STABILITY OF LINEAR INFINITE DIMENSIONAL SYSTEMS REVISITED

by

K. Maciej Przyłuski

University of Technology
Dept. of Mathematics & Computing Science
P.O. Box 513, Eindhoven
The Netherlands
STABILITY OF LINEAR
INFINITE DIMENSIONAL SYSTEMS REVISITED*)

by

K. Maciej Przyłuski **) 

Abstract. The paper is devoted to a study of stability questions for linear infinite-dimensional discrete-time and continuous-time systems. The concepts of power stability and $L^p$-stability for a linear discrete-time system $x_{k+1} = Ax_k$ (here $x_k \in X$, $X$ is a Banach space, $A$ is linear and bounded) are introduced and studied. Relationships between these concepts and the inequality $r(A) < 1$ ($r(A)$ denotes the spectral radius of $A$) are also given. Next, the discrete-time results are used for a simple derivation of some well-known properties of exponentially stable and $L^p$-stable linear continuous-time systems described by $\dot{x}(t) = Ax(t)$ ($A$ generates here a strongly continuous semigroup of linear and bounded operators on $X$). Some remarks on norms related to stable systems are also included.

*) The final version of this paper was written while the author was visiting the Technische Hogeschool at Eindhoven.

**) Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8,
P.O. Box 137, 00-950 Warszawa, Poland.
INTRODUCTION

The paper contains a concise study of stability theory for linear infinite dimensional discrete-time and continuous-time systems. Most of the results presented here are quite classical. The novelty lies mainly in the arrangement, and the proofs are new. In particular, it is shown that stability questions for continuous-time systems can be analysed by discrete-time methods. Thus, the intimate connections between the traditionally separated theories of discrete-time and continuous-time systems are emphasized. Of course, such connections are known and they are more or less implicit in a number of works; see e.g. [10, Chap.II, Sec.3 and Chap.III, Sec.5]. It has been shown in [14] that discrete-time methods are useful in studies of asymptotic stability and $L^p$-stability of infinite dimensional continuous-time systems. We shall follow some ideas of this last paper.

The summary of the contents of the present paper reads as follows: In Section 1, we introduce some notations and basic definitions. In Section 2, we prove that $L^p$-stability and power stability of a discrete-time system $x_{k+1} = A x_k$ (where $A$ is a linear bounded operator on a Banach space $X$) are identical concepts. In fact, we prove that they are both equivalent to the condition $r(A) < 1; r(A)$ stands here for the spectral radius of $A$. The equivalence between the last condition and $L^p$-stability was found by Zabczyk [16, Sec.5]. Among other things we prove that, for an $L^p$-stable discrete-time system, two norms, $\| \cdot \|$ and $| \cdot |_p$, are equivalent. Here for all $x \in X$,

$$|x|_p := (\sum_{k=0}^{\infty} \|A^k x\|^p)^{1/p}$$
and \( \| \cdot \| \) is the original norm on \( X \). In Section 3, we consider continuous-time systems given by \( \dot{x}(t) = A x(t) \) (where \( A \) generates a strongly continuous semigroup \( \{S(t)\}_{t \in \mathbb{R}_0^+} \) of linear and bounded operators on \( X \)).

The concepts of exponential stability and \( L^p \)-stability are defined. It is shown that \textit{exponential stability of a continuous-time system is equivalent with power stability of a (properly defined) discrete-time system}. A similar conclusion holds for \( L^p \)-stable systems. In this case a relation between two norms on \( X, \| \cdot \|_p \) and \( | \cdot |_p \), is given. Here

\[
\| x \|_p := \left( \int_0^\infty \| S(t)x \|^p \, dt \right)^{1/p}
\]

and \( | \cdot |_p \) is defined as before, with \( A = S(t) \) for some positive fixed \( t \).

The norms \( \| \cdot \|_p \) and \( | \cdot |_p \) cannot be (in general) equivalent; nevertheless it is possible to obtain (for every \( \tau \)) the following inequality:

\[
(1/\alpha_\tau)^p \| x \|_p^p \leq \tau | x |_p^p \leq \alpha_\tau^p \| x \|_p^p + \tau \| x \|_p^p.
\]

Here \( \alpha_\tau \) is a positive number depending on \( \tau \).

The discrete-time results of Section 2 are used in Section 3 to obtain a simple derivation of a well-known (see e.g.[4],[6],[14]) relationship between exponential stability and \( L^p \)-stability of a continuous-time system.

We note that in Section 1 a "frequency-domain" machinery to study discrete-time systems is developed and next used, in Section 2, to derive some characterizations of stable discrete-time systems. This machinery can also be used to study other control-theoretic problems (see e.g.[15]).
The importance of the paper is twofold: It provides a simple illustration of "frequency-domain" methods in infinite-dimensional context and, it proves that discrete-time methods simplify the theory of continuous-time systems.

Acknowledgement. The author is indebted to Professor M.L.J. Hautus for many helpful suggestions.
1. DISCRETE-TIME AND CONTINUOUS-TIME SYSTEMS. BASIC RESULTS

The main purpose of this section is to describe the classes of systems which we will consider. We shall be concerned only with the discrete-time systems which can be defined by linear difference equations and with the continuous-time systems which can be defined by linear differential equations in a Banach space. Moreover, the systems considered will be autonomous, i.e., the operators specifying the appropriate difference or differential equations will be constant.

The section starts with the definition of a linear discrete-time system. The topic next dealt with is the formal discrete Laplace transform and its usage to describe linear discrete-time systems. The section contains also a discussion of some spaces of formal power series and germs of holomorphic functions, which will play an important rôle in the next section. Lastly, the definition of a linear continuous-time system is given. The section concludes with a simple but crucial result relating continuous-time systems with discrete-time systems.

Let \( X \) be a Banach space and \( L(X) \) denote the set of all linear and bounded operators \( X \to X \). The norm on \( X \) will be denoted by \( \| \cdot \| \). For the set of all nonnegative integers we shall write \( \mathbb{Z}_0^+ \), i.e., \( \mathbb{Z}_0^+ = \{ 0, 1, 2, \ldots \} \).

Let \( A \in L(X) \) be fixed.

**Definition.** It is said that a linear discrete-time system \( S(A) \) is given, if the linear difference equation

\[
x_{k+1} = A x_k
\]

is considered for all \( k \in \mathbb{Z}_0^+ \).
Trivially, $x_k = A^k x_0$ for all $k \in \mathbb{Z}_0^+$; here $A^0 := I$, the identity operator on $X$.

Now we introduce some new concepts needed in the sequel.

Let $L(X[[z]])$ denote the ring of formal power series in the indeterminate $z$ with coefficients in $L(X)$. Similarly, $X[[z]]$ will denote the left $L(X[[z]])$-module of formal power series in the indeterminate $z$ with coefficients in $X$. For a discussion of formal power series we refer the reader to [3, Chap.III, §2, no.11].

Let $W := \{W_k\}_{k=0}^{\infty}$ be a sequence of elements of $L(X)$ (of $X$). The formal power series in the indeterminate $z$ with coefficients in $L(X)$ (in $X$) defined by

$$\hat{W}(z) = \sum_{k \in \mathbb{Z}_0^+} z^k W_k$$

is called the formal discrete Laplace transform of $W$.

A standard but useful result (whose proof is included here only for completeness) is given by the following

**Proposition 1.1.** Let a linear discrete-time system $G(A)$ be given and $X := \{x_k\}_{k=0}^{\infty}$ be a sequence of elements of $X$. Denote by $\hat{X}(z)$ the formal discrete Laplace transform of $X$. Then

$$(I - zA) \hat{X}(z) = x_0$$

(as an equality of formal power series) if and only if the sequence $X$ satisfies the linear difference equation defining the linear discrete-time system $G(A)$. 
Proof. Note that in the ring $L(X[[z]])$,

$$
\left( \sum_{k \in \mathbb{Z}^+} z^k A^k \right)(I - zA) = (I - zA)\left( \sum_{k \in \mathbb{Z}^+} z^k A^k \right) = I.
$$

Hence, the considered equation $(I - zA) \hat{x}(z) = x_0$ holds if and only if

$$
\sum_{k \in \mathbb{Z}^+} z^k x_k = \left( \sum_{k \in \mathbb{Z}^+} z^k A^k \right)x_0.
$$

It proves that $(I - zA) \hat{x}(z) = x_0$ if and only if, for every $k \in \mathbb{Z}^+$,

$$
x_k = A^k x_0, \text{ i.e., } x_{k+1} = A x_k, \text{ } k \in \mathbb{Z}^+.
$$

It is of special interest for the sequel to study these formal power series which satisfy some additional requirements. Therefore, we shall consider all formal power series $W(z) := \sum_{k \in \mathbb{Z}^+} z^k w_k$ in the indeterminate $z$ with coefficients in $L(X)$ (or $X$) such that the following standard growth condition holds:

There exist $\tilde{M} \geq 1$ and $\rho$, $0 \leq \rho < 1$, such that $\|w_k\| \leq \tilde{M} \rho^k$ for all $k \in \mathbb{Z}^+$.

A formal power series which satisfies the standard growth condition will be called a Hurwitz formal power series. Simple calculations show that the set of all Hurwitz formal power series with coefficients in $L(X)$ is a subring of $L(X[[z]])$; we shall call it the Hurwitz ring of $L(X)$. Similarly, all Hurwitz power series from $X[[z]]$ form a left module over the Hurwitz ring of $L(X)$; it will be called the Hurwitz module of $X$. 

Let \( \overline{D}_1 := \{ s \in \mathbb{C} \mid |s| \leq 1 \} \) be the closed unit disc of \( \mathbb{C} \) and let \( H(\overline{D}_1; L(X)) \) denote the set of all functions \( \overline{D}_1 \to L(X) \) such that for every \( f(\cdot) \in H(\overline{D}_1; L(X)) \) there exist an open neighbourhood \( N_f \) of \( \overline{D}_1 \) and a holomorphic function \( g_f(\cdot) : N_f \to L(X) \) so that \( f(\cdot) \) is the restriction of \( g_f(\cdot) \) to \( \overline{D}_1 \).

Addition and multiplication of functions from \( H(\overline{D}_1; L(X)) \) can be defined in the standard way and \( H(\overline{D}_1; L(X)) \) becomes a ring; it is called the ring of germs of \( L(X) \)-valued holomorphic functions on \( \overline{D}_1 \). Similarly, replacing \( L(X) \) by \( X \), we can define \( H(\overline{D}_1; X) \), this set is a left module over \( H(\overline{D}_1; L(X)) \), if addition and (left) multiplication by elements of \( H(\overline{D}_1; L(X)) \) are defined as usual. \( H(\overline{D}_1; X) \) is called the \( H(\overline{D}_1; L(X)) \)-module of germs of \( X \)-valued holomorphic functions on \( \overline{D}_1 \). For a more general treatment of the concept of germs the reader is referred to [2, Chap.I, §4, no.1].

Let \( W(z) := \sum_{k \in \mathbb{Z}^+} z^k w_k \) be an element of the Hurwitz ring of \( L(X) \). The well-known Cauchy-Hadamard formula for the radius of convergence of a power series enables us to define a function

\[
\overline{D}_1 \ni s \mapsto W(s) := \sum_{k=0}^{\infty} s^k w_k \in L(X)
\]

Actually, as it is easy to see, the function belongs to \( H(\overline{D}_1; L(X)) \). Moreover, simple calculations show that the resulting mapping from the Hurwitz ring of \( L(X) \) into the ring of germs of \( L(X) \)-valued holomorphic functions on \( \overline{D}_1 \) is a homomorphism of rings. The homomorphism is an isomorphism of rings. Indeed, every element of \( H(\overline{D}_1; L(X)) \) is representable (via the Taylor expansion at 0) by a (convergent on some neighbourhood of \( \overline{D}_1 \)) power series which defines a Hurwitz formal power series; thus
the homomorphism considered is surjective. It is also an injective homomorphism as follows from the principle of isolated zeros (see e.g. [7, Chap.IX, Sec.1]). We shall denote the defined above *isomorphism of the Hurwitz ring of* $L(X)$ *onto the ring of germs of* $L(X)$-*valued holomorphic functions on* $\overline{D}_1$ *by* $\varphi$.

In a similar manner as above, replacing the Hurwitz ring of $L(X)$ by the Hurwitz module of $X$, and $H(\overline{D}_1;L(X))$ by $H(\overline{D}_1;X)$, we can define a bijective mapping $\psi$ from the Hurwitz module of $X$ onto $H(\overline{D}_1;X)$. As it is easy to see, the mapping is an *isomorphism of the additive group of the Hurwitz module of* $X$ *onto the additive group of the* $H(\overline{D}_1;L(X))$-*module of germs of* $X$-*valued holomorphic functions on* $\overline{D}_1$.

Now, we can observe that $\psi$ is a bijective semi-linear mapping (relative to the isomorphism $\varphi$) of the Hurwitz module of $X$ onto the $H(\overline{D}_1;L(X))$-module $H(\overline{D}_1;X)$; in other words, the ordered pair $(\varphi, \psi)$ of isomorphisms is a *dimorphism of the Hurwitz module of* $X$ *onto* $H(\overline{D}_1;X)$. Hence, the ordered pair $(\varphi^{-1}, \psi^{-1})$ of isomorphism is a *dimorphism of the* $H(\overline{D}_1;L(X))$-*module* $H(\overline{D}_1;X)$ *onto the Hurwitz module of* $X$. We shall not make the above statements more precise; the detailed definitions of semi-linear mappings and dimorphisms are to be found in standard textbooks (see e.g. [3, Chap.II, §1, no.13]). Here we confine ourselves to the simplest properties of the pair $(\varphi, \psi)$, which can be deduced directly from the definitions of $\varphi$ and $\psi$.

Let us say that a formal power series $W(z) \in L(X)[[z]]$ ($w(z) \in X[[z]]$) defines a germ of holomorphic functions on $\overline{D}_1$, i.e., an element of $H(\overline{D}_1;L(X))$ (of $H(\overline{D}_1;X)$, if the series is a Hurwitz formal power series. In this case the germ $W(\cdot) := \varphi(W(z))$ ($w(\cdot) := \psi(w(z))$) of holomorphic
functions on $\tilde{D}_1$ is said to be defined by the formal power series $W(z)$ ($w(z)$). Similarly, a germ of holomorphic functions on $\tilde{D}_1$, $W(\cdot) \in H(\tilde{D}_1;L(X))$ ($w(\cdot) \in H(\tilde{D}_1;X)$) defines a formal power series by the Taylor expansion at 0. Let denote the series by $W(z)$ ($w(z)$). It is a Hurwitz formal power series. We have $W(z) = \varphi^{-1}(W(\cdot))$ ($w(z) = \psi^{-1}(w(\cdot))$) and the formal power series is said to be defined by the Taylor expansion (at 0) of the germ $W(\cdot)$ ($w(\cdot)$) of holomorphic functions on $\tilde{D}_1$.

Now, the basic properties of the dimorphism $(\varphi,\psi)$ can be summarized as follows. If we have an equality (involving only well-defined sums and products) of formal power series which define some germs of holomorphic functions on $\tilde{D}_1$ (i.e., this equality is an equality of Hurwitz formal power series), the equality holds for the germs of holomorphic functions on $\tilde{D}_1$ defined by the formal power series. In other words, this equality holds for all $s \in \tilde{D}_1$. Conversely, every equality (involving only well-defined sums and products) in $H(\tilde{D}_1;X)$ or $H(\tilde{D}_1;L(X))$ holds also for the formal power series defined by the Taylor expansion (at 0) of the germs of holomorphic functions on $\tilde{D}_1$ occurring at both sides of the considered equality in $H(\tilde{D}_1;X)$ or $H(\tilde{D}_1;L(X))$.

In accordance with the remarks above we note the following

**Proposition 1.2.** Let $W(z) \in L(X)[[z]]$, $V(z) \in L(X)[[z]]$, $w(z) \in X[[z]]$ and $v(z) \in X[[z]]$ be given Hurwitz formal power series. Let $W(\cdot):= \varphi(W(z)) \in H(\tilde{D}_1;L(X))$, $V(\cdot):= \varphi(V(z)) \in H(\tilde{D}_1;L(X))$, $w(\cdot):= \psi(w(z)) \in H(\tilde{D}_1;X)$ and $v(\cdot):= \psi(v(z)) \in H(\tilde{D}_1;X)$ be the germs of holomorphic functions on $\tilde{D}_1$ defined by these formal power series. Then

\begin{equation}
(*) \quad W(z) V(z) v(z) = w(z)
\end{equation}
holds as an equality of formal power series if and only if

\[(**) \quad W(\cdot) V(\cdot) v(\cdot) = w(\cdot),\]

i.e., if and only if for all \(s \in \overline{D}_1\)

\[(***) \quad W(s) V(s) v(s) = w(s).\]

Conversely, let \(W(\cdot) \in H(\overline{D}_1; L(X)), V(\cdot) \in H(\overline{D}_1; L(X)), w(\cdot) \in H(\overline{D}_1; X)\)
and \(v(\cdot) \in H(\overline{D}_1; X)\) be given germs of holomorphic functions on \(\overline{D}_1\). Let
\(W(z) := \psi^{-1}(W(\cdot)) \in L(X)[[z]], V(z) := \psi^{-1}(V(\cdot)) \in L(X)[[z]], w(z) := \psi^{-1}(w(\cdot)) \in X[[z]]\) and \(v(z) := \psi^{-1}(v(\cdot)) \in X[[z]]\) be the formal power series defined by the Taylor expansion (at 0) of these germs of holomorphic functions on \(\overline{D}_1\). Then equality (**) (i.e., equivalently, equality (***) holds if and only if equality (*) takes place.

In the next section spectral properties of the operator \(A\) defining \(G(A)\) will be of some interest. Thus, we shall write \(\sigma(A)\) for the spectrum of \(A\). Let \(r(A)\) denote the spectral radius of \(A\), i.e.,
\(r(A) := \sup\{ |\lambda| \mid \lambda \in \sigma(A) \}\). We denote the following well-known (Beurling-Gelfand) formula:

\[r(A) = \lim_{k \to \infty} (\|A^k\|)^{1/k}.\]

Let \(R_A(z) := (I - zA)^{-1} \in L(X)[[z]]\). An easy consequence of the Beurling-Gelfand formula is the following

**Proposition 1.3.** \(R_A(z)\) defines a germ of holomorphic functions on \(\overline{D}_1\) if and only if \(r(A) < 1\), i.e., \(\sigma(A) \subset D_1\).
Of course, $D_1$ denotes here the (open) unit disc of $\mathbb{C}$.

Now we give some basic facts about continuous-time systems. Let $A$ be a given linear operator: $\text{dom}(A) \rightarrow X$ with $\text{dom}(A) \subset X$. By $\text{dom}(A)$ we shall denote the domain of $A$. Throughout the paper it will be assumed that $A$ generates a strongly continuous semigroup $\{S(t)\}_{t \in \mathbb{R}_0^+}$ of linear and bounded operators from $X$ into $X$. In other words, $A$ is the infinitesimal generator of $\{S(t)\}_{t \in \mathbb{R}_0^+}$. As usual, $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers, i.e., $\mathbb{R}_0^+ := \{t \in \mathbb{R} | t \geq 0\}$.

We shall study a linear differential equation $\dot{x}(t) = Ax(t)$ for $t \in \mathbb{R}_0^+$. It is convenient to assume that the differential equation is satisfied in a weak sense. Therefore we shall consider a Cauchy problem with $x(0)$ not restricted to be in $\text{dom}(A)$ but being from the whole space $X$. All solutions of the Cauchy problem are given by $x(t) = S(t) x(0)$, $t \in \mathbb{R}_0^+$. They are called the mild solutions of $\dot{x}(t) = Ax(t)$, $x(0) \in X$, $t \in \mathbb{R}_0^+$. Details concerning the theory of linear differential equations in Banach spaces are presented, for instance in [4] and [9].

Now we are ready to make the following

**Definition 1.2.** It is said that a linear continuous-time system $\mathcal{G}(A)$ is given, if the mild solutions of the linear differential equation

$$\dot{x}(t) = Ax(t)$$

are considered for all $t \in \mathbb{R}_0^+$.

Using the semigroup property of $\{S(t)\}_{t \in \mathbb{R}_0^+}$, we can easily get the following (see also [14])
Proposition 1.4. Let $\tau > 0$ be a given number and $X := \{x_k\}_{k=0}^\infty$ be a sequence of elements of $X$. Assume $A := S(\tau)$. Then the sequence $X$ satisfies the linear difference equation $x_{k+1} = Ax_k$ for $k \in \mathbb{Z}_+^0$ if and only if $x_k = \bar{x}(k\tau)$ for all $k \in \mathbb{Z}_0^+$, where $\bar{x}(t) = S(t) x(0)$ denotes the (mild) solution of $\dot{\bar{x}}(t) = A \bar{x}(t)$, $t \in \mathbb{R}_0^+$, with $\bar{x}(0) := x_0$. \(\Box\)

The above result describes a natural correspondence between continuous-time and discrete-time systems. The correspondence can be used to study the problem of stability of continuous-time systems; it will be done in Section 3.
2. STABILITY OF DISCRETE-TIME SYSTEMS

This section provides an easily accessible and complete account of some of the most fundamental results in stability theory of linear discrete-time systems. We begin with the important definition of power stability. A theorem characterizing this notion is given. The presented proof of the mentioned theorem makes use of the spaces of germs of holomorphic functions which have been defined in the previous section. Next, the concept of $\ell^p$-stability is introduced and studied. It is shown that $\mathcal{G}(A)$ is $\ell^p$-stable if and only if the spectral radius of $A$ is less than 1. Whereas the result is (at least for $p = 2$) well known, the proof given in the text seems to be new. It is based on the fact that if $r(A)$ is less than 1, one can find a new norm $|\cdot|$ on $X$ which is equivalent to the original norm of $X$ and such that $|A|$ is less than 1. As a consequence of the obtained results we note that power stability and $\ell^p$-stability are equivalent notions. Some remarks supply the main text.

The concept of power stability (which will be defined below) is a discrete-time counterpart of the more common notion of exponential stability of a continuous-time system. Thus, the following definition is quite natural.

**Definition 2.1.** A linear discrete-time system $\mathcal{G}(A)$ is said to be **power stable** if for all $x_0 \in X$ there exist $M \geq 1$ and $0 < r < 1$ such that

$$\|x_k\| \leq Mr^k$$

for all $k \in \mathbb{Z}_0^+$. Here $\{x_k\}_{k=0}^{\infty}$ is defined by $x_{k+1} = Ax_k$, $k \in \mathbb{Z}_0^+$, i.e.,

$$x_k = A^k x_0.$$
The theorem given below links the introduced above concept of power-stability and spectral properties of the operator $A \in L(X)$ which defines $G(A)$. More precisely, we have

**Theorem 2.1.** Let a linear discrete-time system $G(A)$ be given. The system is power stable if and only if $\sigma(A)$, the spectrum of $A$, is contained in the unit disc $D_1$ of $\mathbb{C}$, i.e., the inequality $r(A) < 1$ takes place.

**Proof.** Let $G(A)$ be power stable. Then, as easily can be checked, the linear operator $(I - sA)$ is injective for all $s \in \overline{D}_1$. We wish to show that it is a surjective operator. For this, let $\hat{X}(z)$ be the formal Laplace transform of a sequence $X := \{x_k\}_{k=0}^{\infty}$ where, for every $k \in \mathbb{Z}_0^+$, $x_k$ is defined by the difference equation $x_{k+1} = Ax_k$. In accordance with Proposition 1.1, we have the following equality of formal power series:

$$(I - zA) \hat{X}(z) = x_0.$$ 

Having assumed $G(A)$ to be power-stable we can note that $\hat{X}(z)$ defines an element of $H(\overline{D}_1; X)$. Trivially, the formal power series $(I-zA)$ defines an element of $H(\overline{D}_1; L(X))$. Thus by Proposition 1.2, for all $s \in \overline{D}_1$ we have

$$(I - sA) \left( \sum_{k=0}^{\infty} s^k x_k \right) = x_0.$$ 

Of course, we have the same equality as above for all $x_0 \in X$. Hence, $(I - sA)$ is surjective for all $s \in \overline{D}_1$. Now the desired inclusion $\sigma(A) \subset D_1$ is obvious.

To prove that $r(A) < 1$ implies that $G(A)$ is power stable note that, by Proposition 1.3, $R_A(z) := (I - zA)^{-1} \in L(X)[[z]]$ defines a germ of
holomorphic functions on $D_1$. Thus, for all $x_0 \in X$, $R_A(z)x_0$ is a Hurwitz formal power series, i.e., for every $x_0 \in X$ there exist $M \geq 1$ and $r$, $0 \leq r < 1$, such that $\|x_k\| (= \|A^k x_0\|) \leq Mr^k$ for all $k \in \mathbb{Z}_0^+$. \hfill \Box

Remark 2.1. It is possible to give a simple direct proof of the fact that the condition $r(A) < 1$ implies power stability of $S(A)$. However, we feel that the reasoning given above is more natural in the context of the rest of the proof of Theorem 2.1.

In order to state an important lemma we need to define $\ell^p(X)$ for $1 \leq p < \infty$. Thus, by $\ell^p(X)$ we shall denote the Banach space of all sequences $(x_k)_{k=0}^{\infty}$ such that for all $k \in \mathbb{Z}_0^+$, $x_k \in X$ and the sequence of numbers $(\|x_k\|^p)_{k=0}^{\infty}$ is summable. The norm of $X = \{x_k\}_{k=0}^{\infty} \in \ell^p(X)$ is

$$
\|x\|_{\ell^p} := \left( \sum_{k=0}^{\infty} \|x_k\|^p \right)^{1/p}.
$$

We have the following

Lemma 2.1. The conditions below are equivalent:

(i) For every $p$, $1 \leq p < \infty$, and for every $x \in X$, $(A^kx)_{k=0}^{\infty} \in \ell^p(X)$.

(ii) There exists $p$, $1 \leq p < \infty$, such that for every $x \in X$, $(A^kx)_{k=0}^{\infty} \in \ell^p(X)$.

(iii) The spectral radius of $A$, $r(A)$, is less than 1.

(iv) There exist $M \geq 1$ and $0 \leq r < 1$ such that $\|A^k\| \leq Mr^k$ for every $k \in \mathbb{Z}_0^+$.

Proof. The only nontrivial step is to show that (ii) implies (iii). As it is well known, for any given $\varepsilon > 0$ we can find a new norm $\| \cdot \|$ on $X$
so that the inequality $|A| \leq r(A) + \varepsilon$ takes place and the norms $\| \cdot \|$ and $| \cdot |$ are equivalent. Since $r(A)$ is not greater than $|A|$, we shall try to find a new norm $| \cdot |$ (equivalent with $\| \cdot \|$) such that the inequality $|A| < 1$ will hold.

Assuming (ii), we have fixed a number $p$, $1 \leq p < \infty$, such that for every $x \in X$, $(A^k x)_{k=0}^{\infty} \in \ell^p(X)$. Let

$$|x|_p := \left( \sum_{k=0}^{\infty} \| A^k x \|^p \right)^{1/p}$$

It is a well-defined norm on $X$. In fact, $X$ equipped with the norm $| \cdot |_p$ is linearly isometric with the linear subspace $L_A$ of $\ell^p(X)$, where

$$L_A := \left\{ x \in \ell^p(X) \mid x = (A^k x)_{k=0}^{\infty} \text{ for some } x \in X \right\}.$$

We want to show that $L_A$ is closed in $\ell^p(X)$. To prove this assertion assume that $x := (x_k)_{k=0}^{\infty} \in \ell^p(X)$ is in the closure of $L_A$. Then there exists a sequence $(x^j_k)_{j=0}^{\infty}, x^j \in L_A$, such that $\lim_{j \to \infty} \| x^j - x \| = 0$. Of course, $x^j = (A^k x^j)_{k=0}^{\infty}$ for some $x^j \in X$.

Hence, $\lim_{j \to \infty} \left( \sum_{k=0}^{\infty} \| A^k x^j - x_k \|^p \right) = 0$. In particular, for all $k \in \mathbb{Z}_0^+$,

$$\lim_{j \to \infty} \| A^k x^j - x_k \| = 0.$$

For $k = 0$ we have $\lim_{j \to \infty} \| x^j - x_0 \| = 0$. This equality implies $\lim_{j \to \infty} \| A^k x^j - A^k x_0 \| = 0$ for every $k \in \mathbb{Z}_0^+$. Now,

$$\| A^k x_0 - x_k \| = \| A^k x_0 - A^k x^j + A^k x^j - x_k \| \leq \| A^k x_0 - A^k x^j \| +$$

$$+ \| A^k x^j - x_k \|.$$

It implies that $x_k = A^k x_0$ for all $k \in \mathbb{Z}_0^+$, i.e., $x = (x_k)_{k=0}^{\infty} \in L_A$ and $L_A$ is a closed subspace of $\ell^p(X)$. 
To end the proof note that for every $x \in X$ we have the obvious inequality $\|x\| \leq |x|_p$. By the interior mapping principle (see e.g. [8, Chap. II, Sec. 2]) there exists a positive number $a$ such that, for every $x \in X$, $|x|_p \leq a\|x\|$. Evidently, it can be assumed that $a$ is greater than 1. Now, we have

$$|Ax|_p = |x|_p - \|x\| \leq |x|_p - a^{-1}|x|_p = (1 - a^{-1})|x|_p.$$ 

The above inequality implies $|A|_p < 1$ and, in consequence, $r(A) < 1$. 

**Remark 2.2.** There is another interesting way to prove Lemma 2.1. It proceeds as follows. Assuming (ii), we have that $S : X \to L^p(X)$, $Sx := \{A^k x\}_{k=0}^\infty$, is a well defined linear mapping. It can be checked that $S$ is a closed operator. Since the domain of $S$ is the whole space $X$, $S \in L(X)$ by the closed graph theorem. Thereby, there exists a positive number $a$ such that, $\|Sx\|_p \leq a\|x\|$. Since $\|Sx\| = |x|_p$, we get $|x|_p \leq a\|x\|$, i.e., the same inequality as in the final part of the proof of Lemma 2.1. It should be noted that the calculations which prove that $S$ is a closed operator are exactly the same as these given in the proof of Lemma 2.1 to show that $L_A$ is a closed subspace of $L^p(X)$.

Use, in a similar context, of an operator which plays an analogous role in a continuous-time system theory as $S$ for discrete-time systems has been suggested by A. Pazy (see [12, p. 293]).

The concept of $L^p$-stability (which will be defined below) is a discrete-time counterpart of the widely used notion of $L^p$-stability of a continuous-time system. For $p = 2$ and $X$ being a Hilbert space, the importance of the notion of $L^p$-stability is well recognized; see e.g. [11], [14], [16].
Definition 2.2. Let $p, 1 \leq p < \infty$, be given. A linear discrete-time system $G(A)$ is said to be $\ell^p$-stable if for all $x_0 \in X$ the sequence $(x_k)_{k=0}^{\infty}$ belongs to $\ell^p(X)$. Here $(x_k)_{k=0}^{\infty}$ is defined by $x_{k+1} = A x_k$, $k \in \mathbb{Z}_0^+$, i.e., $x_k = A^k x_0$.

As a trivial consequence of Lemma 2.1 and the definition of $\ell^p$-stability we note the following

Theorem 2.2. The conditions below are equivalent:

(i) For every $p, 1 \leq p < \infty$, $G(A)$ is $\ell^p$-stable.

(ii) There exists $p, 1 \leq p < \infty$, such that $G(A)$ is $\ell^p$-stable.

(iii) The spectral radius of $A$, $r(A)$, is less than 1.

(iv) There exist $M \geq 1$ and $0 \leq r < 1$ such that $\|A^k\| \leq Mr^k$ for every $k \in \mathbb{Z}_0^+$.

Remark 2.3. The equivalence between $\ell^p$-stability of $G(A)$ and the inequality $r(A) < 1$ has been established by J. Zabczyk (see [16, Sec.5]). His proof of the result is based on a technique quite different from ours and makes use of the Banach-Steinhaus theorem. It seems that the method presented in the proof of Lemma 2.1 is not only simpler, but also a more natural one. However, it should be noted that Zabczyk was able to prove a more general stability result.

By Theorems 2.1 and 2.2 we immediately get

Corollary 2.1. Let a linear discrete-time system $G(A)$ be given. It is power stable if and only if it is $\ell^p$-stable for some (for every) $p$, $1 \leq p < \infty$. 

\[\square\]
Remark 2.4. It is unnecessary to use the whole power of Theorem 2.1 to get the above corollary. In fact, a simple direct calculation shows that power stability implies $L^p$-stability. Hence, by Theorem 2.2, power stability implies the inequality $r(A) < 1$. Thus, to get Corollary 2.1 it is sufficient to use only the part of Theorem 2.1 which says that the inequality $r(A) < 1$ implies power stability. Note that the above reasoning provides a proof of the remaining (and more difficult) part of Theorem 2.1. However, the proof of Theorem 2.1 presented in the paper has its own value. The "frequency-domain" method suggested by this proof can be used to study various control-theoretic questions. In particular, a solution of the so-called stabilizability problem can be obtained by use of the mentioned method; see e.g. [15].

The proof of Lemma 2.1 shows that the corollary below is true.

**Corollary 2.2.** Let a linear discrete-time system $\mathcal{G}(A)$, which is $L^p$-stable, be given. Then the system defines a new norm $\| \cdot \|_p$ on $X$ which is given (for all $x \in X$) by

$$\| x \|_p := \left( \sum_{k=0}^{\infty} \| A^k x \|_p \right)^{1/p}$$

Moreover, the norms $\| \cdot \|$ and $\| \cdot \|_p$, are equivalent.

Remark 2.5. The statement of Corollary 2.2 is rather surprising in view of a result by A. Pazy. He showed (see [12, Thm.2]) that, without additional assumptions, an analogous result for continuous-time systems is false. More precisely, it is shown in [12] that the following result
holds: Let \( \{S(t)\} \) be a strongly continuous semigroup of linear and bounded operators from \( X \) into \( X \), \( 1 \leq p < \infty \) be fixed, and

\[
\|x\|_p := \left( \int_0^\infty \|S(t)x\|^p \, dt \right)^{1/p}
\]

be finite for every \( x \in X \). Then \( \| \cdot \|_p \) is a new norm on \( X \). The norms, \( \| \cdot \| \) and \( \| \cdot \|_p \), are equivalent if and only if there exist \( t_0 > 0 \) and \( c_0 > 0 \) such that

\[
\|S(t_0)x\| \geq c_0 \|x\|
\]

for all \( x \in X \). To prove that the last inequality is necessary for \( \| \cdot \| \) and \( \| \cdot \|_p \) to be equivalent norms, some estimates of the norm \( \| \cdot \|_p \) are made in [12]. They involve an analysis of small time intervals, \([0, t] \subset \mathbb{R}_0^+\). Clearly, we are not able to make something similar to it, if discrete-time systems are considered. This is a reason for which our Corollary 2.2 holds.
3. STABILITY OF CONTINUOUS-TIME SYSTEMS

In the previous section we presented stability theory for linear discrete-time systems. It turns out that this theory provides a very natural approach to the study of stability problems for continuous-time systems. These problems are to be studied in the present section.

We start with the well-known definition of exponential stability of a linear continuous-time system. Our first theorem of this section proves that exponential stability of a continuous-time system is equivalent with power stability of a suitable discrete-time system. Next, the important concept of $L^p$-stability is defined. In order to study $L^p$-stability we derive some inequalities between some norms on $X$; the norms are related to a discrete-time system. A theorem is given that expresses the fact that $L^p$-stability of a continuous-time system can be explored by an analysis of $\ell^p$-stability of a discrete-time system. Using Corollary 2.1 we get equivalence between exponential stability and $L^p$-stability for linear continuous-time systems.

A number of papers have been devoted to stability theory for linear continuous-time system; we mention only [1], [4], [5], [6], [12] and [13].

We begin with the following

**Definition 3.1.** A linear continuous-time system $G(A)$ is said to be *exponentially stable* if for all $x(0) \in X$ there exist $M \geq 1$ and $\gamma < 0$ such that

$$\|x(t)\| \leq M \exp(\gamma t)$$
for all \( t \in \mathbb{R}_0^+ \). Here \( z(t) \) is the corresponding to \( z(0) \) (mild) solution of \( \dot{z}(t) = Az(t) \), \( t \in \mathbb{R}_0^+ \), i.e., \( z(t) = S(t) z(0) \).

In order to prove Theorem 3.1 (to be given below) we need some notations. By \( \tau \) we shall denote a given positive number.

Now, let

\[
\alpha_\tau := \sup \{ \| S(0) \| \mid 0 \in [0, \tau] \}.
\]

It is well known that \( \alpha_\tau \) is finite. (The result follows from the principle of uniform boundedness [8, Chap.II, Sec. 1] and strong continuity of \( \{ S(t) \} \).) We shall also use the notation \( [t]_\tau \) for the entire part of \( t/\tau \). Note that \( t \in \mathbb{R}_0^+ \) implies \( [t]_\tau \in \mathbb{Z}_0^+ \). Let \( \Theta_\tau \in [0, \tau] \) be given by

\[
\Theta_\tau := t - \tau [t]_\tau.
\]

We will prove the following

**Theorem 3.1.** Let a linear continuous-time system \( \mathcal{S}(A) \) be given and \( \tau \) be a fixed positive number. Let \( A := S(\tau) \). Then the continuous-time system \( \mathcal{S}(A) \) is exponentially stable if and only if the discrete-time system \( \mathcal{S}(A) \) is power stable.

**Proof.** Using Proposition 1.4 we have the equality \( x_k = z(k\tau) \) for all \( k \in \mathbb{Z}_0^+ \); here \( z(t) = S(t) z(0) \) is the (mild) solution of \( \dot{z}(t) = Az(t) \), \( t \in \mathbb{R}_0^+ \), and \( x_{k+1} = Ax_k \), \( k \in \mathbb{Z}_0^+ \), with \( z(0) = x_0 \).

Assume \( \mathcal{S}(A) \) to be exponentially stable. Then, the inequality \( \| z(t) \| \leq M \exp(\xi t) \) is fulfilled for all \( t \in \mathbb{R}_0^+ \) and some \( M \geq 1, \xi < 0 \). Since the relation
holds for all \( k \in \mathbb{Z}_0^+ \), the condition of Definition 2.1 is satisfied with \( M := M \) and \( r := \exp(\xi) \). The evident inequality \( 0 \leq \exp(\xi t) < 1 \) proves that \( \mathcal{G}(A) \) is power stable.

Conversely, let \( \mathcal{G}(A) \) be power stable. Note that

\[
\|x(t)\| = \|x(t)\| = \|S(t)\| \leq a \|x(t)\|.
\]

Now the power stability of \( \mathcal{G}(A) \) enables us to write the inequality

\[
\|x(t)\| \leq a \|x(t)\| = M \exp(\xi t)
\]

which is valid for all \( t \in \mathbb{R}_0^+ \) and some \( M \geq 1, 0 \leq r < 1 \). We may assume that \( r \neq 0 \). Then, for all \( t \in \mathbb{R}_0^+ \),

\[
\|x(t)\| \leq M \exp(\xi t)
\]

where \( M := (a/\tau)M \) and \( \xi := (1/\tau) \ln(r) \). Clearly, the inequalities \( M \geq 1 \) and \( \xi < 0 \) are satisfied, and \( \mathcal{G}(A) \) is exponentially stable.

\[\square\]

**Remark 3.1.** The proof presented above suggests a simple way to prove the following well-known result: \( \mathcal{G}(A) \) is exponentially stable if and only if there exist \( M \geq 1 \) and \( \xi < 0 \) such that, for all \( t \in \mathbb{R}_0^+ \),

\[
\|S(t)\| \leq M \exp(\xi t).
\]

To show the nontrivial part of this result it is sufficient to note that, by Theorem 3.1 and Corollary 2.1, we may use condition (iv) of Theorem 2.2 for \( A := S(t) \). Then

\[
\|S(t)\| = \|S(\tau[t] + \theta[t])\| = \|S(\theta[t])\| \leq a \|x(t)\| \leq M \exp(\xi t)
\]

with \( M \) and \( \xi \) expressed by the same formulas as in the final part of the proof of Theorem 3.1. Hence, the desired estimate of \( \|S(t)\| \) is obtained.
The important concept of $L^p$-stability is defined as follows.

**Definition 3.2.** Let $p$, $1 \leq p < \infty$ be given. A linear continuous-time system $\mathcal{G}(A)$ is said to be $L^p$-stable if for all $\chi(0) \in \mathcal{X}$,

$$\int_0^\infty \|\chi(t)\|^p \, dt$$

is finite.

Here $\chi(t)$ is the corresponding to $\chi(0)$ (mild) solution of $\dot{\chi}(t) = A\chi(t)$, $t \in \mathbb{R}_0^+$, i.e., $\chi(t) = S(t) \chi(0)$.

The proposition and theorem given below are a slight generalization of [14, Prop. 9]. The notations are exactly the same as introduced before Theorem 3.1.

**Proposition 3.1.** Let a linear continuous-time system $\mathcal{G}(A)$ be given and, for a fixed positive number $\tau$, $A := S(\tau)$. For a given $x \in \mathcal{X}$, let $\{x_k\}_{k=0}^\infty$ denote the corresponding to $x_0 = x$ solution of the difference equation which describes the discrete-time system $\mathcal{G}(A)$, and let $\chi(t)$, $t \in \mathbb{R}_0^+$, be the corresponding to $\chi(0) = x$ (mild) solution of the differential equation which defines $\mathcal{G}(A)$. Then, the numbers $\int_0^\infty \|\chi(t)\|^p \, dt$ and $\sum_{k=0}^\infty \|x_k\|^p$ are simultaneously finite or infinite. Moreover, the following inequality holds. Here $p$, $1 \leq p < \infty$, is a fixed number.

$$(1/\alpha \tau)^p \int_0^\infty \|\chi(t)\|^p \, dt \leq \tau \sum_{k=0}^\infty \|x_k\|^p \leq \alpha \tau^p \int_0^\infty \|\chi(t)\|^p \, dt + \tau \|x\|^p$$

An immediate consequence of Proposition 3.1 is the following interesting
Theorem 3.2. Let a linear continuous-time system $\mathcal{G}(\mathcal{A})$ be given and $\tau$ be a fixed positive number. Let $A := \mathcal{S}(\tau)$. Then the continuous-time system $\mathcal{G}(\mathcal{A})$ is $L^p$-stable if and only if the discrete-time system $\mathcal{G}(\mathcal{A})$ is $L^p$-stable. Here $p$, $1 \leq p < \infty$, is a fixed number.

Proof of Proposition 3.1. From Proposition 1.4 we have the equality

$$x_k = x(k\tau) \text{ for all } k \in \mathbb{Z}_0^+.$$  

Now, note that $x(t) = \mathcal{S}(\theta_t) \mathcal{S}(t\tau) x(0) = \mathcal{S}(\theta_t) A^{t\tau} x$. Hence, for all $t \in \mathbb{R}^+$, we have the inequality $(1/\alpha_{\tau})^p \|x(t)\|^p \leq \|A^{t\tau} x\|^p$. Assuming that $\sum_{k=0}^{\infty} \|x_k\|^p$ is finite and noting that

$$\int_0^{\infty} \|A^{t\tau} x\|^p \, dt = \tau \sum_{k=0}^{\infty} \|x_k\|^p$$

we get immediately the desired inequality

$$(1/\alpha_{\tau})^p \int_0^{\infty} \|x(t)\|^p \, dt \leq \tau \sum_{k=0}^{\infty} \|x_k\|^p.$$  

Clearly, the number $\int_0^{\infty} \|x(t)\|^p \, dt$ is finite.

To prove the remaining part of Proposition 3.1 note that for every $t \in \mathbb{R}_0^+$ we can write the following equality: $\tau(t\tau + 1) = (\tau - \theta_t) + t$. Noting that $(\tau - \theta_t) \in [0, \tau]$ we get for all $t \in \mathbb{R}_0^+$

$$\|A^{(t\tau + 1)} x\|^p = \|\mathcal{S}(\tau(t\tau + 1)) x\|^p = \|\mathcal{S}(\tau - \theta_t) \mathcal{S}(t) x\|^p \leq \alpha_{\tau}^p \|x(t)\|^p.$$  

Now, assuming that $\int_0^{\infty} \|x(t)\|^p \, dt$ is finite we have the inequality

$$\tau \sum_{k=1}^{\infty} \|x_k\|^p \leq \alpha_{\tau}^p \int_0^{\infty} \|x(t)\|^p \, dt.$$
Evidently, the number $\sum_{k=0}^{\infty} \|x_k\|^p$ is finite.

**Remark 3.1.** Let $\mathcal{G}(A)$ be $L^p$-stable and $A := S(\tau)$. Let, for every $x \in X$,

$$\|x\| := \left( \int_0^\infty \|S(t)x\|^p \, dt \right)^{1/p}.$$  

It is a new norm on $X$. Using Proposition 3.1 and Theorem 3.2 we have the following inequality

$$(1/\alpha)^p \|x\|^p \leq \tau |x|^p \leq \alpha^p \|x\|^p + \tau \|\|x\|^p$$

between the norms $\| \cdot \|$, $\| \cdot \|^p$ and $| \cdot |^p$. The last norm depends on $\tau$. The inequality above suggests the following question:

*Are the norms $\| \cdot \|$ and $| \cdot |^p$ equivalent norms on $X$?* To answer this question note that by Corollary 2.2, the norms $\| \cdot \|$ and $\| \cdot \|^p$, are equivalent. On the other hand, a result by Pazy (see our Remark 2.5) says that the norms $\| \cdot \|$ and $\| \cdot \|^p$ are equivalent if and only if there exist $t_0 > 0$ and $c_0 > 0$ such that $\|S(t_0)x\| \geq c_0 \|x\|$ for all $x \in X$. Thus the following result holds: For a given $L^p$-stable continuous-time system $\mathcal{G}(A)$, the norms $\| \cdot \|^p$ and $| \cdot |^p$ are equivalent if and only if there exist $t_0 > 0$ and $c_0 > 0$ such that

$$\|S(t_0)x\| \geq c_0 \|x\|$$

for all $x \in X$.

As a consequence of Corollary 2.1, Theorem 3.1 and Theorem 3.2 we note the following well-known (see [6], [12] and [14])
Corollary 3.1. Let a linear continuous-time system $G(s)$ be given. It is exponentially stable if and only if it is $L^p$-stable for some (for every) $p, 1 \leq p < \infty$. \qed
References


