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Note on the spatial distance of lines, line bisectors and certain related linear complexes

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by

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Note on the spatial distance of lines, line bisectors and certain related linear complexes.

An inquiry is made after the locus of lines with equal spatial distances of two, and the number of lines with equal spatial distances of three given lines. The results appear slightly different from those obtained by Bottema and Roth ([1] p. 51, Examples 3,4).

In the sequel several special cases are considered which may be bypassed, marked with *. 

0. Introduction. We consider threedimensional affine space $E^3$; threedimensional vectorspace is denoted $E^3_*$; since an origin $0$ will be chosen we may denote points $a$ by vectors $\mathbf{a} := 0 \to a$; $0 := 0 \to 0$; the line $l := \{x = \mathbf{a} + \lambda \mathbf{p} \mid \lambda \in \mathbb{R}\}$ with $\mathbf{p} \neq 0$ can alternatively be written $l = [\mathbf{p},q]$, that is, with Plücker coordinates; here $q = \mathbf{a} \times \mathbf{p}$, independent of the choice of $\mathbf{a}$ on $l$, dependent on the choice of $0$, which, when made, will be fixed; $l$ passes through $0$ iff $q = 0$; a Plücker representation $[0,q]$ denotes the common infinite line of the planes perpendicular to $q$.

If $0$, $\mathbf{p}$, $q$ are given, with $\mathbf{p} \neq 0$ and $(\mathbf{p},q) = 0$, then $[\mathbf{p},q]$ determines uniquely a finite line, since $\mathbf{a} = (\mathbf{p},p)^{-1} p \times q + q p$. Lines $[\mathbf{p}, q]$ and $[\mathbf{r},s]$ are coplanar iff $(\mathbf{p},s) + (q,r) = 0$; they have, then, a common point, finite or infinite; the expression $(\mathbf{p}, s) + (q,r)$ is invariant under isometric transformations.

Since parts of the investigation expand in the infinite plane $H^*_0$, we occasionally use homogeneous coordinates. The finite point $a$ can then be written $(a_0, \tilde{a})$ with a number $a_0$ and a vector $\tilde{a} \in E^3_*$; the quadruple $(a_0, \tilde{a})$ can be identified with the vector $\tilde{a} \in E^4_*$; if $\tilde{a}$ and $\tilde{a}$ denote the same point, then $\tilde{a} = a_0^{-1} \tilde{a}$ if $a_0 \neq 0$; if $a_0 = 0$ then $\tilde{a}$ denotes a point in $H^*_0$ (whose equation is $x_0 = 0$). Hence $(0,p)$ are homogeneous coordinates of the infinite point of $[\mathbf{p},q]$. The relation $q = \tilde{a} \times \mathbf{p}$ will read in homogeneous coordinates $a_0 q = \tilde{a} \times \mathbf{p}$, and then applies also to infinite points and lines.

We call the augmented space $P^3 := E^3 \cup H^*_0$ threedimensional projective space.

For details see [3 ] or any text on Projective Geometry of $P^3$. 
In computations we consider $\mathbf{a}$, $\mathbf{\alpha}$ and $\mathbf{\alpha}$ as column vectors and write the corresponding rows as $\mathbf{a}^T$, $\mathbf{\alpha}^T$ and $\mathbf{\alpha}^T$ respectively.

1. The case of two lines.

1.1. Let two lines $l_1 = \{x = a_1 + \lambda \mathbf{p}_1 \mid \lambda \in \mathbb{R}\}$ be given with $\mathbf{p}_1$ and $\mathbf{p}_2$ linearly independent. Then a unique orthogonal transversal $n$ exists with direction $\mathbf{p}_1 \times \mathbf{p}_2$. The projection of $a_2 - a_1$ on $n$ will be

$$\frac{1}{|\mathbf{p}_1 \times \mathbf{p}_2|^2} (a_2 - a_1, \mathbf{p}_1 \times \mathbf{p}_2, \mathbf{p}_1 \times \mathbf{p}_2);$$

observing that $(a_2 - a_1, \mathbf{p}_1 \times \mathbf{p}_2) = -(\mathbf{p}_1, \mathbf{q}_2) - (\mathbf{p}_2, \mathbf{q}_1)$ and substituting $e_3 := |\mathbf{p}_1 \times \mathbf{p}_2|^{-1} \mathbf{p}_1 \times \mathbf{p}_2$ gives

$$|\mathbf{p}_1 \times \mathbf{p}_2|^{-1} \{- (\mathbf{p}_1, \mathbf{q}_2) - (\mathbf{p}_2, \mathbf{q}_1)\} e_3$$

for this projection, the expression being not only independent of the choice of $a_1$ and $a_2$ on $l_1$ and $l_2$ respectively, but, moreover, independent of the choice of (euclidean) coordinates.

We define the algebraic distance $D(l_1, l_2)$ of $l_1$ and $l_2$ as

$$D(l_1, l_2) := -|\mathbf{p}_1 \times \mathbf{p}_2|^{-1} \{(\mathbf{p}_1, \mathbf{q}_2) + (\mathbf{p}_2, \mathbf{q}_1)\}.$$  

The distance $d(l_1, l_2)$ in the usual sense will then be $d(l_1, l_2) = |D(l_1, l_2)|$.

The angle between $l_1$ and $l_2$ is, ambiguously,

$$\delta(l_1, l_2) := \arccos\left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1| |\mathbf{p}_2|}\right) \in \left(0, \pi\right)$$

or its supplement; anyway, $0 < \delta(l_1, l_2) < \pi$.

The spatial distance $\text{spd}(l_1, l_2)$ is defined to be

$$\text{spd}(l_1, l_2) := (d(l_1, l_2), \delta(l_1, l_2)).$$

Our first problem is: determine the locus of (finite) lines $m$ with
1.2. We introduce the following conventions, not restricting generality:

- $a_i$ is the common point of $l_i$ and $n$, ($i = 1, 2$); $0 = a_1 + a_2$;
- $\vec{a}_1 = -\alpha_3 e_3$, $\vec{a}_2 = \alpha_3 e_3$; $\vec{p}_1 = e_1 + \beta e_2$, $\vec{p}_2 = e_1 - \beta e_2$, $\beta \neq 0$; \{e_1, e_2, e_3\} is an orthonormal base in $E_3$.

The lines $b' := b'_{12} := [\vec{e}_1, 0]$, $b' := b''_{12} := [\vec{e}_2, 0]$ are called the line bisectors of $l_1$ and $l_2$.

Now $q_1 = \alpha \beta e_1 - \alpha e_2$, $q_2 = \alpha \beta e_1 + \alpha e_2$.

Let $m$ be denoted $[x, s]$, $M_{12} := \{m \mid \text{spd}(l_1, m) = \text{spd}(l_2, m)\}$.

Then $M$ is determined by the equations

\begin{align*}
(1) & \quad |(x, p_1)| = |(x, p_2)|, \\
(2) & \quad |(x, q_1) + (s, p_1)| = |(x, q_2) + (s, p_2)|.
\end{align*}

We define

\begin{align*}
(3a) & \quad L'_{12} := \{m \mid (x, p_1 + p_2) = 0\} = \{m \mid (x, e_1) = 0\}, \\
(3b) & \quad L''_{12} := \{m \mid (x, p_1 - p_2) = 0\} = \{m \mid (x, e_2) = 0\}, \\
(3c) & \quad K'_{12} := \{m \mid (x, q_1 + q_2) + (p_1 + p_2, s) = 0\} = \{m \mid (x, \alpha \beta e_1) + (s, e_1) = 0\}, \\
(3d) & \quad K''_{12} := \{m \mid (x, q_1 - q_2) + (p_1 - p_2, s) = 0\} = \{m \mid (x, \alpha e_2) - (s, \beta e_2) = 0\}.
\end{align*}

It is obvious then that $M$, apart from infinite lines, is the union of $L' \cap K'$, $L'' \cap K''$, $L' \cap K'$ and $L'' \cap K''$.

1.3. $L'$, $L''$, $K'$ and $K''$ are linear complexes.

$L'$ and $L''$ are special ones, since, e.g., $0 = (x, e_1) = (x, e_1) + (s, 0)$ reveals that $L'$ consists of the transversals of $[q, e_1]$; consequently $L'$ consists (as far as finite lines are concerned) of all lines orthogonal to $b'$; and correspondingly $L''$ consists of all lines orthogonal to $b''$. 
$K'$ and $K''$ are special only iff $a = 0$; this case is considered in §1.5.

If, in general, a linear complex $A$ is given by the equation

$$(x, b) + (a, a) = 0,$$

$a$ and $b$ denoting arbitrary vectors, then a skew $4 \times 4$-matrix $[A]$ can be associated with $A$; to this purpose we introduce the notation $\Omega(v)$ for the linear operator of $E^3_4$ defined by

$$\Omega(v)x := v \times x \quad (v \in E^3_4, \ x \in E^3_4)$$

and use the same symbol for the matrix of $\Omega(v)$. Then

$$[A] := \begin{bmatrix} 0 & -a^T \\ a & \Omega(b) \end{bmatrix}$$

Interchanging the role of $a$ and $b$ we get the complex $A^\tau$, with

$$[A]^\tau := [A^\tau] = \begin{bmatrix} 0 & -b^T \\ b & \Omega(a) \end{bmatrix}$$

It is easy to check that $[A][A^\tau] = [A^\tau][A] = -(a, b) I_4$, that $A$ is special iff $(a, b) = 0$ and that, apart from a numerical factor, $[A]$ and $[A^\tau]$ are inverses if $A$ is not special.

The lines of $A$ which pass through a given point $\hat{\xi} \in P^3$ are coplanar in the plane $[A]^\tau \hat{\xi}$, through $\hat{\xi}$, called the polar plane of $\hat{\xi}$.

The lines of $A$ which lie in a given plane $\hat{\nu}$ are concurrent in the point $[A] \hat{\nu}$, in $\hat{\nu}$, called the pole of $\hat{\nu}$.

For proofs see [2], chapter II.

We apply these considerations on $L'$ and $K'$. 
\[ [L']^\tau = \begin{bmatrix} 0 & -\vec{e}_1 \\ \vec{e}_1 & \Omega(0) \end{bmatrix} \]

and the polar plane of \( \xi \) will be

\[ [L']^\tau \xi = \begin{bmatrix} -(\vec{e}_1, \xi) \\ \xi \times \vec{e}_1 \end{bmatrix} \] with equation \( \xi_0 (\vec{e}_1, \vec{x}) = (\vec{e}_1, \xi) x_0 \).

If \( \xi \) is finite then the equation reads \( x_1 = \xi_1 \), and \( [L']^\tau \xi \) is the plane through \( \xi \) perpendicular to \( b' \) (as was to be expected). If \( \xi \in H_\infty \), that is, if \( \xi_0 = 0 \), then, if \( (\vec{e}_1, \xi) \neq 0 \), \( [L']^\tau \xi = H_\infty \). If \( \xi_0 = 0 \wedge (\vec{e}_1, \xi) = 0 \) then \( [L']^\tau \xi = H_\infty \), not, due to the fact that \( L' \) is special, determining a plane at all. But these conditions express that \( \xi \in [\vec{e}_0, \vec{e}_1] \), hence all lines through \( \xi \) belong to \( L' \) (singular case).

\[ [K']^\tau = \begin{bmatrix} 0 & -\alpha \vec{e}_1^T \\ \alpha \vec{e}_1 & \Omega(\vec{e}_1) \end{bmatrix} \] and the polar plane of \( \xi \) will be

\[ [K']^\tau \xi = \begin{bmatrix} -\alpha (\vec{e}_1, \xi) \\ \alpha \xi \times \vec{e}_1 + \vec{e}_1 \times \xi \end{bmatrix} \] with equation

\[ \alpha \xi_0 (\vec{e}_1, \vec{x}) + \det[\vec{e}_1, \vec{e}_1, \vec{x}] = \alpha \beta (\vec{e}_1, \xi) x_0 \tag{4} \]

for every \( \xi \) representing a plane.

If \( \xi_0 = 0 \) this reads \( \det[\vec{e}_1, \vec{e}_1, \vec{x}] = \alpha \beta (\vec{e}_1, \xi) x_0 \) which represents \( H_\infty \) iff \( \vec{e}_1 \times \xi = 0 \), or \( \xi \equiv \vec{e}_1 \).

The behaviour of \( L'' \) and \( K'' \) is similar, and obtained by interchanging \( \vec{e}_1 \) and \( \vec{e}_2 \) and replacing \( \beta \) by \( -\beta^{-1} \).

1.4. \( L' \cap K' \) and \( L'' \cap K'' \) are easily obtained from (3).

\[ B' := B'_{12} := L' \cap K' = \{ m \mid (\xi, \vec{e}_1) = (\xi, \vec{e}_1) = 0 \}, \]

the 1-1-congruence of lines that intersect \( b' \) orthogonally; and \( B'' := B''_{12} := L'' \cap K'' \) is the 1-1-congruence of lines that intersect \( b'' \) orthogonally. As for \( C' := C'_{12} := L' \cap K' \), we consider points \( \xi \) in \( H_\infty \). Ignoring infinite lines we look for \( \xi \) with \( (\vec{e}_1, \xi) = 0 \); then the lines of \( C' \) through \( \xi \) are the lines of \( K'' \) through \( \xi \), c.q. the lines in the polar plane \( [K'' \tau] \xi \) with equation

\[ -\beta \det[\vec{e}_2, \vec{e}_2, \vec{x}] = \alpha (\vec{e}_2, \xi) x_0 \tag{5} \]
or, if \( \xi = \lambda \mathbf{e}_2 + \mathbf{e}_3 \neq \mathbf{e}_2 \),
\[-\beta(\mathbf{e}_1, \mathbf{x}) = \alpha \lambda x_0,\]
in nonhomogeneous coordinates
\[-\beta x_1 = \alpha \lambda .\]

The intersection of this plane with \( b' \) is the point \( -\beta^{-1} \alpha \lambda \mathbf{e}_1 \), the line through this point and \( \xi \) is \( [\xi, -\beta^{-1} \alpha \lambda \mathbf{e}_1, \mathbf{x}] \) and this line is coplanar with \( l_1 \) as well as with \( l_2 \), we have, for instance, \( l_1 = [\mathbf{e}_1, \mathbf{e}_2] \), and
\[
(\xi, \mathbf{e}_1) - \beta^{-1} \alpha \lambda (\mathbf{e}_1 \times \mathbf{x}, \mathbf{e}_2) = \text{det}[\lambda \mathbf{e}_2 + \mathbf{e}_3, -\mathbf{e}_2, \mathbf{e}_1] - \beta^{-1} \alpha \lambda \text{det}[\mathbf{e}_1, \lambda \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2] =
\]
\[= -\alpha \lambda - \beta^{-1} \alpha \lambda (-\beta) = 0.\]

Hence the conclusion: \( C'_{12} \) consists of planar bundles of parallel lines; the planes are perpendicular to \( b' \) and the direction of the lines in each plane is determined by its collinear intersections with \( l_1, b' \) and \( l_2 \). And \( C''_{12} \) consists of planar bundles of parallel lines; the planes are perpendicular to \( b'' \) and the direction of the lines in each plane is determined by its collinear intersections with \( l_1, b'' \) and \( l_2 \).

* 1.5. The case \( \alpha = 0 \), or \( l_1 \) and \( l_2 \) intersect.

\( K' = \{ \mathbf{m} \mid (\mathbf{e}_1, \mathbf{e}_1) = 0 \} \), \( K'' = \{ \mathbf{m} \mid (\mathbf{e}_2, \mathbf{e}_2) = 0 \} \).

From \( 0 = (\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_1) + (\mathbf{e}_1, \mathbf{e}_2) \) we see that lines of \( K' \) are intersecting \( b' = [\mathbf{e}_1, \mathbf{0}] \).

We infer that \( B' \) and \( B'' \) do not change in this case. However, \( C' \) will now be the collection of lines intersecting \( b'' = [\mathbf{e}_2, \mathbf{0}] \) and \( b' = [\mathbf{e}_1, \mathbf{0}] \); these lines have the common point \( \mathbf{e}_2 = [0, \mathbf{e}_2] \); hence the conclusion: \( C'_{12} \) consists of the lines either in the plane through \( b'' \) and perpendicular to \( b' \), or parallel to \( b'' \); and \( C''_{12} \) consists of the lines either parallel to \( b' \), or in the plane through \( b' \) and perpendicular
to b". See fig. 1 (in Perspective).

fig. 1

\[ \text{\textbullet} \]

1.6. The case \( l_1 \parallel l_2 \), or, equivalently, \( e_1 = e_2 = e_1' \).

Now all finite lines have equal angles with \( l_1 \) and \( l_2 \). The equations (3c,d) for \( K' \) and \( K'' \) depend on equation (2) which is invalidated if \( e \) and \( e_1 \) are linearly dependent.

If we take an arbitrary orthogonal transversal \( j \) of \( l_1 \) and \( l_2 \) with intersections \( a_1 \) and \( a_2 \), \( 0 := a_1 + a_2 \), \( a_1 := -a_2 \) and \( \{e_1, e_2, e_3\} \) orthonormal then

\[ K' = \{m \mid (e, e_1) = 0\} \quad K'' = \{m \mid (e, e_3) = 0\} \]

or, with \( b' := [e_1, 0] \), \( b' := [0, e_3] \) and \( h' := [0, e_1] \), \( m \in K' \) iff \( m \) intersects \( b' \), \( m \in K'' \) iff \( m \) intersects \( b'' \), \( b'' \) now being the infinite line of the plane through \( l_1 \) and \( l_2 \).

Take \( z := \{m \mid (e, e_2) = (e, e_3) = 0\} \), the hitherto excluded congruence of lines parallel to \( b' \); these lines have equal distances to \( l_1 \) and \( l_2 \) iff they intersect \( k := [e_3, 0] \), that is, iff \( (e, e_3) = 0 \).

Hence the conclusion: In this case \( M \) consists of the lines that either intersect \( b' \), or intersect \( b'' \) but, if both in their common point, also intersect \( k \). See fig. 2 (in Perspective).
1.7. Most of these results can be obtained by rather simple descriptive geometrical methods.

2. The case of three lines

2.1. Let three lines \( l_1 = \{ x = a_1 + \lambda p_1 \mid \lambda \in \mathbb{R} \} \) be given; for the moment we suppose them to be in general position, implying that \( p_1, p_2, p_3 \) are linearly independent and no two of them are coplanar.

The second problem is: determine the set of lines \( m \) with \( \text{spd}(l_1, m) = \text{spd}(l_2, m) = \text{spd}(l_3, m) \). It is obvious that this set is \( M_{12} \cap M_{23} \) (where the notation \( M_{23} \) will be clear by analogy).

Now \( M_{12} = B_{12} \cup B''_{12} \cup C_{12} \cup C''_{12} \) and consequently \( M_{12} \cap M_{23} \) is the union of the following 16 sets, each consisting of one element:
(i) \( B_{12} \cap B_{23} : m \) intersects \( b_{12}' \) and \( b_{23}' \) orthogonally.
(ii) \( B_{12} \cap B_{23}' : m \) intersects \( b_{12}' \) and \( b_{23}' \) orthogonally.
(iii) \( B_{12} \cap C_{23} : m \) is orthogonal to \( b_{12}' \), lies in a plane perpendicular to \( b_{23}' \) and intersects \( b_{12}' \).
(iv) \( B_{12} \cap C_{23}' : m \) is orthogonal to \( b_{12}' \), lies in a plane perpendicular to \( b_{23}' \) and intersects \( b_{12}' \).
(v) \( B_{12}' \cap B_{23} : \) analogous to (i).
(vi) \( B_{12}' \cap B_{23}' : \) analogous to (i).
(vii) \( B_{12}' \cap C_{23} : \) analogous to (iii).
(viii) \( B_{12}' \cap C_{23}' : \) analogous to (iii).
(ix) \( C_{12} \cap B_{23} : \) analogous to (iii).
(x) \( C_{12} \cap B_{23}' : \) analogous to (iii).
(xi) \( C_{12} \cap C_{23} : m \) is orthogonal to \( b_{12}' \) and \( b_{23}' \) and is the intersection of the corresponding planes perpendicular to \( b_{12}' \) and \( b_{23}' \).
(xii) \( C_{12}' \cap C_{23} : \) analogous to (xi).
(xiii) \( C_{12}' \cap B_{23} : \) analogous to (iii).
(xiv) \( C_{12}' \cap B_{23}' : \) analogous to (iii).
(xv) \( C_{12}' \cap C_{23} : \) analogous to (xi).
(xvi) \( C_{12}' \cap C_{23}' : \) analogous to (xi).

Hence there are 16 of these lines, divided in 4 classes of 4 parallel ones each: \{ (i), (iii), (ix), (xi) \}, \{ (ii), (iv), (x), (xii) \}, \{ (v), (vii), (xiii), (xv) \} and \{ (vi), (viii), (xiv), (xvi) \}.
If one looks at the configuration in the direction of class number one, then the projection will be as in fig. 3, where the direction is $\xi$ and the planes of the $K$-type complexes are denoted $K', K''$, etc.

2.2. Several special cases can be considered, from which we choose the case that $l_1', l_2', l_3'$ are concurrent; cf. [1], p. 51, Example 5. Let the common point of $l_1', l_2', l_3'$ be $a$. The case splits up in subcases.

2.2.1. $a \not\in H_\infty, P_1, P_2, P_3$ linearly independent. We have $M_{12}$ and $M_{23}$ as in §1.5; the congruences of type $B$ are not altered as compared to the general case, but those of type $C$ are. We find (see fig. 4)

(iii) $B'_{12} \cap C'_{23} = \{m \mid m \text{ intersects } b'_{12} \text{ orthogonally } \wedge [m \parallel b''_{23} \vee m \text{ in plane, through } b''_{23} \text{ and } l b'_{23}' \}.$

If $m \parallel b'_{12}$ and $m \parallel b''_{23}$ then $b'_{12} \parallel b''_{23}$, which in general is not true. If $V$ is a plane $\parallel b'_{23}$ then $V \parallel b'_{12}$ would imply $b'_{23} \parallel b'_{12}$ which, under the stated conditions is not true. We find, in general, the unique line in the unique plane $V$, through $b''_{23}$ and $l b'_{23}'$ that intersects $b'_{12}$ orthogonally. This line, however, intersects $b'_{23}$ orthogonally and is the same line as (i), in $B'_{12} \cap B'_{23}$.

(ix) $C'_{12} \cap C'_{23}$, analogous to (iii) we derive at (i).

(xi) $C'_{12} \cap C'_{23} = \{m \mid [m \parallel b''_{12} \vee m \text{ in } V_{12}, \text{ through } b''_{12} \text{ and } l b'_{12}] \wedge [m \parallel b''_{23} \vee m \text{ in } V_{23}, \text{ through } b''_{23} \text{ and } l b'_{23}]\}.$

Now $m \parallel b''_{12} \parallel b''_{23}$ is impossible; if $m \parallel b'_{12}$ and $m \parallel b'_{23}$ then $b'_{12} \parallel b'_{23}$, in general false; we are left with $m = V_{12} \cap V_{23}$, and, since $b'_{12}$ and $b'_{23}$ are again not parallel, $V_{12}$ and $V_{23}$ are different planes, through $a$; $m$ is their intersection, both $b'_{12}$ and $b'_{23}$ intersecting orthogonally; hence $m$ is the same line as in (i).

We conclude that the class $\{(i), (iii), (ix), (xi)\}$ collapses into only one line; so do the others; only 4 lines remain.
2.2.2. \( \mathbf{a} \notin H_m \) and \( E_1, E_2, E_3 \) linearly dependent. It is easily seen that there is 1 line \( m \) now, the perpendicular in \( \mathbf{a} \) on the plane through \( l_1, l_2, l_3 \).

2.2.3. \( \mathbf{a} \in H_m \) and \( l_1, l_2, l_3 \) not coplanar.

Recall that in \( \S \) 1.6 we arrived at

\[
M_{12} = \{m \mid [m \text{ intersects } b'_{12} \lor m \text{ intersects } b''_{12}] \wedge [m, b'_{12}, b''_{12} \text{ concurrent}] \Rightarrow m \text{ intersects } k_{12} \}\}
\]

combining this with the analogon for \( M_{23} \) and observing that \( b'_{12}, b'_{23}, b''_{12} \) and \( b''_{23} \) are concurrent, we see at one that there is only line \( m \) through the intersection of \( k_{12} \) and \( k_{23} \) (which purposely have been taken in one plane \( J \) (see fig. 5).

As for lines not through \( \mathbf{a} \), the lines intersecting \( b'_{12} \) and \( b'_{23} \) lie in the plane \( W_{13} \) through \( b''_{13} \), those intersecting \( b'_{12} \) and \( b''_{23} \) lie in the plane \( W_{23} \) through \( b'_{13} \), and those intersecting \( b''_{12} \) and \( b'_{23} \) lie in the
plane $W_{12}$ through $b_{13}$.
So we get all the lines in these planes, except those through $\hat{a}$; and one line through $\hat{a}$.

2.2.4. $\hat{a} \in H_{\infty}$ and $l_1, l_2, l_3$ coplanar, in the plane $V$. It is easily seen that $m_0$ becomes infinite, $k_{12}$ and $k_{23}$ being parallel; that the 3 planes coincide with $V$; and that the intersecting lines of $b''_{12}$ and $b''_{23}$ now are all lines intersecting $b''_{12}$, since $b''_{12}$ and $b''_{23}$ coincide.
$M_{12} \cap M_{23}$ consists of all lines parallel to $V$, except those who are parallel to $b_{12}$ (and to $l_1, l_2, l_3$, for that matter).

3. Remarks

3.1. The conclusions of [1], p. 51-55, about directed lines or spears are not altered by the results mentioned here.

3.2. Comparing these results with the assertion of Bottema and Roth ([1], p. 51, Ex. 3) that, in our notation, $M_{12} = B'_{12} \cup B''_{12}$, it seems that the discrepancy is reducible to equations (1) and (2); more specific, their set $n$, corresponding with our $B''_{12}$ say, is determined by equations (3b) and (3d), equivalent to

(1') $(x, p_1) = (x, p_2)$,

(2') $(x, q_1) + (s, p_1) = (x, q_2) + (s, p_2)$.

The latter of these is in accordance with their formula 7.4, taken from [4] p. 32-34, corresponding with our formula for $D(l_1, l_2)$ in §1. But the definition of the spatial distance in [1] is the same as in this note, that is, with $d(l_1, l_2) = |D(l_1, l_2)|$. 
4. References


