandrov [1] proved a theorem, stating that every autobijection of $\mathbb{R}^4$ (for simplicity we assume that space-time is flat) preserving light cones is a Lorentz transformation, a translation or a scalar multiplication (and hence a similarity).

In fact Aleksandrov's theorem states much more, but we restrict ourselves to the case in question $^*)$.

Assume that in every point, in every moment of time, we emit light beams in all possible directions and we also test whether those beams have reached any other point. After all these (infinitely many!) experiments one possesses a list of pairs of events that can be connected by a light ray.

Due to the fact that space-time is supposed to be flat, there exist coordinates, i.e. a one-one mapping $\mathbb{c} = (c_0, c_1, c_2, c_3)$ of the set $S$ of all events onto $\mathbb{R}^4$. Hence we are able to express that a light ray from $a$ can reach $b$ ($a < b$) in the following way ($a, b \in S$)

$$a < b \iff \begin{cases} c_0(a) < c_0(b) \text{ and } \\ [c_0(a) - c_0(b)] = \frac{2}{3} \sum_{i=1}^{3} [c_i(a) - c_i(b)]^2 \end{cases}$$

We try to find such a function $\mathbb{c}$.

Assume that some computer program led us to some $\mathbb{c}'$, satisfying (1), but perhaps some other computer procedure will lead us to $\mathbb{c}''$.

Because $\mathbb{c}'$ and $\mathbb{c}''$ are bijections, the function $f = \mathbb{c}''(\mathbb{c}')^{-1}$ exists and is an autobijection of $\mathbb{R}^4$, preserving light cones i.e.

$$Q(x - y) = 0 \iff Q(fx - fy) = 0.$$ 

Therefore, due to the main theorem of chronogeometry, we conclude that $\mathbb{c}'$ and $\mathbb{c}''$ are connected by a scalar multiplication, a translation or a Lorentz transformation.

Assume moreover that we have chosen two fixed events $a$ and $b$ (e.g. the

$^*)$ All details of the theorem are to be found in [2]
the positions of the two ends of a standard meter in some fixed moment of time) and that we look only for such coordinates in which those events are simultaneous, i.e.

\[ c_0(a) = c_0(b) \]

and the spatial distance between them equals to some fixed number e.g.

\[ \sum_{i=1}^{3} [c_1(a) - c_1(b)]^2 = 1 \]

Now the class of possible transformations is greatly narrowed. Scalar multiplications are excluded (they violate (2)). Similarly proper Lorentz transformations are impossible (they violate (3)). Hence, only movements (rotations and translations) are left and that means that the conditions (1), (2) and (3) determine \( c \) modulo a movement.

As is well-known, all geometrical quantities (lengths, angles and so on) are preserved by movements and so we finally conclude that purely chronogeometrical observations are allowed to measure lengths and angles.

3. Conclusions and remarks

3.1. Of course, Aleksandrov's theorem does not give a real solution of the above-described experimental problem, because it requires that all light beams, emitted from all trajectories of events, are known. In reality only finitely many pairs of events about which we know that they are connected by light rays, are known. Therefore, it is necessary to use some additional information in order to reconstruct lengths and angles uniquely.

This information is supplied by a kind of apparatus that is neglected in the main theorem; namely clocks. Clocks allow us to measure time delays (and hence derivative functions) and this information turns out to be really sufficient. For details the reader is referred to the methods of Very Long Baseline Interferometry (VLBI) as described in [3].
3.2. It is important to investigate whether conservation, not of all light cones but of subsets of them, implies that the mapping $f$ in question is a similarity. E.g. Kuz'minyh states in [4] that a one to one transformation of space-time onto itself that preserves the constancy of the speed of light, emitted by sixteen sources at rest in some initial system, is a Lorentz transformation.

3.3. Another important item is whether local conservation of cones implies that the corresponding autobijections are linear. Probably it turns out that this is not true but that in this case we are led to conformal maps.

3.4. In [5] one of the authors proved that the main theorem, as quoted in this note, remains valid if vector spaces over arbitrary commutative fields $K$ are considered, in stead of $\mathbb{R}^4$. The physical meaning of that generalization is that the light relation is conserved if subsets of $\mathbb{R}^4$ are considered, formed by points whose coordinates belong to some subfield of $\mathbb{R}$ generated by (or even formed by) numbers that do appear in real experiments.

3.5. Due to the fact that all real observations are very approximate, it is necessary to investigate whether the main theorem is stable, that is, if light cones are transformed into small neighbourhoods of light cones, this implies that such a transformation is close to a similarity.
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