Yield criteria for anisotropic elasto-plastic metals

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Chapter 1

Introduction

In classical plasticity theory, it is assumed that plastic deformation can only take place when the stress tensor satisfies a certain condition. This condition or yield criterion can formally be written as:

\[
\Phi(\sigma) = C \quad \rightarrow \quad \text{elasto-plastic behaviour}
\]
\[
\Phi(\sigma) < C \quad \text{or} \quad \Phi(\sigma) = C \land \dot{\Phi}(\sigma) < 0 \quad \rightarrow \quad \text{elastic behaviour}
\]

(1.1)

where \( \Phi \) is the yield function, \( \sigma \) is the stress tensor and \( C \) is a constant. A dot over a symbol denotes a material time derivative. The situation \( \Phi > C \) has no physical meaning. The equation \( \Phi(\sigma) = C \) defines a surface in the six-dimensional stress space. This surface is usually termed the yield surface.

A common assumption in plasticity theory is the maximum-dissipation postulate [21]. The details of this postulate are beyond the scope of this report, but two consequences are relevant here: (i) the yield surface in stress space must be convex (Fig. 1.1), (ii) the plastic strain rate tensor must be directed along the outside normal of the yield surface in the stress-space. The second condition is satisfied if the strain rate tensor is related to the stress tensor via a so-called associated flow rule, which is given by:

\[
\dot{\epsilon}^p = \gamma \frac{\partial \Phi}{\partial \sigma}
\]

(1.2)

Figure 1.1: Schematic drawing of a convex yield surfaces (left) and a non-convex yield surface (right) in a 2-dimensional stress space.
where \( \dot{\gamma} \) is a constant, equal to zero for elastic deformations and greater than zero for elasto-plastic deformations, and \( \varepsilon^p \) is the plastic strain tensor. In this situation, the yield function is often called the plastic potential.

This report gives an overview of criteria that can be used to model materials with anisotropic yield properties, i.e., the plastic properties are a function of orientation. Two types of plastic anisotropy can be discriminated: initial anisotropy and induced anisotropy. The first type is found in materials that are structurally anisotropic, even before plastic deformation has taken place. The second type appears as a result of plastic deformation and can also be found in initially isotropic materials.

The overview given here is not intended to be complete. Only the most common criteria are mentioned. Moreover, the overview is limited to yield criteria that are independent of mean stresses (pressure). These criteria are generally used for modelling metals. Criteria dependent on mean stresses are necessary when plasticity theory is applied to rock, soil and concrete materials. Furthermore, only phenomenological yield functions will be considered. These criteria do not result directly from microstructure-based calculations. Their advantage over the yield surfaces calculated from microstructure is that they are easy to implement in numerical (FEM) codes and lead to relatively fast computations.

Although the main attention is given to the anisotropic criteria, the most familiar isotropic criteria will be summarised for convenience in Chapter 2. These criteria are often special cases of the anisotropic criteria which will be treated in Chapter 3.
Chapter 2

Isotropic yield criteria

Tresca

One of the earliest yield criteria is proposed by Tresca [21]. He assumed that plastic yield occurs when a critical value of the shear stress is reached:

$$\Phi = |\max(\sigma_1, \sigma_2, \sigma_3) - \min(\sigma_1, \sigma_2, \sigma_3)| = \sigma_0$$  \ (2.1)

where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are the principal stresses, and $\sigma_0$ is the equivalent uniaxial yield stress. In the 3-dimensional principal stress space, the yield surface of this criterion is represented by a hexagonal cylinder, parallel to the $\sigma_1=\sigma_2=\sigma_3$-axis. The projection of this cylinder on the $\pi$-plane (plane for which $\sigma_1 + \sigma_2 + \sigma_3 = 0$) is schematically drawn in Fig. 2.1. The Tresca criterion can be convenient for analytical calculations if the stress state is simple and the principal stresses can be obtained easily.

Von Mises

Von Mises proposed a criterion which states that plastic yielding occurs when the following condition is satisfied:

$$\Phi = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2$$  \ (2.2)

This criterion gives better predictions for most polycrystalline metals than the Tresca criterion. Moreover, it is more suitable in numerical models, since sharp corners in the yield surface are absent (Fig. 2.1).

A few well known pressure dependent criteria which are generalisations of the previous criteria are the Coulomb-Mohr (generalised Tresca) and Drucker-Prager (generalised Von Mises) criteria. These criteria are often used for modelling of rock and soil materials.

Hershey/Hosford

A generalisation of the previous criteria for pressure independent yielding is proposed by Hershey et al. [11] and Hosford [17]:

$$\Phi = |\sigma_1 - \sigma_2|^n + |\sigma_2 - \sigma_3|^n + |\sigma_3 - \sigma_1|^n = 2\sigma_0^n$$  \ (2.3)
For \( n = 2 \) or \( n = 4 \) this criterion leads to the Von Mises criterion and for \( n = 1 \) and \( n \to \infty \) to the Tresca criterion. For other values of \( n \) the yield surface lies between the yield surface of Tresca and Von Mises (Fig. 2.1). According to Hershey this criterion gives good predictions of experiments on polycrystalline aggregates of face-centred cubic (FCC) crystals for \( n = 6 \).

Figure 2.1: Schematic drawing of the yield surfaces of the Tresca (Eq. (2.1)), Von Mises (Eq. (2.2)), and Hershey/Hosford (Eq. (2.3)) criteria in the \( \pi \)-plane.

**Drucker**

The modified Drucker [6] criterion states that yielding occurs when the following condition is satisfied:

\[
\Phi = 27 \left( J_2^3 - c J_3^2 \right) \left( 1 - \frac{4c}{27} \right)^{-1} = \sigma_0^6
\]

(2.4)

where \(-3.374 \leq c \leq 2.25 \) [4] defines the curvature of the yield function and \( J_2 \) and \( J_3 \) are the second and third invariant of the stress deviator:

\[
J_2 = \frac{1}{2} S_{ij} S_{ij} \quad J_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki}
\]

(2.5)

Here, \( S_{pq} \) are the components of the stress deviator:

\[
S = \sigma - \frac{1}{3} (I : \sigma) \sigma
\]

(2.6)

For comparison with the anisotropic criteria in the later sections, the four isotropic criteria are plotted in Fig. 2.2 for a plane stress state. Note that the shear component \( \sigma_{xy} \) of the stress tensor is set to zero.
Figure 2.2: Yield surfaces of Tresca (Eq. (2.1), dash-dot), Von Mises (Eq. (2.2), dashed), Hershey/Hosford (Eq. (2.3), $m = 6$, dotted), and Drucker (Eq. (2.4), $c = 2.25$, solid) for a plane stress state. $S_{rx} = \sigma_x/\sigma_0$, $S_{ry} = \sigma_y/\sigma_0$, $\sigma_{xy} = 0$. 
Chapter 3

Anisotropic yield criteria

Before summarising the anisotropic yield criteria, a few general notes about them are made. Often, the parameters in these models are determined from measurements on uniaxial tensile tests on samples cut out at different angles to a direction of anisotropy. In these tests, quantities like yield stresses and so-called Lankford coefficients $R_\alpha$ are measured. The latter ones are defined as:

$$R_\alpha = \frac{\varepsilon_{\text{trans}}^p}{\varepsilon_{\text{thick}}^p}$$  \hspace{1cm} (3.1)

where $\alpha$ is the angle from an anisotropy-direction to the tensile-direction of the sample, $\varepsilon_{\text{trans}}^p$ is the plastic strain in the direction transverse to the tensile direction and $\varepsilon_{\text{thick}}^p$ is the plastic strain in the thickness-direction of the sample. Anisotropic materials will show a variation of both the Lankford coefficient and the yield stress with the orientation angle $\alpha$.

In experiments, it can be observed that criteria, quadratic in the components of the stress tensor, give a relatively inaccurate approximation of the yield surface of polycrystalline materials with an FCC (Face Centred Cubic) or BCC (Body Centred Cubic) micro-structure. The angular dependence of both the yield stress and the Lankford coefficient cannot be described within experimental accuracy by these criteria. Woodthrope et al. [25] reported these phenomena for aluminium alloys and called this the 'anomalous' behaviour of the materials.

To obtain better approximations several non-quadratic yield criteria are proposed. A quadratic and a non-quadratic criterion are drawn schematically in Fig. 3.1 in the principal stress space for a plane stress situation. The non-quadratic criterion has a small curvature on the uniaxial and equibiaxial axes, giving it a 'rounded off' Tresca look. This is often confirmed by experiments on FCC and BCC materials.

The criteria in this chapter are categorised in two groups. The first group embodies the criteria that can only be applied to plane stress states. These will be summarised in Section 3.1. The second group comprises the criteria which can be applied to general stress states. These will be treated in Section 3.2. An extensive overview of anisotropic yield criteria is given in the book of Zyczkowski [27]. Anisotropic yield criteria for sheet metals are reviewed by Szczepinski [23]. A review of anisotropic yield criteria for aluminium alloys is given by Habraken [10].
3.1 Plane stress

The practical use of the criteria summarised in this section is limited by the following two factors: (i) the type of anisotropy they can describe, (ii) the stress state to which they can be applied. The most general criteria mentioned here are able to describe orthotropic plasticity under general plane stress states. Orthotropic materials have three mutually perpendicular planes of symmetry. This type of anisotropy is often assumed for metal sheets. These sheets are usually fabricated by rolling the material in several steps. The orthotropy directions are: (1) the rolling direction, (2) the direction transverse to the rolling direction in the plane of the plate, and (3) the direction perpendicular to the plate.

Bassani

Bassani [3] proposed a criterion which is limited to planar isotropy. A material is planar isotropic if its properties are direction independent in the plane of the plate and differ from the properties in the direction perpendicular to this plane. The criterion is given by:

$$\Phi = \left| \frac{\sigma_1 + \sigma_2}{2\sigma_b} \right|^m + \left| \frac{\sigma_1 - \sigma_2}{2\tau_0} \right|^m = 1$$

(3.2)

where $\sigma_1$ and $\sigma_2$ are the principal stresses in the plane of the plate, $\sigma_b$ is the equibiaxial yield stress, $\tau_0$ is the yield stress in simple shear in the plane of the plate, and $m$ and $n$ are two dimensionless parameters. A criterion with the same restriction (planar isotropy) is proposed by Budianski [5].
Logan

The orthotropic criterion of Logan et al. [20] states that yielding occurs if the following condition is satisfied:

$$\Phi = g |\sigma_1|^m + f |\sigma_2|^m + h |\sigma_1 - \sigma_2|^m = \sigma_0^m$$  \hspace{1cm} (3.3)

Here $f$, $g$, and $h$ are positive parameters satisfying $f + g = 1$. The parameter $m$ can have two discrete values: $m = 6$ for BCC crystal grains and $m = 8$ for FCC crystal grains. This criterion has no shear stress components. This means that the directions of the principal stresses must coincide with the orthotropy-directions. The criterion is not valid for general stress states, which is a severe restriction.

Hill (1)

Hill [15] proposed a criterion which allows anisotropic behaviour of the material (Bauschinger effect). A material is anisotropic if the behaviour changes when the signs of all stress components are altered. The criterion is given by:

$$\Phi = \frac{\sigma_0^2}{\sigma^2_0} - \frac{c \sigma_1 \sigma_2}{\sigma_0 \sigma_{90}} + \frac{\sigma_{90}^2}{\sigma_{90}^2} + \left\{ (p + q) - \frac{p \sigma_1 + q \sigma_2}{\sigma_b} \right\} \frac{\sigma_1 \sigma_2}{\sigma_0 \sigma_{90}} = 1$$  \hspace{1cm} (3.4)

where $\sigma_0$ and $\sigma_{90}$ are the yield stresses in the rolling direction and the direction transverse to the rolling direction of a sheet, and $c, p$ and $q$ are dimensionless parameters. Like the previous criterion, this criterion is limited to stress states in which the directions of the principal stresses are parallel to the directions of orthotropy, since it includes no shear stresses.

Gotoh

One of the orthotropic criteria that is applicable to general plane stress states, is proposed by Gotoh [8, 9]:

$$\Phi = \sigma_2^2 + A_2 \sigma_2^2 \sigma_0 + A_3 \sigma_2^2 \sigma_{90} + A_4 \sigma_2^2 \sigma_y^2 + A_5 \sigma_0^2 \sigma_y^2 \sigma_{90} + \left\{ A_6 \sigma_2^2 + A_7 \sigma_2 \sigma_y + A_8 \sigma_2 \right\} \sigma_{xy}^2 + A_9 \sigma_{xy}^4 = \sigma_0^4$$  \hspace{1cm} (3.5)

where $A_2$ to $A_9$ are material parameters. The indices $x$ and $y$ denote the stress components in a coordinate system parallel to the directions of orthotropy. The curvature near points of uniaxial stress cannot be influenced by variation of parameters. The parameters are determined from $R_0$, $R_{22.5}$, $R_{45}$, $R_{90}$, $\sigma_{22.5}/\sigma_0$, $\sigma_0/\sigma_{90}$, $\sigma_0/\sigma_y$. The parameters ensure a good correlation with experimental $R_\alpha$, $\sigma_\alpha$ (yield stress as a function of orientation) and the ratios of yield stresses in principal directions. The yield surface is shown in Fig. 3.2 for a plane stress state.

Barlat (1)

Barlat et al. [2] proposed the following function for orthotropic anisotropy:

$$\Phi = a |K_1 + K_2|^m + a |K_1 - K_2|^m + (2 - a)|2K_2|^m = 2\sigma_0^m$$  \hspace{1cm} (3.6)
Figure 3.2: Gotoh [8, 9] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $A_2 = -2.60, A_3 = 3.75, A_4 = -2.79, A_5 = 0.991, A_6 = 6.29, A_7 = -7.72, A_8 = 6.33, A_9 = 8.96$ (form [9]). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{rx} = \sigma_x/\sigma_0, S_{ry} = \sigma_y/\sigma_0$.

where:

\[
K_1 = \frac{\sigma_x + h\sigma_y}{2}, \quad K_2 = \sqrt{\left(\frac{\sigma_x - h\sigma_y}{2}\right)^2 + p^2\sigma_{xy}^2}
\] (3.7)

Here, $a$, $h$, and $p$ are material constants and $m$ is a parameter comparable to $n$ in Eq. (2.3). This function is shown to be convex for $m > 1$. For $m = 2$, the criterion reduces to the Hill-criterion [12] (see Eq. (3.18)). The parameter $m$ also determines the curvature near points of deviatoric uniaxial tensile and compressive stress. $R_0$, $R_{90}$, and $R_{45}$ can be used to fit the parameters $a$, $h$, and $p$. The yield surface for a plane stress situation is shown in Fig. 3.3.

Hill (2)

In addition to a criterion which is only valid if the directions of principal stress are parallel to the directions of anisotropy, Hill [14] also proposed a criterion which can be used in general plane stress states:

\[
\Phi = \sigma_x^m + \sigma_y^m + \left(\sigma_{xy}^m / \tau_0^m\right)\left(\sigma_x - \sigma_y\right)^2 + 4\sigma_{xy}^2 \left(\sigma_x - \sigma_y\right)^2
\] (3.8)

\[
\left|\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}^2 \right|^{(m/2)-1} \left\{-2a(\sigma_x^2 - \sigma_y^2) + b(\sigma_x - \sigma_y)^2\right\} = 2\sigma_0^m
\]

where $a$, $b$, and $m$ are dimensionless parameters. The convexity of this yield function is not always guaranteed. Varying the parameter $m$ does not result in yield surfaces that
Figure 3.3: Barlat [2] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $a = 1.24, h = 1.15, p = 1.02, m = 8$ (from [2]). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{xx} = \sigma_x/\sigma_0, S_{xy} = \sigma_y/\sigma_0$.

are similar to those obtained from experimental data for polycrystalline materials. The yield surface is shown in Fig. 3.4.

Montheillet

Montheillet et al. [22] proposed the following yield function:

$$\Phi = c |a_{1}\sigma_x + a_{2}\sigma_y|^m + h |\sigma_x - \sigma_y|^m + 2n |\tau_{xy}|^m = \sigma_0^m$$  \hspace{1cm} (3.9)

where $c$, $h$, $n$, $a_{1}$, and $a_{2}$ are independent parameters. This criterion is capable of describing the 'anomalous' behaviour of several polycrystalline materials. The yield surface is shown in Fig. 3.5

Zhou

A yield function proposed by Zhou [26] states that yield occurs if:

$$\Phi = c \left[ (\sigma_x + \alpha \sigma_y)^2 \right]^{m/2} + h \left[ (\sigma_x - \sigma_y)^2 + b \tau_{xy}^2 \right]^{m/2} = \sigma_0^m$$  \hspace{1cm} (3.10)

where $c$, $\alpha$, $m$, $h$, and $b$ are parameters. Fig. 3.6 shows the yield surface of this criterion.
Figure 3.4: Hill [14] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $a = 1.2, b = 1.3, m = 2.3, \tau_0 = 0.85\sigma_0, \sigma_b = 1.1\sigma_0$ (arbitrary choice). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{rx} = \sigma_x/\sigma_0, S_{ry} = \sigma_y/\sigma_0$.

Ferron

This orthotropic criterion [7] is based on the Drucker (Eq. (2.4)) isotropic criterion. In contrast to the previous criteria, this criterion is represented in a polar coordinate system. Let $\sigma_1$ and $\sigma_2$ be the principal stresses in the plane of the plate, making an angle $\alpha$ with the orthotropy directions. The principal stresses are written in the following polar form:

\[ \frac{\sigma_1 + \sigma_2}{2\sigma_b} = g(\theta, \alpha) \cos(\theta), \quad \frac{\sigma_1 - \sigma_2}{2\sigma_b} = g(\theta, \alpha) \sin(\theta) \]  \hspace{1cm} (3.11)

The Drucker (Eq. (2.4)) criterion can be given in polar coordinate form by:

\[ (1 - k)g(\theta)^{-6} = \Gamma(\theta) = \left(\cos^2(\theta) + A \sin^2(\theta)\right)^3 - k \cos^2(\theta) \left(\cos^2(\theta) - B \sin^2(\theta)\right)^2 \]  \hspace{1cm} (3.12)

where $k = 4c/27$ and $c$ is the parameter in the Drucker criterion. For $A = 3$ and $B = 9$ the isotropic Drucker criterion is obtained. If $A$ and $B$ deviate from 3 and 9, a planar isotropic criterion is obtained. In-plane orthotropy is introduced by making $g$ dependent on $\alpha$:

\[ (1 - k)^{m/6}g(\theta, \alpha) = F(\theta)^{m/6} + 2a \sin(\theta) \cos^{2n-1}(\theta) \cos(2\alpha) + b \sin^{2p}(\theta) \cos^{2q}(2\alpha) \]  \hspace{1cm} (3.13)

According to Ferron [7], choosing $m = 2, n = 2, p = 2,$ and $q = 1$ gives the best correlation with experiments. The parameters $A, B, a,$ and $b$ can be determined from $R_0, R_{45}, R_{90},$ and $\sigma_{45}/\sigma_0.$ The parameter $k$ determines the curvature of the yield function. Convexity of the yield function is not always guaranteed and should be verified for all parameter
Figure 3.5: Montheillet [22] yield criterion in plane stress. Curves are drawn for constant values of the shear stress \( \sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5 \) (inner curve). Parameter values: \( c = 1.2, h = 1.3, n = 1.0, \alpha_1 = 1.05, \alpha_2 = 0.95, m = 1.5 \) (arbitrary choice). Coordinate axes of stresses coincide with the directions of anisotropy: \( S_{rx} = \sigma_x/\sigma_0, S_{ry} = \sigma_y/\sigma_0 \).

If the 'observed' yield function \( g \) determined from Eq. (3.11) is smaller than the yield function \( g \) calculated from Eq. (3.12) and Eq. (3.13), no plastic yielding will occur. The criterion is drawn for a plane stress state in Fig. 3.7.
Figure 3.6: Zhou [26] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $a = 0.8$, $b = 2.0$, $c = 0.8$, $h = 0.9$, $m = 2.4$ (arbitrary choice). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{rx} = \sigma_x/\sigma_0$, $S_{ry} = \sigma_y/\sigma_0$.

Figure 3.7: Ferron [7] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $k = 0.55$, $A = 2.2$, $B = 3.71$, $m = n = p = 2.0$, $q = 1.0$, $a = 0.24$, $b = 0.47$, $\sigma_0 = 2\sigma_0$ (from Ferron [7]). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{rx} = \sigma_x/\sigma_0$, $S_{ry} = \sigma_y/\sigma_0$. 

13
3.2 General stress states

General quadratic yield criterion

A general yield function for anisotropic materials which is a quadratic function of the components of the stress tensor is given by [24]:

\[ \Phi = F_i \sigma_i + F_{ij} \sigma_i \sigma_j = 1 \] (3.14)

where the stress components are defined as:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix} = \begin{bmatrix}
\sigma_x^2 \\
\sigma_y^2 \\
\sigma_z^2 \\
\sigma_{yz}^2 \\
\sigma_{xz}^2 \\
\sigma_{xy}^2 \\
\end{bmatrix}
\] (3.15)

The linear terms in Eq. (3.14) take into account the difference between positive and negative yield stresses (Bauschinger effect). The quadratic terms define an ellipsoid in the stress space. This general form is unwieldy for practical applications, because it has many parameters that have to be determined from experiments. From this general equation a few better known criteria can be derived.

Hoffman

One of these criteria is proposed by Hoffman [16]. This criterion is valid for orthotropic materials which show anisotropic behavior. It is given by:

\[ \Phi = C_1 (\sigma_y - \sigma_z)^2 + C_2 (\sigma_z - \sigma_x)^2 + C_3 (\sigma_x - \sigma_y)^2 + C_4 \sigma_x + C_5 \sigma_y + C_6 \sigma_z + C_7 \tau_{yz}^2 + C_8 \tau_{xz}^2 + C_9 \tau_{xy}^2 = 1 \] (3.16)

The parameters \( C_1 \ldots C_9 \) can be determined uniquely from nine basic strength data:

- The three uniaxial tensile strengths: \( T_x, T_y, T_z \).
- The three uniaxial compressive strengths: \( C_x, C_y, C_z \).
- The three pure shearing strengths: \( S_{yz}, S_{xz}, S_{xy} \).

The relations are given by:

\[
\begin{align*}
C_1 &= \frac{1}{2} \left[ (T_y C_y)^{-1} + (T_z C_z)^{-1} - (T_x C_x)^{-1} \right] \\
C_2 \text{ and } C_3 &= \text{by permutation of } x, y, z \\
C_4 &= T_x^{-1} - C_x^{-1} \\
C_5 \text{ and } C_6 &= \text{by permutation of } x, y, z \\
C_7 &= S_{yz}^{-2} \\
C_8 \text{ and } C_9 &= \text{by permutation of } x, y, z
\end{align*}
\] (3.17)

Note that the criterion is only pressure independent if \( C_4 + C_5 + C_6 = 0 \) [23]. If \( C_1 \ldots C_9 \) are determined with Eq. (3.17), this condition is generally not satisfied.
Hill (3)

If the material behaves isosensitive Eq. (3.16) can be simplified by eliminating the terms which are linear in the stress components. This is achieved by setting the yield stresses for compression equal to the corresponding yield stresses for tension in Eq. (3.17). A general form of this criterion is proposed by Hill [12]:

$$\Phi = F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\tau_{yx}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1$$ (3.18)

where $F$, $G$, $H$, $L$, $M$, and $N$ are parameters that can be determined from six basic strength data by Eq. (3.17) (tensile and compressive strengths are equal now). This criterion is quite often implemented in Finite Element codes. For $F = G = H = (2\sigma_0^2)^{-1}$ and $L = M = N = \frac{3}{2}F$ Eq. (3.18) leads to the isotropic Von Mises criterion Eq. (2.2). The yield surface is shown for a plane stress situation in Fig. 3.8.

![Yield surface](image)

Figure 3.8: Hill [12] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\tau_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ (inner curve). Parameter values: $F = 0.64, G = 1.42, H = 0.58, N = 3.91$ (from Habraken [10]). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{rx} = \sigma_x/\sigma_0$, $S_{ry} = \sigma_y/\sigma_0$.

Hu

A generalisation of the Tresca criterion Eq. (2.1) to anisotropic materials is derived by Hu [18]. This generalisation is however only valid in a very particular case: the anisotropy is restricted to orthotropy and the principal directions of the stress state coincide with the directions of orthotropy. The criterion is given by a system of six linear equations:

$$\Phi_{xx'} = \frac{\sigma_x}{T_x} - \left( \frac{1}{T_x} - \frac{1}{C_x} \right) \sigma_y - \frac{\sigma_z}{C_z} = 1$$ (3.19)
\[ \Phi_{xz} = \frac{-\sigma_x}{C_x} + \left( \frac{1}{C_x} - \frac{1}{I_x} \right) \sigma_y + \frac{\sigma_z}{I_x} = 1 \]

The other four equations follow from cyclic permutation of the indices.

**Hill (4)**

A criterion suitable for planar orthotropy is proposed by Hill [13]:

\[ \Phi = f|\sigma_2 - \sigma_3|^m + g|\sigma_3 - \sigma_1|^m + h|\sigma_1 - \sigma_2|^m + a|2\sigma_1 - \sigma_2 - \sigma_3|^m \]
\[ + b|2\sigma_2 - \sigma_3 - \sigma_1|^m + c|2\sigma_3 - \sigma_1 - \sigma_2|^m = \sigma^m \]  

(3.20)

where \( a, b, c, f, g, \) and \( h \) are positive parameters, the exponential parameter \( m \) is chosen greater than 1 and \( \sigma \) regulates the unit of stress. The criterion has no shear components. Hence, it is only valid when the directions of principal stresses are aligned to the directions of orthotropy.

**Barlat (2)**

Barlat et al. [1] proposed a criterion which is valid for arbitrary stress states in orthotropic materials. The yield function is defined as:

\[ \Phi = |S_1 - S_2|^m + |S_2 - S_3|^m + |S_3 - S_1|^m = 2\sigma_0^m \]  

(3.21)

where \( S_{i=1,2,3} \) are the components of a symmetric matrix \( S_{ij} \) defined with the components of the stress tensor \( \sigma \):

\[
\begin{align*}
S_{xx} &= \frac{c(\sigma_{xx} - \sigma_{yy}) - b(\sigma_{zz} - \sigma_{xx})}{3} \\
S_{yy} &= \frac{a(\sigma_{yy} - \sigma_{zz}) - c(\sigma_{xx} - \sigma_{yy})}{3} \\
S_{zz} &= \frac{b(\sigma_{zz} - \sigma_{xx}) - a(\sigma_{yy} - \sigma_{zz})}{3} \\
S_{yz} &= f\sigma_{yz} \\
S_{xz} &= g\sigma_{xz} \\
S_{xy} &= h\sigma_{xy}
\end{align*}
\]  

(3.22)

The Cauchy stresses are expressed in coordinates that are parallel to the orthotropy axes. For \( a = b = c = f = g = h = 1 \) this criterion reduces to the isotropic Tresca or Von Mises criterion, depending on the value of \( m \). In fact, it is a generalisation of the Hershey/Hosford criterion Eq. (2.3). It does not reduce to the Hill Eq. (3.18) criterion, not even for \( m = 2 \). The criterion is convex for \( m > 1 \). The parameter \( m \) can be used to influence the curvature of the yield surface near the uniaxial and balanced biaxial tension ranges as is observed in polycrystalline materials (Fig. 3.9). The criterion is not able to describe the Bauschinger-effect.
Figure 3.9: Barlat [1] yield criterion in plane stress. Curves are drawn for constant values of the shear stress $\sigma_{xy}/\sigma_0 = 0.0, 0.1, 0.2, 0.3, 0.4$ (inner curve). Parameter values: $a = 1.0$, $b = 1.19$, $c = 0.98$, $h = 1.15$, $m = 12$ (from Habraken [10]). Coordinate axes of stresses coincide with the directions of anisotropy: $S_{xx} = \sigma_x/\sigma_0$, $S_{xy} = \sigma_y/\sigma_0$.

Karafillis

Karafillis et al. [19] proposed a criterion which is based on a linear transformation of the stress tensor. First, they introduced a general isotropic criterion, obtained by mathematically mixing of two criteria. Second, the linear transformation of stresses is applied which results in a criterion for anisotropic materials. It is assumed that the isotropic yield function lies between a lower (Tresca) and upper bound as is indicated in Fig. 3.10. Isotropic criteria lying between the lower bound and the Von Mises yield surface are described by

Figure 3.10: Upper and lower bounds of the general isotropic yield criterion in the $\pi$-plane. $S_i$ are the principal values of the deviatoric stress tensor.
the Hershey/Hosford (Eq. (2.3)) criterion:

\[ \Phi_1 = (S_1 - S_2)^{2k} + (S_2 - S_3)^{2k} + (S_3 - S_1)^{2k} = 2\sigma_0^{2k} \]  

(3.23)

where \( S_i \) are the principal values of the stress deviator. The parameter \( k \) can vary from 1 to \( \infty \). For \( k = 1 \) the Von Mises surface is obtained and for \( k \rightarrow \infty \) the lower bound. Yield surfaces between the Von Mises and the upper bound are described by the following yield condition:

\[ \Phi_2 = S_1^{2k} + S_2^{2k} + S_3^{2k} = \frac{2k^2 + 2}{3k^2} \sigma_0^{2k} \]  

(3.24)

For \( k = 1 \) the Von Mises criterion is obtained, while for \( k \rightarrow \infty \) the upper bound is recovered.

The general isotropic yield function is obtained by mathematically mixing the previous criteria:

\[ \Phi(S) = (1 - c)\Phi_1(S) + c \frac{3^{2k}}{2k - 1 + 1} \Phi_2(S) = 2\sigma_0^{2k} \]  

(3.25)

where \( c \in [0, 1] \). This yield surface lies between the lower and upper bound and preserves convexity.

To obtain an anisotropic yield surface, a linear transformation of the stress tensor is applied as indicated in Fig. 3.11. The transformation tensor \( S^{IPE} \) is called the ‘isotropy plasticity equivalent’ (IPE) deviatoric stress tensor and is used as an argument in Eq. (3.25). The type of anisotropy is given in the fourth order tensor \( L \). For an orthotropic material and \( c = 0 \) the criterion reduces to the Barlat (Eq. (3.21)) criterion. The Bauschinger effect can be included by adjusting the transformation:

\[ S^{IPE} = L : (\sigma - B) \]  

(3.26)

where \( B \) is a second order tensor with \( I : B = 0 \).

Figure 3.11: Linear transformation of stresses. Left: anisotropic material. Right: isotropy plasticity equivalent (IPE).
Chapter 4

Concluding remarks

For general stress states the Hill-criterion (Eq. (3.18)) is the most widely used criterion. It is relatively simple, and the parameters can be determined easily using standard tensile tests. Anisosensitive behaviour can be incorporated (see Eq. (3.16)). However, it has some drawbacks when applied to metals like aluminium. The yield surface observed in experiments (a 'rounded-off' Tresca surface) cannot be described accurately by this criterion. Furthermore, if Lankford-coefficients $R_\alpha$ are used to determine the parameters, this criterion tends to overestimate the variation of the yield stress as a function of orientation to the direction of anisotropy. The Barlat criterion Eq. (3.21) and the Karafillis Eq. (3.25) give better results in the case of general stress states. The Barlat criterion is only valid for orthotropic materials while the Karafillis criterion is valid for more general forms of anisotropy.

For plane stress states there are more criteria that are able to give better predictions than the Hill (Eq. (3.18)) criterion for aluminium alloys: Gotoh (Eq. (3.5)), Barlat (Eq. (3.6)), Montheillet (Eq. (3.9)), and Ferron (Eq. (3.13)). The other criteria for plane stress are somewhat restricted by their assumptions: principal directions of stress parallel to the directions of orthotropy (Eq. (3.4), Eq. (3.3)) and planar isotropy (Eq. (3.2)).
Bibliography


