Control of dynamic 
sampled-data systems 
with frequency aliasing

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Abstract

In control of dynamic systems more and more use is made of sampled-data control loops. Controls are implemented in digital computers, which are also used to digitize signals. As a result of this digitalization, frequency aliasing can occur in the signals used in the control loop. This report focusses on the theoretical background of aliasing and the influence of aliased signals in dynamic sampled-data control systems.

A theoretical examination of the sampling of signals and frequency aliasing is given. Next several new methods to detect and reconstruct aliased signals are briefly mentioned. A method is introduced to derive a discrete model for the continuous part of a sampled-data control system. This discrete model takes the aliasing error, which occurs in the control system, into account. Properties of this model are shown, using an example system. Finally this model is integrated in a model for the complete sampled-data control system.

The model developed here can be used for further research in the area of frequency aliasing and control of dynamic sampled-data systems.
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Chapter 1

Introduction

When dealing with control of dynamical systems, signals are of great importance. Signal analysis, for example, plays an important role in frequency response measurement, to get a better view of the characteristics of a system. The last decades more and more use has been made of digital equipment to control dynamical systems. Digital computers are used to sample and digitize analog signals. These signals are then led into a control-loop to control a dynamical analog system.

In digital signal analysis there are several error-sources due to sampling of analog signals. One source of errors is related to the so-called Aliasing. The goal of this report is to give more insight in the theoretical background of aliasing and the influence of aliased signals in sampled-data control systems.

First, a basic sampled-data control system will be discussed. The theory for sampling of signals is summarized. Next the occurrence and background of aliasing is described. Several methods to detect and reconstruct aliased signals are briefly mentioned. Furthermore the relationship between aliasing and digital control systems is discussed. After that a method is introduced to derive a discrete model for the continuous part of a digital control loop. This so-called aliased frequency response function (AFRF) includes aliased frequency effects into the system description. An example system is used to show some of the properties of the AFRF. Finally a transfer function for a digital control loop is derived in which the AFRF is taken into account. This transfer function can be used in further research to investigate the influence of aliasing on controllers based on loopshaping.
Chapter 2

Theory for Frequency Aliasing

2.1 Basic sampled-data control system

In figure 2.1 a block scheme of a basic sampled-data control system is drawn. In this digital control loop two parts can be distinguished. The D/A converter, analog system and sensor form the analog, continuous-time part of the system. The A/D converter and digital controller form the digital, discrete-time part of the system. A discrete-time signal $u(n\Delta T)$ is transformed into a continuous-time signal $u(t)$ by the D/A converter. This continuous signal is then send as an input to an analog mechanical system.
The continuous output of the system $y(t)$ is measured with a sensor and sent to the digital, discrete-time part of the control loop. The continuous signal is transformed into a discrete signal $y(n\Delta T)$ by an A/D converter. After comparison with an input $r(n\Delta T)$, which is the reference signal, a control action is computed by the digital controller and this control action is sent again into the analog, continuous-time part of the system. In order to transform a continuous-time signal into a discrete-time signal, use is made of the so-called sampling of signals. The theory for discrete-time signal analysis and the sampling operation will be treated in the next paragraph. In sampled-data control systems the A/D and D/A conversion is part of the digital controller, which is implemented in a digital computer.

2.2 Discrete-time Signal Analysis

2.2.1 Fourier Transformation

When dealing with periodic signals one usually uses a Fourier series approach to write the signal as an (infinite) sum of harmonic signals with angular frequency $\omega_n := n(2\pi/T)$. This can be written as:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t} \quad (n = 0, \pm 1, \pm 2, ...)
$$

with $x(t)$ a periodic signal with period $T$. A derivation of this result is given in [1] and [2].

In practice however usually non-periodic signals are encountered. In that case one should use the Fourier-integral transformation. An expression for this transformation is given in [1] and [2]. The transformation can be seen as the limit of the Fourier-series approach for a period time going to $\infty$. The Fourier-integral transformation is defined as:

$$X(f) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-2\pi jft} dt
$$

$$x(t) = F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) e^{2\pi jft} df$$

(2.2)
In the above described equations use has been made of a signal \( x(t) \) which is continuous in time. This signal is defined for \(-\infty < t < \infty\) but only the part \(0 \leq t \leq T\) is taken into account. In a sampled-data system however the signals are often digitized by an analog-to-digital converter. In that case only a discretised version of the signal is used. In other words, in sampling of a system one only uses the signal values or amplitudes of a continuous signal at a limited number of equidistant points of time (with fixed timestep \( \Delta T \)).

In the Discrete Fourier Transformation (DFT) the discretization of the signal is taken into account. The DFT for the discretised function \( x(t) \) is defined as:

\[
X[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-2\pi jkn/N}, \quad n = 0, 1, 2, \ldots N - 1 \tag{2.3}
\]

This result is also given in [2]. From the above definitions one can conclude that there are two major sources of errors in the numerical calculation of Fourier transforms. Firstly a signal \( x(t) \) is used which is defined for \(-\infty < t < \infty\) but in the Discrete Fourier Transformation only a (small) portion of the signal in the window \(0 \leq t \leq T\) is considered. It is obvious to conclude that this can cause errors in the digital result. This kind of error is called \textit{Signal Leakage}. Secondly a discretised sample of a continuous signal part is used in the DFT calculation. This sample only consists of a limited number of discrete function-values. As a result of this an error will occur which is called \textit{Aliasing}. In the remaining part of this report aliasing will be treated in more detail. In [1] and [2] derivations of the equations used in this paragraph are given.
2.2.2 Sampling of a signal

It is mentioned earlier that an originally continuous-time signal $x(t)$ can be sampled on the discrete time points $t_i = n\Delta T$. One can define a sampling frequency $f_s$:

$$f_s = \frac{1}{\Delta T} \text{[Hz]} \quad (2.4)$$

This definition is also given in [1] and [2]. After sampling the original signal $x(t)$ has become a sampled signal $x_b(t)$. This digitally sampled signal $x_b(t)$ can be seen as a so-called pulse-train with variable intensity:

$$x_b(t) = \sum_{n=-\infty}^{\infty} [x(t)\tau]\delta(t - n\Delta T) \quad (2.5)$$

with $\tau$ the time-duration of one sample and $\Delta T$ the time between two samples, as defined in [1].

In this equation the pulse-train is defined as:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \quad (2.6)$$

This equation can also be written in the following form:

$$p(t) = \sum_{k=-N}^{N} \frac{1}{\Delta T} e^{j2\pi k f_s t} \quad (2.7)$$

In [1] a detailed derivation of this result is given.

The digitally sampled signal can now be written as:

$$x_b(t) \approx x(t)\tau \sum_{k=-N}^{N} \frac{1}{\Delta T} e^{j2\pi k f_s t} \quad (2.8)$$

or using the Fourier transformation:

$$X_b(f) = \sum_{k=-N}^{N} \frac{\tau}{\Delta T} X(f - kf_s) \quad (2.9)$$

More details on the subject of sampling of digital signals can be found in [1] and [2].
2.3 Aliasing

In the previous paragraph the Fourier transform of a sampled signal is derived. This transform consists of a summation of exact Fourier transforms each time shifted by $k f_s$. The effect of this operation can be shown. Consider an exact Fourier transform $X(f)$ for which holds that $|X(f)| = 0$ for $f > f_{max}$. One can now distinguish the following situations:

**Situation 1:**

This situation is sketched in the figure above. The solid line in this figure represents the theoretical Fourier transform $X(f)$ and the dotted lines represent the shifted functions $X(f - k f_s)$. $X_b(f)$ is the summation of a set of these functions according to (2.9). One can conclude from this figure that in the interval $-f_s/2 \leq f \leq f_s/2$ the function $X_b(f)$ is exactly equal to $X(f)$.

**Situation 2:**

Figure 2.3: Situation with Aliasing, $f_{max} > f_s/2$
\( f_{\text{max}} > f_5/2 \). A graphical interpretation of this situation is shown in the figure above. In this figure the resulting summation of all shifted functions is shown by the red line. If one again looks at the interval \(-f_5/2 \leq f \leq f_5/2\) it can be seen that the final result \(X_4(f)\) differs considerably from the searched function \(X(f)\). The Fourier components with a frequency \(f > f_5/2\) are folded back to the interval \(-f_5/2 \leq f \leq f_5/2\). This type of error is also called **Aliasing**.

The only way to avoid aliasing is to create a situation where:

\[
f_{\text{fold}} = f_5/2 > f_{\text{max}} \quad \text{or} \quad \frac{1}{2\Delta T} > f_{\text{max}}
\]

(2.10)

\(1/2\Delta T\) is also called the **Nyquist-frequency**. Based on expression (2.10) Shannon's well-known sampling theorem can be defined. This theorem states that frequency components greater than one-half the sampling frequency, known as the folding or Nyquist frequency, will be aliased to a lower frequency.

In sampled-data systems aliasing causes errors in the characterization of the system. Because of the sampling operation the high-frequency content of the system will falsely appear as lower-frequency components. Aliased frequency components become indistinguishable from the system's response in the lower frequency range.

The background of aliasing can be illustrated by a simple example: Suppose one wishes to sample a 5 [Hz] sinusoidal signal. This signal is shown in the first plot of figure 2.4. In the second plot the situation is shown for the 5 Hertz signal, which is sampled every 0.125 seconds \((f_5 = 8 \text{ [Hz]}\)). In the third plot of figure 2.4 this sampling frequency is used to sample a 3 Hertz sinusoidal signal. One can conclude immediately that this results in exactly the same sampling points as was the case for the 5 Hertz signal. In other words, when sampling a 5 Hertz signal at a sampling frequency of \(f_5 = 8 \text{ [Hz]}\), the sampled signal will appear as a sine wave of only 3 [Hz]. The 5 [Hz] signal has been folded or aliased around the Nyquist frequency of 4 [Hz] to a 3 [Hz] signal by the sampling operation. In figure 2.5 an illustration of folding in the frequency domain is given for this example.

Aliasing is discussed in more detail in [1] and [2].
Figure 2.4: Sampling of a 5 [Hz] harmonic signal at a sampling frequency of 8 [Hz].

Figure 2.5: Aliasing shown as Folding in the Frequency Domain.
2.4 Anti-aliasing

In literature and recent research several attempts have been made to reduce the influence of aliasing errors. The classical solution for problems caused by aliasing is to place an analog anti-alias prefilter between the sensor and the A/D converter. This method, however, has several drawbacks. Therefore it is obvious that research is conducted into the area of detection and reconstruction of aliased components in signals.

2.4.1 Anti-alias prefilter

As mentioned above anti-alias prefilters are often placed between the sensor and the A/D converter in a digital control loop. In most cases a simple first-order low-pass filter is used to reduce the high-frequency content of an analog signal before it is sampled. For this purpose the cut-off frequency of the low-pass filter is selected to be lower than the Nyquist frequency in order to attenuate frequencies which are greater than the Nyquist frequency. In this way the noise above the Nyquist frequency will not affect the control system performance when it is aliased to lower frequencies.

It is mentioned before that there are several drawbacks to this method. First it is not possible to eliminate aliasing completely. However, the magnitude of the aliased noise can be reduced to an acceptable level. A second drawback is that devices equipped with anti-alias filters are unable to provide any information about the total signal energy or the energy of the aliased components. Furthermore, filtering of vibration modes that have significant frequency content above the Nyquist frequency renders those modes either uncontrollable or unobservable.

A major disadvantage of low-pass filter methods is the high order of the filters. If sample rates are to be changed, the cut-off frequency of these filters has to be changed as well. So a dedicated filter for each sample rate or a digital filter method has to be used. Mainly due to costs and complexity it is unusual to use filters in digital equipment. Recently, however, single-chip filters with adjustable cut-off frequency have become available, which reduces the need for complex and expensive high-order filters. Use of high-order low-pass filters can also result in phase-nonlinearities and possibly in an amplitude-ripple, which is another disadvantage.
2.4.2 Detection and reconstruction of aliased signal components

In digital equipment oversampling techniques are commonly used as an aliasing detection method. The signal is then sampled at the highest possible rate. The samples are only partly stored, the other samples are neglected. Oversampling techniques use these surplus samples to detect aliasing or to minimize the effects of aliasing. Oversampling is also used to reduce the order of the analog anti-alias prefilters. In [3] and [4] several oversampling techniques are treated in more detail.

Oversampling just like filtering also has disadvantages. Oversampling methods are unable to provide any information about the total signal energy or the energy of the aliased components. By using oversampling one can only achieve an indication of faulty measurements. However the method does not provide any information about correction of the measurements. The major disadvantage of oversampling techniques is that their function is only ensured for high oversampling rates.

Recently other methods have been published in order to detect and reconstruct aliased signal components from a measured signal.

In [5], for example, a technique is described, which has been developed to retrieve signals many thousand times greater than the limitations, which are caused by the Nyquist criterion. The method used here is to leave the incoming signal unfiltered and then interpreting the change in indicated frequency, which results from different sampling rates. In this way, by sampling the signal at two or three different sampling rates, one can detect aliased components in a signal. It is even possible to recover the actual frequency of the signal.

A method, which is closely related to this, is described in [3] and [4]. This method is also based on the sampling of a signal at different sampling rates. The basic system of this method consists of two ADC's working at slightly different sampling rates. The frequency components occurring in the resulting spectra are also slightly shifted to each other. By calculating the differences between the two spectra an aliased frequency component will give a contribution to the difference. A non-aliased component will not. It is even possible for certain ratios of the sample-rates to derive the original frequency and phase of an aliased component.

Another alias detection methodology is discussed in [6]. This method is also based on sampling of the signal at two or more different rates over the same
total time. Then the corresponding DFTs for these sampling rates are computed. Depending on which sampling rate is applied, aliased frequencies will normally appear in each spectrum at different frequencies. Real frequencies will appear at the same frequency in both spectra. So by comparing both spectra one can detect aliased components. A problem occurs when aliased components are folded to the same frequency for different sampling rates.

2.5 Aliasing and sampled-data control systems

Aliasing can have substantial consequences on digital control systems. In continuous systems, high-frequency noise components, which are much higher than the control-system bandwidth will normally have a marginal effect because the system will not respond at the high frequencies. In digital systems however, the high-frequency noise components can be aliased to lower frequencies. The closed-loop system will then respond to the noise. Or in other words, in a poorly designed digital control system, noise can have a greater effect than if an analog control system is used. As a result aliasing causes errors in the characterization of systems and plants. These errors can particularly become severe if the sampled signal contains many high-frequency components.

In the next chapter of this report a method will be explained to take aliasing into account in a sampled-data control system. In order to do this a discrete model for the continuous part of the sampled-data system is derived. Some of the properties of this discrete model will be shown for an example-system. Finally the discrete model for the continuous part of the sampled-data system is added to a transfer function model for the complete sampled-data system. This model can be used to evaluate the influence of aliasing on controllers based on loopshaping.
Chapter 3

Aliased Frequency Response Function (AFRF)

3.1 Discrete model of a sampled-data system

In the analysis of continuous systems the Laplace transform is often used. In discrete systems a similar procedure is available. This so-called z-transform is defined by

\[ Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k} \]  

(3.1)

where \( f(k) \) is a sampled version of a continuous function \( f(t) \) and \( k = 0,1,2,3,... \) refers to discrete sample times.

Figure 3.1: Discrete model of the continuous part of the sampled-data control loop

In order to study the influence of aliasing in control systems one wishes to find a model for the continuous part of the sampled-data system. In figure 3.1 this continuous part is drawn. A discrete transfer function can be found between the input samples \( u(n\Delta T) \) and the output \( y(t) \). As discussed in paragraph 2.1, the input \( u(n\Delta T) \) comes from a digital controller, which
is implemented in a computer. The actual continuous dynamical system is described by the transfer function $G_p(s)$. In between the digital controller and the analog system a D/A conversion takes place. This conversion consists of a so-called hold operation. Each sample entering the D/A converter is held at a constant value during one sample period until the next sample enters the converter. In this way a continuous signal can be composed of successive samples. In equation (2.5) it has been shown that the output of a sampler can be seen as a string of pulses. To complete the description of the physical sample-and-hold a model for the hold operation has to be derived. The hold is defined as the means whereby the sample-pulses are extrapolated to the piecewise constant signal $x_h$, defined as

\[ x_h(t) = x(n\Delta T), \quad n\Delta T \leq t < n\Delta T + T \]

(3.2)

In figure 3.2 the typical signals for sample and hold are sketched. First the input signal $x$ is plotted. This is the same signal as was used before in figure 2.4. In figure 3.2(b) the sampled signal $x_b$ is plotted. In contrast with the example of figure 2.4 a sampling frequency of 25 [Hz] is chosen. Figure 3.2(c) shows the output signal $x_h$.

![Figure 3.2: Typical signals for sample and hold. (a) Input signal $x$; (b) Sampled signal $x_b$; (c) Output signal $x_h$](image)
In this case, the hold operation is assumed to be a zero-order hold (ZOH) because $x_h$ is composed of zero-order hold polynomials passing through the samples of $x(n\Delta T)$. The impulse response of the ZOH can be defined as $1(t) - 1(t - \Delta T)$. With this result the required transfer function of the ZOH is

$$ZOH(s) = \mathcal{L}\{1(t) - 1(t - \Delta T)\} = \int_0^\infty [1(t) - 1(t - \Delta T)e^{-st}]dt = (1 - e^{-s\Delta T})/s \quad (3.3)$$

![Figure 3.3: Bode-diagram ZOH(s) for $\Delta T = 0.001$ [s]](image)

In figure 3.3 the Bode-diagram of this transfer function is plotted. When looking at the Bode-diagram one can see several properties of the ZOH. The ZOH acts like a low-pass filter with a cut-off frequency at 1000 [Hz]. Notice that the sampling frequency is also 1000 [Hz]. One can now conclude that the cut-off frequency of the ZOH is well beyond the Nyquist frequency (500 [Hz]) thus aliasing of high-frequency components to lower frequency parts can still occur.
In the $s$-domain the response of the system can be expressed as

$$Y(s) = (1 - e^{-s\Delta T}) \frac{G_p(s)}{s}$$  \hspace{1cm} (3.4)$$

Now the required transfer function is the $z$-transform of the samples of the inverse of $Y(s)$.

$$H(z) = Z\{y(n\Delta T)\}$$

$$= Z\{\mathcal{L}^{-1}\{Y(s)\}\}$$

$$= \mathcal{Z}\left\{(1 - e^{-s\Delta T}) \frac{G_p(s)}{s}\right\}$$  \hspace{1cm} (3.5)$$

This transfer function can be rewritten as

$$H(z) = (1 - z^{-1}) \mathcal{Z}\left\{ \frac{G_p(s)}{s}\right\}$$  \hspace{1cm} (3.6)$$

where

$$z = e^{s\Delta T} = e^{j\omega\Delta T} = e^{j2\pi f \Delta T}$$  \hspace{1cm} (3.7)$$

$H(z)$ is a discrete model of the continuous part of the sampled-data system. The response of the analog system at a given frequency $f$ is the sum of its primary band response, plus the aliased components from all of its higher frequency counterparts. So, by means of equation (3.6), the aliased components of the signal are taken into account in the transfer function. This equation can also be called the aliased frequency response function (AFRF). In [7] and [8] the derivation of the AFRF and its use in controller design is discussed in more detail.
3.2 Practical approach

In the previous paragraph an aliased frequency response function has been derived. In this paragraph the theory for the derivation of the aliased frequency response function (AFRF) is used in practice. The AFRF is determined for a fourth-order example system with two rotating masses, a spring and some damping. In figure 3.4 a schematic view of the system is sketched. The specific AFRF for this example system is used to show some of the properties of the AFRF.

![Schematic view of the system](image)

Figure 3.4: Schematic view of the system

$M_{\text{motor}}$ is the torque applied by a motor. $\theta_1$ and $\theta_2$ are the rotations of respectively the first and second mass. $J$ is the mass moment of inertia of both rotating masses, $k$ is the stiffness of the spring connecting both masses and $b$ represents the damping.

First a transfer function for this model must be derived from the dynamics of the system. A derivation of the transfer functions for both first and second mass can be found in appendix A. From the system’s dynamics two transfer functions follow:

$$G_p(s) = \begin{bmatrix} \frac{s^2J_1+sb+k}{s^4(2s^2J+2sb+2k)J} \\ \frac{s^2J_2+sb+k}{s^4(2s^2J+2sb+2k)J} \end{bmatrix}$$ (3.8)

In order to get a realistic description of the system, measurements have been done. From these measurements one can find parameter values for the transfer functions in equation (3.8). For the data acquisition the TUEDACS system has been used. The MATLAB-based application QadScope is used to on-line compute frf’s from the measured data.
In figure 3.5 the measured magnitudes of both first and second mass of the system are plotted. The sampling frequency of this measurement is 1000 [Hz]. A random signal with a bandwidth up to 450 [Hz] is used as system input. This signal is between minus 2.5 and plus 2.5 volts. The output of the system is the rotation of respectively the first and second mass. These angle-rotations are measured with encoderdisks.

![Figure 3.5: Magnitudes of both first and second mass](image)

In the remaining part of this report only the transfer function of the first mass will be used. In figure 3.6 the bode diagram of this transfer function is plotted.

![Figure 3.6: Bodediagram](image)
As the transfer function of the system $G_p(s)$ is now known, the knowledge of the previous paragraph can be used to compute the aliased frequency response function (AFRF) for this specific case. This computation can easily be done by making use of the 'c2d' command in MATLAB. With MATLAB Bode-diagrams for the AFRF can be plotted.

![Bode-diagram AFRF](image)

Figure 3.7: Bode-diagram AFRF. (a) for $f_S = 10$ [Hz]; (b) for $f_S = 20$ [Hz]

In figure 3.7(a) and 3.7(b) the Bode-diagrams for sampling frequencies of respectively 10 and 20 [Hz] are shown. It is clear that the AFRF function is not able to describe the system correctly anymore due to the extremely low sampling frequencies. One can see that the AFRF indeed contains aliasing because the Bode-diagrams of the AFRF seem to represent a complex summation of aliased components.

In figure 3.6 one can see that the system has a resonance at 50 [Hz]. If the AFRF function describes the system properly, the AFRF must also have a resonance at 50 [Hz]. However if the sampling frequency of the AFRF is chosen to low, this must result in other resonance-frequencies due to the aliasing effect. The resonance-frequency of a transfer function can be found by computing the poles of the system with MATLAB. The AFRF can be computed for an increasing sampling frequency from 0 up to 200 [Hz]. The resonance-frequencies of the resulting system-descriptions can be found from the poles. The results of this theoretical experiment are plotted in figure 3.8.
Figure 3.8: Resonance of the AFRF system description for increasing $f_S$

From this figure one can conclude that the AFRF describes the actual system properly for sampling frequencies of 100 [Hz] and higher. However if the sampling frequency is chosen to be lower than 100 [Hz] aliasing will occur in the model. This can easily be understood by making use of the earlier defined Shannon’s sampling theorem. If the resonance of 50 [Hz] is sampled with 200 [Hz] the spectrum of the signal does not contain any frequency content above half the sampling rate (100 [Hz]) and no aliasing of the resonance frequency will occur. However if a sampling frequency of, for example, 80 [Hz] is chosen, the spectrum of the signal with a resonance at 50 [Hz] contains a frequency, which is higher than half the sampling frequency (40 [Hz]). Consequently aliasing will occur and the resonance at 50 [Hz] will be folded back to 30 [Hz].

In this way, figure 3.7(a) and 3.7(b) can also be explained. If a sampling frequency of 10 [Hz] is chosen the system’s resonance at 50 [Hz] will be folded around 5 [Hz] to a frequency of -40 [Hz]. In a bodediagram, the resonance will then show up at 40 [Hz]. Next the aliased resonance frequency of 40 [Hz] is again folded around 5 [Hz] and a new resonance frequency can be found at 30 [Hz]. The AFRF thus represents a complex summation of aliased components as shown in figure 3.7(a) and 3.7(b).
3.3 Model development for a sampled-data control system

The discrete model of the continuous part of the sampled-data system (the AFRF) will now be used to derive a transfer function for the system model, which is shown in figure 3.9. This is another representation for the basic sampled-data control system, which has been described in paragraph 2.1. The method discussed here is based on [9].

\[ R(z) \]
\[ + \]
\[ E(z) \]
\[ \rightarrow \]
\[ C(z) \]
\[ \rightarrow \]
\[ H(z) \]
\[ \rightarrow \]
\[ Y(z) \]

Figure 3.9: Sampled-data system control loop

For the remaining part of the derivation discussed here \( H(z) \) is the aliased frequency response function (AFRF) as defined earlier. \( C(z) \) is the control law.

The following two relations can be derived from figure 3.9:

\[ Y(z) = C(z)H(z)E(z) \]  \hspace{1cm} (3.9)

\[ E(z) = R(z) - Y(z) \]  \hspace{1cm} (3.10)

Equation (3.10) can be substituted in (3.9), which results in

\[ Y(z) = C(z)H(z)[R(z) - Y(z)] \]
\[ = C(z)H(z)R(z) - C(z)H(z)Y(z) \]  \hspace{1cm} (3.11)

Rearranging of this equation gives

\[ Y(z) = \frac{H(z)C(z)}{1 + H(z)C(z)} R(z) \]  \hspace{1cm} (3.12)
Equation (3.12) suggests a transfer function in the form of:

$$\frac{Y(z)}{R(z)} = \frac{C(z)H(z)}{1 + C(z)H(z)}$$

(3.13)

In this derivation $Y(z)$ is the $z$-transform of the output of the control loop $Y(s)$.

The model of the sampled-data control system derived here can now be used to evaluate the influence of aliasing in control systems. By means of the AFRF function $H(z)$ one can ensure that aliasing occurs in the control loop.
Chapter 4

Conclusion

This report focusses on frequency aliasing, which results from digitaliza­
tion of continuous signals in sampled-data control systems. The theoretical
background of this error source is summarized. New methods to detect and
reconstruct frequency aliasing in signals are briefly mentioned.

With the theoretical background of frequency aliasing in mind, the second
part of the report is focussed on the development of a theoretical model
for dynamic sampled-data control systems in which frequency aliasing is
allowed.

The model of the sampled-data control system derived here can be used in
following research projects to design controllers which are based on loop­
shaping, allowing for the sample rate to be a design parameter. One of the
goals is to study the influence of frequency aliasing on controllers. Another
goal is to investigate whether a controller must be designed by taking the
aliasing error into account or not. The model described here is a powerful
tool for further research activities in this area.

In order to proceed with this research project, one should start with an
evaluation of the influence of frequency aliasing on the stability of controllers
for the open loop case. A controller based on loopshaping must be designed
for the system in which aliasing is allowed. The stability of the resulting
system should be evaluated. Furthermore a system in which no aliasing
occurs should be investigated. Next the same can be done for the closed
loop case. One should also proceed with practical experiments to investigate
the influence of frequency aliasing in real control systems. Another topic
of interest is the improvement of detection and reconstruction methods for
frequency aliasing.
Appendix A

System Description

Lagrange:

\[ \frac{d}{dt} (T, \dot{q}) - T, q + V, q = (Q^{nc})^T \]

\[
\begin{align*}
\mathbf{q} &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, & \dot{\mathbf{q}} &= \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
T &= \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 \\
v &= \frac{1}{2} k (\theta_1 - \theta_2)^2 \quad \Rightarrow \quad V = \frac{1}{2} k (\theta_1^2 - 2 \theta_1 \theta_2 + \theta_2^2) \\
T, q = 0, & T, \dot{\mathbf{q}} &= \begin{bmatrix} J_1 \dot{\theta}_1 \\ J_2 \dot{\theta}_2 \end{bmatrix}, & \frac{d}{dt} (T, \dot{q}) &= \begin{bmatrix} J_1 \dot{\theta}_1 \\ J_2 \dot{\theta}_2 \end{bmatrix}
\end{align*}
\]
Equations of motion:

\[
\begin{bmatrix}
J_1 \ddot{\theta}_1 \\
J_2 \ddot{\theta}_2
\end{bmatrix} + \begin{bmatrix}
k(\theta_1 - \theta_2) \\
k(\theta_2 - \theta_1)
\end{bmatrix} = \begin{bmatrix}
-M_{motor} - b(\dot{\theta}_1 - \dot{\theta}_2) \\
-b(\dot{\theta}_2 - \dot{\theta}_1)
\end{bmatrix}
\]

\[J_1 \ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = M_{motor}\]

\[J_2 \ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0\]

These Equations of motion can be rewritten as:

\[M \ddot{q} + D \dot{q} + K q = Q\]

with

\[
M = \begin{bmatrix}
J_1 & 0 \\
0 & J_2
\end{bmatrix}, \quad D = \begin{bmatrix}
b & -b \\
-b & b
\end{bmatrix}, \quad D = \begin{bmatrix}
k & -k \\
-k & k
\end{bmatrix}, \quad Q = \begin{bmatrix}
M_{motor} \\
0
\end{bmatrix}
\]

\[\Rightarrow \ddot{\theta}_1 + \frac{b}{J_1} \dot{\theta}_1 - \frac{b}{J_1} \dot{\theta}_2 + \frac{k}{J_1} \theta_1 - \frac{k}{J_1} \theta_2 = \frac{M_{motor}}{J_1}\]

\[\Rightarrow \ddot{\theta}_2 + \frac{b}{J_2} \dot{\theta}_2 - \frac{b}{J_2} \dot{\theta}_1 + \frac{k}{J_2} \theta_2 - \frac{k}{J_2} \theta_1 = 0\]

This results in the following state-variable form:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{k}{J_1} & -\frac{b}{J_1} & \frac{k}{J_1} & \frac{b}{J_1} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{k}{J_2} & \frac{b}{J_2} & -\frac{k}{J_2} & -\frac{b}{J_2} & 0 & 0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
\frac{1}{J_1} \\
0 \\
0
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
0
\end{bmatrix} u
\]

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with
\[
\mathbf{x} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}
\]

Now the Transfer-matrix can be computed by
\[
G_p(s) = C(sI - A)^{-1}B + D
\]

with
\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_1} & -\frac{b}{J_1} & \frac{b}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_2} & \frac{b}{J_2} & -\frac{k}{J_2} & -\frac{b}{J_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

This finally results in the following Transfer-matrix:
\[
G_p(s) = \begin{bmatrix} \frac{s^2 J_1 + sb + k}{s^2 (s^2 J_1 J_2 + 2s(J_1 b + k J_1 + sb J_2 + k J_2))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J_1 J_2 + 2s(J_1 b + k J_1 + sb J_2 + k J_2))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J_1 J_2 + 2s(J_1 b + k J_1 + sb J_2 + k J_2))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J_1 J_2 + 2s(J_1 b + k J_1 + sb J_2 + k J_2))} \end{bmatrix}
\]

or with \( J = J_1 = J_2 \):
\[
G_p(s) = \begin{bmatrix} \frac{s^2 J_1 + sb + k}{s^2 (s^2 J + 2sb + 2kd(J))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J + 2sb + 2kd(J))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J + 2sb + 2kd(J))} \\ \frac{s^2 J_1 + sb + k}{s^2 (s^2 J + 2sb + 2kd(J))} \end{bmatrix}
\]

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Bibliography

1. J.J. Kok, M.J.G. van de Molengraft; Signaalanalyse W1.2, University of Technology Eindhoven, Department of Mechanical Engineering, Section Control Systems Technology, Lecture Notes 4811, 2003


