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Remark on a paper by J.W.P. Hirschfeld

by

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1. Introduction

In the paper: Ovals in Desarguesian Planes of Even Order [1] Hirschfeld shows (theorem 4, corollary 1) that there exist no more projectively distinct ovals with representation $D(k)$ than $D(2), D(4)$ and $D(6)$. Below we shall show that the ovals, thus represented, are indeed mutually projectively distinct.

2. Preliminary considerations

In this paper we shall restrict ourselves to ovals $O$ in $PG(2,32)$ which have a representation of the form $D(k)$ (for definitions and notation see [1], p. 79).

Representations $D(k)$ of an oval $O$ in $PG(2,32)$ depend completely on the frame used to define a coordinate system in $PG(2,32)$. We shall only use frames $(P,Q,R,S)$ with $P = (0,0,1), Q = (0,1,0), R = (1,0,0), S = (1,1,1)$ where $P, Q, R$ and $S$ all are points on the oval $O$ under consideration.

Let $O$ be an oval with representation $D(k)$ for a suitable frame $(P,Q,R,S)$. Let $a \in \gamma_0$ ($\gamma = GF(32)$). Since

$$\{(1,t,t^k) \mid t \in \gamma\} = \{(1,a^{-1}t,(a^{-1}t)^k) \mid t \in \gamma\} = \{(1,a^{-1}t,a^{-k}t^k) \mid t \in \gamma\}.$$

$D(k)$ is also the representation of $O$ for any frame $(P,Q,R,S')$ with $S' \not\in \{P,Q,R\}$, $S'$ on $O$ (for the frame $(P,Q,R,S)$: $S' = (1,a,a^k)$).

If $O$ has a representation as a translation oval and this representation is $D(k)$ then by definition ([1], p. 79) we have for $a \in \gamma$:

$$\{(1,t,t^k) \mid t \in \gamma\} = \{(1,t+a),(t+a)^k) \mid t \in \gamma\} = \{(1,t+a,t^ka^k) \mid t \in \gamma\}$$

so if $D(k)$ is the representation of $O$ for some frame $(P,Q,R,S)$ it is also the representation of $O$ for any frame $(P,Q,R',S)$ with $R' \not\in O\{P,Q,S\}$ and with (1) for any frame $(P,Q,R',S')$ with $\{R',S'\} \subset O\{P,Q\}$ ($R' = (1,a,a^k)$).

If $O$ has representation $D(2)$ for some frame $(P,Q,R,S)$ then $O$ has also representation $D(2)$ for the frame $(R,Q,P,S)$ since

$$\{(1,t,t^2) \mid t \in \gamma_0\} = \{(1,t^{-1},t^{-2}) \mid t \in \gamma_0\} = \{(t^2,t,1) \mid t \in \gamma_0\}$$

and the transformation $(x_0,x_1,x_2) \rightarrow (x_2,x_1,x_0)$ belongs to the transition

*) field automorphisms $\phi$ of $\gamma$ have the form $\phi: t \rightarrow t^j, 1 \leq j \leq 30$, and

$$D(k) = \{(1,t,t^k) \mid t \in \gamma\} \cup \{(010),(001)\} = \{(1,t^j,(t^j)^k) \mid t \in \gamma\} \cup \{(010),(001)\} = \{(1,t^j,t^{k-j}) \mid t \in \gamma\} \cup \{(010),(001)\}.$$
from frame \((P,Q,R,S)\) to frame \((R,Q,P,S)\). From (1) and (2) and the fact that 
\[(x+y)^2 = x^2 + y^2\] for all \(x\) and \(y\) in \(\gamma\) we can conclude that \(O\) has represen-
tation \(D(2)\) for any frame \((P',Q,R',S')\) on \(O\).  
\[(3)\]
A transformation will always work on coordinates, not on the points them-
selves.

3. **Representations equivalent to \(D(2)\)**

Let \(O\) be an oval with representation \(D(k)\) for the frame \((P_k,Q_k,R_k,S_k)\) on \(O\) and representation \(D(2)\) for the frame \((P_2,Q_2,R_2,S_2)\) on \(O\). We shall show that exactly one of the following three cases holds

\[(4)\]

i) \(k = 2, Q_k = Q_2\) and \(D(k)\) is the representation of \(O\) for any frame 
\((P'_k,Q'_k,R'_k,S'_k)\) on \(O\),

ii) \(k = 16, R_k = Q_2\) and \(D(k)\) is the representation of \(O\) for any frame 
\((P'_k,Q'_k,R'_k,S'_k)\) on \(O\),

iii) \(k = 30, P_k = Q_2\) and \(D(k)\) is the representation of \(O\) for any frame 
\((P'_k,Q'_k,R'_k,S'_k)\) on \(O\).

Now assume \(Q_k \neq Q_2\). Let \(Q'_k \in O \setminus \{P_k,R_k,Q_2\}\). We can choose \(R\) and \(S\) on \(O\) and, if necessary, change \(S_k\) (with (1)) in such a way that \(\{P_k,R_k,S_k\} = \{Q_2,R,S\}\). Now of course \(\{Q_k,Q'_k,Q_2\} \cap \{R,S\} = \emptyset\). We have: For the frames 
\((Q_k,Q_2,R,S)\) and \((P_k,Q_k,R_k,S_k)\) the oval \(O\) has representations \(D(2)\) and \(D(k)\), respectively.

The transformation \(T\) which satisfies for the frame \((Q_k,Q_2,R,S)\) 
\(T_p = (001), T_q = (010), T_R = (100), T_S = (111)\) has also \(TD(2) = D(k)\) 
since \(O\) has representation \(D(k)\) for the frame \((P_k,Q_k,R_k,S_k)\).

For the frame \((Q'_k,Q_2,R,S)\) the oval \(O\) has representation \(D(2)\) (by (3)) and 
\(P_k, R_k\) and \(S_k\) have the same coordinates as for the frame \((Q_k,Q_2,R,S)\). So, 
for \((Q'_k,Q_2,R,S)\), \(T(P_k,Q'_k,R_k,S_k) = ((001),(010),(100),(111))\). For this frame 
\(O\) has representation \(TD(2) = D(k)\). So we can conclude:

If \(O\) has representation \(D(k)\) for some frame \((P_k,Q_k,R_k,S_k)\) and \(Q_k \neq Q_2\) 
then \(O\) has this representation for any frame \((P_k,Q'_k,R_k,S_k)\) on \(O\) with 
\(Q'_k \neq Q_2\).

The same argument for the cases \(P_k \neq Q_2\) and \(R_k \neq Q_2\) leads to a similar con-
clusion for \(P_k\) and \(R_k\).
Now consider the following cases:

i) $P_k \neq Q_2$ and $R_k \neq Q_2$. Now we are free to choose $P_k' = R_k$ and $R_k' = P_k$ and know that $0$ has representation $D(k)$ for the frame $(P_k', Q_k', R_k', S_k')$. This yields:

$$\forall t \in Y_0 \exists s \in Y_0 [t = (s^k, s, 1)], \quad \forall s \in Y_0 [s^k = s^{k}]$$

so $k(1-k) \equiv -k (mod 31)$, so $k = 0$ or $k = 2$.

Since $D(0)$ represents no oval we find $k = 2$.

ii) $P_k \neq Q_2$ and $Q_k \neq Q_2$. We choose $P_k' = Q_k$ and $Q_k' = P_k$ and get:

$$\forall t \in Y \exists s \in Y [t = (s^k, s, 1)], \quad \forall s \in Y [s^2 = s]$$

so $k^2 \equiv 1 (mod 31)$ so $k = 1$ or $k = 30$.

Since $D(1)$ represents no oval we find $k = 30$ (with [1], theorem 1, p. 81).

But this result excludes that of i) so we may conclude: $R_k = Q_2$ and in case i): $Q_k = Q_2$.

Now the only other possibility is:

iii) $P_k = Q_2$. We choose $Q_k = R_k$ on $R_k' = Q_k$ we find:

$$\forall t \in Y_0 \exists s \in Y_0 [t = (s^k, s, 1)], \quad \forall s \in Y_0 [s^{k-1} = s^{k-1}]$$

so $-k \equiv k - 1 (mod 31)$ so $k = 16$.

To prove (4) we still have to show that $D(16)$ and $D(30)$ are indeed representations of $0$. This follows in exactly the same manner:

$$\{(1, t, t^2) | t \in \gamma\} = \{(1, t^{16}, t) | t \in \gamma\}$$

so if $D(2)$ is the representation of $0$ for $(P, Q, R, S)$ then $D(16)$ is the representation for $(Q, P, R, S)$ and:

$$\{(1, t, t^2) | t \in Y_0\} = \{(t^{-1}, 1, t) | t \in Y_0\} = \{(t^{30}, 1, t) | t \in Y_0\}$$

so if $D(2)$ is the representation of $0$ for the frame $(P, Q, R, S)$ then $D(30)$ is the representation of $0$ for the frame $(R, P, Q, S)$. q.e.d.

With (4) we have directly: The only representations $D(k)$ equivalent with $D(2)$ are: $D(2)$, $D(16)$ and $D(30)$. 
4. The relation between the representations $D(4)$ and $D(6)$

In this section we shall show that the assumption $D(4) \sim D(6)$ leads to a contradiction.

Assume that there exists an oval $\mathcal{O}$ with representations $D(4)$ and $D(6)$ for the frames $(P_4,Q_4,R_4,S_4)$ and $(P_6,Q_6,R_6,S_6)$ respectively. We distinguish two cases:

i) $\{P_4,Q_4\} \cap \{P_6,Q_6,R_6\} \neq \emptyset$.

With (1) and (4) we can choose $R_4$, $S_4$ and $S_6$ such that $\{P_4,Q_4,R_4,S_4\} = \{P_6,Q_6,R_6,S_6\}$.

Let $T$ be the transformation with respect to $(P_4,Q_4,R_4,S_4)$ with $TP_6 = P_4$, $TQ_6 = Q_4$, $TR_6 = R_4$, $TS_6 = S_4$ and $TD(4) = D(6)$. Now $R_4 \neq S_6$ or $S_4 \neq S_6$, say $R_4 \neq S_6$.

Then $P_6 = R_4$ or $Q_6 = R_4$ or $R_6 = R_4$, say $P_6 = P_4$.

Let $P_6' \in \mathcal{O} \setminus \{P_6,Q_6,R_6,S_6\}$. Now $(P_4,Q_4,P_6',S_4)$ is a frame for which $\mathcal{O}$ has representation $D(4)$. Also for this frame $T(P_6',Q_6,R_6,S_6) = (001),(101),(100),(111)$ and $\mathcal{O}$ has representation $TD(4) = D(6)$ for this frame. This implies:

If $P_6 = P_4$ then for any frame $(P_6',Q_6,R_6,S_6)$ $\mathcal{O}$ has representation $D(6)$.

By applying the same argument we get:

If $P_6 (Q_6$ or $R_6) \in \{R_4,S_4\}$ then $\mathcal{O}$ has representation $D(6)$ for any frame $(P_6',Q_6,R_6,S_6)$ on $\mathcal{O} ((P_4,Q_4,P_6',S_4)$ or $(P_6,Q_6,R_6',S_6)$ resp.). The conclusion is that in the frame that yields representation $D(6)$ for $\mathcal{O}$ we can choose besides $S_6$ (with (1)) at least one other point arbitrarily.

i) If this point is $R_6$ then we can transform via $(P_6,Q_6,R_6,S_6) \leftrightarrow (P_6,Q_6,R_6',S_6)$ and we get:

$$\forall_{t \in Y_{01}} \exists_{s \in Y_{01}} [(1,t,t^6) = (1,1+s,1+s^6)], \text{ so } \forall_{s \in Y_{01}} [(1+s)^6 = 1+s^6],$$

so $\forall_{s \in Y_{01}} [s^2 + s^4 = 0]$ and this is not true.

i2) If this point is $P_6$, we transform $(P_6,Q_6,R_6,S_6) \leftrightarrow (S_6,Q_6,R_6,P_6)$ and find:

$$\forall_{t \in Y_{1}} \exists_{s \in Y_{1}} [(1,t,t^6) = (1+s^6,s+s^6,s^6)],$$

so $\forall_{s \in Y_{1}} [(1+s^5)^6 = (1+s^5)^6]$.

But since $(1+s^5)^6 + (1+s^6)^5$ is a polynomial of degree less than 31 and $(1+s^5)^6 + (1+s^6)^5 = s^6 + s^{10} + s^{20} + s^{24} \neq 0$ this cannot be true.
i3) If this point is $Q_6$ then we transform via $(P_6, Q_6, R_6, S_6) \rightarrow (P_6, S_6, R_6, Q_6)$ and find:

$$v \in \gamma \exists s \in \gamma \ [(1, t, t^6) = (1+s,s,s+s^6)], \text{ so } v \in \gamma \ [(s)_{66}^6 = \frac{s+s^6}{1+s}].$$

So we find: the assumption $(P_4, Q_4) \cap (P_6, Q_6, R_6) \neq \emptyset$ leads to a contradiction.

ii) $(P_4, Q_4) \cap (P_6, Q_6, R_6) = \emptyset$.

We choose $S_6 = P_4$, $S_4 = P_6$, $R_4 = R_6$ and have, for the frame $(P_4, Q_4, R_4, S_4)$:

$Q_6 = (1, a, a^4)$ for some $a \in \gamma_01$. The transformation $T$:

$$(x_0, x_1, x_2) \rightarrow ((a+a^2+a^3)x_0 + (1+a+a^2+a^3)x_1 + x_2, x_1 + x_2, x_1 a^3 + x_2)$$

transforms $P_6$ in $(0,0,1)$, $R_6$ in $(1,0,0), Q_6$ in $(0,1,0)$ and $S_6$ in $(1,1,1)$.

So it must transform $D(4)$ in $D(6)$. But then we must have $TQ_4 \in D(6)$, so:

$$TQ_4 = (1+a+a^2+a^3, 1, a^3) = (1, s, s^6), \text{ for some } a \in \gamma, \text{ so}$$

$$\left(\frac{1}{1+a^4+a^3}\right)^6 = a^3, \text{ so } \frac{1+a}{1+a^4} = \frac{a^3(1+a)}{1+a}$$

so $(1+a^4)^5 a^3 = (1+a)^5$, so $(1+a^4)^4 a^3 = 1+a$, so $(1+a^4)^3 a^6 = (1+a)^2$, so $a^6 = 1+a$.

But $a^6 + a + 1$ is an irreducible polynomial over $GF(2)$ so $a \in \gamma \cap GF(2^6) = GF(2)$ and this leads to the desired contradiction since $a^6 + a = 0$ for $a \in GF(2)$.

We have now that $D(4) \sim D(6)$ leads to a contradiction and conclude, with [2], theorem 12, corollary 1, p. 790 that there are exactly three projectively distinct $D(k)$ over $GF(32)$ i.e. $D(2), D(4)$ and $D(6)$.

References
