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DIRECT CRCTC OF A FLEXIBLE TT-MANIPULATOR

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# Direct CRCTC of a Flexible TT-Manipulator

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1 Introduction

Until recently, most industrial manipulators were designed to obtain stiff constructions in order to reduce deformation and, hence, to ease control. These manipulators are quite heavy and have large inertia, so large torques and forces are required to control the end-effector. The desire for high accelerations and low energy consumption has resulted in light, but flexible constructions. Obviously, precision is not very good with conventional controllers. In 1993, a new approach to the control of flexible manipulators, called Computed Reference Computed Torque Control (CRCTC), was developed by Lammerts (lit.7). Both flexibility in joints and flexibility in links are supported in this theory where it is not tried to suppress vibrations, but to generate inputs such that the elastic deformations remain bounded. Computer simulations with this theory show promising results when applied to joint-level control. In practice, we are interested in end-effector motion, so feedback of the end-effector coordinates is to be used: the so-called direct Computed Reference Computed Torque Control.

This report shows the results of a case study of a flexible two-dimensional translation manipulator, controlled by direct Computed Reference Computed Torque Control (CRCTC).

2 Set-Up of the Experiment

As mentioned before, a flexible two-dimensional translation manipulator with two motors is used. This manipulator, a so-called XY-table, was introduced by Heeren (lit.4). It consists of two slideways in X-direction, and a transverse (with end-effector) in Y-direction. Motor 1 is connected to the transverse at the right slideway and motor 2 is connected to the end-effector (see figure 1). The left side of the transverse is also driven: motor 1 is, via a flexible spindle, also connected to the left side of the transverse. The stiffness of the spindle can be changed within a broad range. As a consequence, the transverse will not always be perfectly perpendicular to the slideways, and a movement of the end-effector along the transverse in general will not be a movement in the Y-direction.

![Figure 1: The flexible TT-manipulator](image-url)
3 Equations of Motion of the XY-table

Because the spindle is flexible, an extra degree of freedom is introduced. This leads to the three degree of freedom model, as derived by v.d. Molengraft (lit.8) and Willems (lit.11). They used the angles of rotation $\varphi_1$, $\varphi_2$ and $\varphi_3$ of, respectively, the belt wheel connected to motor 1 (at the right hand side of the flexible spindle), the belt wheel connected to motor 2 and the belt wheel at the left hand side of the flexible spindle (see figure 1) as degrees of freedom. With these angles, seen as the components of a column $q = [\varphi_1 \ \varphi_2 \ \varphi_3]^T$, the mathematical model is given by

$$M\ddot{q} + C\dot{q} + Kq + k = Hu$$

(1)

where $M = M(q)$ is the symmetric, positive definite mass matrix, $C = C(q,\dot{q})$ is the Coriolis matrix, $K$ is the constant, symmetric, positive definite stiffness matrix and $H$ is the distribution matrix. Furthermore, $u$ is the column with control inputs, i.e. the torques exerted by motor 1 and motor 2 on the belt wheels. Finally, the column $w = w(q,\dot{q})$ contains the friction torques, external disturbances etc. Explicit relations for the entries of $M$, $C$, $K$, $H$ and $w$ are given in Appendix A of this report. In general, the matrix $C$ is not unique and can be chosen, such that $M - 2C$ is skew-symmetrical (lit.5). This is the case for the matrix $C$ as given in Appendix A of this report. For simplicity of notation, the combination $Kq + w$ is denoted by $k$, so

$$M\ddot{q} + C\dot{q} + k = Hu$$

(2)

The output $y$ of the system consists of the $x$- and $y$-coordinates of the end-effector, i.e.

$$y = [x \ y]^T$$

These coordinates are non-linear functions of the degrees of freedom $\varphi_1$, $\varphi_2$ and $\varphi_3$. Explicit relations for these functions are given in Appendix B.

The objective of the research in this report is to control the end-effector position. Therefore, it is advantageous to rewrite the equations of motion (2) in terms of the end-effector coordinates $x$ and $y$. For that purpose, a column $p^*$ is introduced. The first and second component of this column are proportional to $x$, respectively $y$. The third component is proportional to the torsional deformation $\varphi_1 - \varphi_3$ of the flexible spindle (see Appendix B). The original degrees of freedom (the "motor coordinates") $q$ are a non-linear, invertible function of the newly introduced degrees of freedom (the "end-effector coordinates") $p^*$, i.e.

$$q = \psi(p^*)$$

(3)

The derivative of $\psi$ with respect to $p^*$ is denoted by $\Psi$, so

$$\dot{q} = \Psi \dot{p}^* ; \ \Psi_{ij} = \frac{\partial \psi_i}{\partial p_j} \text{ for } i, j = 1, 2, 3$$

Explicit relations for the components of $\psi$ and $\Psi$ are given in Appendix B. With the transformation (3), the equations of motion (2) can be written as

$$M^*\ddot{p}^* + C^*\dot{p}^* + k^* = H^*u$$

(4)

where, according to Lammerts (lit.7), the following relations hold:

$$M^* = \Psi^T M \Psi$$

$$C^* = \Psi^T C \Psi$$

$$k^* = \Psi^T k$$

$$H^* = \Psi^T H$$

The inertia matrix $M^*$ and the Coriolis matrix $C^*$ satisfy $\dot{M}^* - 2C^* + (M^* - 2C^*)^T = 0$, so $M^* - 2C^*$ is skew-symmetrical as is the case for the original matrix $M - 2C$. 

3
4 Summary of the CRCTC Theory

The desired trajectory for the end-effector coordinates \( x \) and \( y \) is specified by the known functions \( x_d = x_d(t) \), respectively \( y_d = y_d(t) \). Hence, the desired trajectories \( p_{1d}^* \) and \( p_{2d}^* \) for the first two components of the column \( p^* \) are known. However, this is not the case for the third component of \( p^* \), i.e. for the torsional deformation of the flexible spindle. For the moment, it is assumed that it will be possible, in one way or another, to determine a desired trajectory \( p_{3d}^* \) for this component.

As a consequence, the desired trajectory \( p_{3d}^* \) for \( p^* \) is known.

In the CRCTC theory as proposed by Lammerts (lit.7), the following strategy is used. In the first step, a column \( p_i^* \) and the tracking error \( e_i^* \) are defined by

\[
\dot{p}_i^* = p_d^* + \Delta e^* \quad e_i^* = p_i^* - p_d^*
\]

where \( e = p_d^* - p^* \) and \( \Delta \) is a constant, symmetric, positive definite matrix.

In the second step, the desired error equation of the closed loop system is specified. Here, we only consider desired error equations of the form

\[
M^* \ddot{e}^* + C^* \dot{e}^* + K_D^* \dot{e}^* + K_P e^* = 0
\]  

(5)

where the matrices \( K_D^* \) and \( K_P^* \) are chosen such that their symmetrical part is positive definite, i.e. \( K_D^* + (K_D^*)^T > 0 \) and \( K_P^* + (K_P^*)^T > 0 \). Furthermore, \( K_P^* \) has to be constant but this is not required with respect to the matrix \( K_D^* \). It is easily seen that \( e^*(t) = 0 \) is an equilibrium point of equation (5). To investigate the stability properties of this equilibrium point the candidate Lyapunov function

\[
V = \frac{1}{2} \dot{e}^T M^* e^* + \frac{1}{2} \dot{e}^T K_P e^*
\]

is used. Then, with the closed loop error equation (5) and symmetry of \( M^* \) and the skew-symmetry of \( M^* - 2C^* \), it is seen that

\[
\dot{V} = -\dot{e}^T K_D^* \dot{e}^*
\]

This implies \( \dot{e}^*(t) \to 0 \) for \( t \to \infty \). From the closed loop error equation and from the fact that \( K_P^* \) is regular, it follows that \( e^*(t) \to 0 \) for \( t \to \infty \). Hence, if it is possible to choose a control law such that the error equation (5) holds for the closed loop system, then the tracking error \( e^* \) converges to 0.

In the third step of the CRCTC strategy the equations of motion (4) and the closed loop error equations (5) are combined, yielding

\[
M^* \ddot{p}_r^* + C^* \dot{p}_r^* + K_D^* \dot{p}_r^* + K_P e^* + k^* = H^* y
\]

(6)

The remaining problem is to determine the two unknown components of the input \( y \) and the unknown third component of \( p_r^* \) from this set of three equations.

To simplify the analysis, equation (6) is premultiplied with \( \dot{\Psi}^{-T} \). This results in

\[
A \ddot{p}_r^* + \dot{b} = H u
\]

(7)

where \( A \) and \( \dot{b} \) are given by

\[
A = M \dot{\Psi} \quad ; \quad \dot{b} = (M \dot{\Psi} + C \dot{\Psi}) \ddot{p}_r^* - D \dot{e}^* - Pe^* + k
\]

with \( D = \dot{\Psi}^{-T} K_D^* \) and \( P = \dot{\Psi}^{-T} K_P^* \). In further analysis, \( M \dot{\Psi} + C \dot{\Psi} \) is denoted by \( \dot{G} \).
The components of $A$ depend on $p^*$ whereas the components of $h$ are functions of $p^*$, $\dot{p}^*$, $\ddot{p}^*$, and $\dddot{p}^*$.

The column $p^*$ is partitioned in a column $\vartheta$ with the two so-called rigid body coordinates (proportional to the end-effector coordinates) and one component $\rho$, equal to the flexible coordinate. The matrix $A$ and the column $h$ are partitioned in a similar way using the special structure of the distribution matrix $H$ and the column $N = [0 \ 0 \ 1]^T$, i.e. (see also Lammerts (lit.7))

$$ p^* = \begin{bmatrix} \vartheta \\ \rho \end{bmatrix} = H\vartheta + N\rho $$

$$ A = \begin{bmatrix} A_{\vartheta\vartheta} & A_{\vartheta\rho} \\ A_{\rho\vartheta} & A_{\rho\rho} \end{bmatrix} = \begin{bmatrix} H^T A H & H^T A N \\ N^T A' & N^T A N \end{bmatrix} ; \quad h = \begin{bmatrix} h_\vartheta \\ h_\rho \end{bmatrix} = \begin{bmatrix} H^T h \\ N^T h \end{bmatrix} $$

Therefore, the equations in (7) can be rewritten into

$$ y = H^T \begin{bmatrix} A H \ddot{\vartheta}_r + A N \ddot{\rho}_r + h \end{bmatrix} $$

$$ N^T A N \ddot{\rho}_r = -N^T \begin{bmatrix} h + A H \ddot{\vartheta}_r \end{bmatrix} \quad (9) $$

From the scalar differential equation (9), a bounded solution $\rho_r$ for the reference trajectory of the flexible coordinate must be determined. In the next section, this is done with straightforward integration. For the moment it is assumed that, in one way or another, it is possible to determine on-line a bounded solution $\rho_r$ from equation (9). Then, the input $y$ follows from

$$ y = H^T \begin{bmatrix} (A H - A N (N^T A N)^{-1} N^T A H) \ddot{\vartheta}_r + (I - A N (N^T A N)^{-1} N^T) h \end{bmatrix} \quad (10) $$

**Figure 2:** Global block scheme of the CRCTC theory
5 MATLAB Simulations

5.1 Introduction

Several MATLAB programs have been written to simulate the behaviour of the controlled XY-table. In the simulations, the desired trajectory is a circle with a radius of 0.1 [m] and an angular velocity of $\pi$ [rad/s]. The first two programs solve the differential equations with an explicit Euler scheme while the third and fourth program use fourth order Runge-Kutta with variable step-size. The Euler scheme is chosen primarily because of its simplicity, the Runge-Kutta algorithm for its accuracy.

In the first and third program, a simplified problem is simulated: desired and actual trajectory are initially supposed to be identical. This is the most simple trajectory tracking problem. A bad performance in this experiment will mean that no high expectations will be reached. In the second and fourth program, the real problem is simulated: a desired trajectory has been defined and a reference trajectory has to be calculated while the actual trajectory initially doesn't equal the desired trajectory.

In all simulations, the spindle flexibility $k$ is set to $k = 0.46$ [N/m] and a sample frequency of $f = 100$ [Hz] is used.
5.2 Euler Integration Scheme Without Initial Tracking Error

In this first experiment, it is supposed that initially $\dot{\vartheta}_e = \dot{\vartheta}_d$, so the controller can concentrate on tracking the desired trajectory instead of first forcing the tracking error to decline.

In this experiment, an explicit first order Euler integration scheme is used to solve the differential equations. The Euler scheme has been chosen primarily because of its simplicity and speed. Due to its simplicity, high accuracy should not be expected.

![Graphs showing results for Euler integration scheme]

When examining the results, we see that a tracking error occurs for large negative $x$ and a smaller error for large positive $x$. A part of this tracking error can be explained when it is realised that for large $x$, the acceleration of the transverse is pointed opposite to the velocity, and the flexible spindle will be "winding up". This effect can be seen in all experiments presented here. The fact that the tracking error is larger for negative $x$ than for positive $x$ will be due to numerical problems.

The tracking error $\vartheta - \vartheta_d$ does not asymptotically approach zero, as it should considering the theory in Section 4 of this report. Although stability cannot be guaranteed, results are still promising with a maximum (periodical) tracking error of 5 [mm].

The system input remains very small, so from the actuator side it seems to be possible to improve tracking by tuning the controller.
5.3 Euler Integration Scheme With Initial Tracking Error

As opposed to the previous experiment where desired trajectory and actual trajectory are supposed to be identical, here the results are discussed from an experiment where an initial tracking error is present. This implies that the desired trajectory and the reference trajectory are not identical, so initially, $\dot{\theta}_r - \dot{\theta}_d \neq 0$ if $\dot{\theta}_d - \dot{\theta} \neq 0$. Because $\dot{\theta}_r = \dot{\theta}_d + \Delta(p_d' - p')$, the reference trajectory will not equal the desired trajectory until the tracking error $y_d - y = 0$. This implies that a reference trajectory has to be calculated at each time.

Of course, it could be expected that this experiment would show an inferior performance compared to the first experiment. However, with an acceptable controller layout, this inferior performance should be limited to the first few seconds.

\[ \text{Figure 4: Results for Euler integration scheme} \]

In the presented results, it is seen that after three seconds, the point $(x, y) = (-0.1, 0)$ is passed which is part of the desired trajectory. After passing this point, an approximated circle is described with centerpoint $(x, y) = (0.75, 0)$ and radius 0.1 [m] which is not the desired trajectory. This error is entirely due to bad tracking of the $x$-coordinate; the $y$-coordinate is tracked perfectly.

The presented results could not be improved by tuning because of instability occurring when the proportional controller parameters are increased in order to reduce the tracking error. The system is very sensitive for parameter fluctuation, so robustness is not good.
5.4 Runge-Kutta Integration Scheme Without Initial Tracking Error

In the next experiment, the Euler integration scheme is replaced by a fourth order Runge-Kutta scheme with variable step-size. Theoretically, this integration routine should perform better than the Euler scheme as it uses a smaller stepsize if useful while the Euler scheme uses a fixed step-size. In the used implementation, the integration interval for the Runge-Kutta routine is equal to the Euler stepsize.

In practice however, a worse performance can be seen than in experiment one. The negative x-coordinate has a larger amplitude than the positive which can't be explained on physical grounds, so it must be a numerical problem.

It may be concluded that for this experiment, using the (very time-consuming) Runge-Kutta algorithm did not boost the performance. In order to achieve this trajectory, large fluctuating torques are needed if x is positive and y negative. Again, there seems to be no physical ground for this phenomenon.

Figure 5: Results for Runge-Kutta integration scheme
5.5 Runge-Kutta Integration Scheme With Initial Tracking Error

In the previous experiment, we only saw a slight improvement compared to the first experiment. In the second experiment, a bad performance was found by combining the Euler algorithm with an initial tracking error. Reminding the small improvement in the third experiment, it is not realistic to expect perfect tracking here.

![Figure 6: Results for Runge-Kutta integration scheme](image)

Indeed, the improvement gained by using the Runge-Kutta algorithm is only small. The realised trajectory is practically identical to the trajectory in the second experiment. Again, the y-coordinate is tracked perfectly while the x-coordinate becomes smoother, but is not tracked better.

As in the third experiment, a sudden increase in the input can be seen when positive x-coordinate and negative y-coordinate are combined. It has to be doubted whether the "bang-bang"-control can be delivered by the used actuator. For sure, non-modelled higher-order dynamics will be excited, so the simulations will not present a reliable picture of the real system.

As in the second experiment, a point of the desired trajectory is passed. After 2.5 seconds, the point \((x, y) = (0, 0.1)\) is passed with a velocity near the desired velocity. Therefore, it is surprising that the controller is not able to keep the tracking error that small: it increases rapidly. Because of the (numerical) instability mentioned before, this could not be improved by tuning.
5.6 Discussion of the Results

It can be concluded that if no initial tracking error is present (such as in experiment one and three), the presented CRCTC-theory does yield an acceptable performance. However, if an initial tracking error is present (experiment two and four), numerical instability makes it impossible to tune the controller so that the desired trajectory is actually tracked. Here, it is remarkable that while the y-coordinate is hardly used to improve tracking by changing the y-coordinate; the numerical instability is mainly because of instability in x-direction.

Tracking is achieved with certain parameters (first and third experiment) when no initial tracking error is present, but with these parameters the initial tracking error does not fade away. On the other hand, with parameters that force the initial tracking error to fade away, no tracking is achieved. This controversy could be solved with some "mixed parameters". A CRCTC-controller with non-constant parameters could improve the performance because at each point, some appropriate parameters can be chosen. This so-called Adaptive Computed Reference Computed Torque Control is described by Lammerts (lit.7). The idea of Adaptive CRCTC is that one set of parameters could be used to decrease the tracking error while other sets will be tuned for tracking the desired trajectory after the tracking error is decreased enough. This possibility is not studied in this report; it could be part of future investigation.

Examining the results of the different integration routines, it might be expected that not the integration routine is to blame for the bad performance, but that the differential equation (9) itself causes the problem. The stability of equation (9) will be investigated in section 7.

6 Research on the Reference Equation

In the previous section, it was suggested that the bad performance was due to numerical instability of the reference equation (9). This equation can be rewritten into

\[
N^T AN_\rho r + N^T A \dot{H}_\rho r + N^T G N_\rho r + N G H \dot{\theta}_r + N^T k - N^T D N_\varepsilon^* - N^T P N_\varepsilon^* = 0
\]  

(11)

The matrix entries of \( A \) and \( G \) can be found in Appendix D.

For a further investigation of equation (11), a special case is investigated. First, it is assumed that the elastic deformation \( \rho = \frac{\rho}{\rho} (\varphi_1 - \varphi_2) \) remains small. In Appendix D, it is shown that for small \( \rho \) the following approximations hold:

\[
N^T A N_\rho r = a_{33}(\varphi); \quad N^T A H_\rho r = \begin{bmatrix} a_{31}(\varphi) & a_{32}(\varphi, \rho) \end{bmatrix}
\]

\[
N^T G N_\rho r = g_{33}(\varphi, \dot{\varphi}); \quad N^T G H_\rho r = \begin{bmatrix} 0 & g_{32}(\varphi, \dot{\varphi}) \end{bmatrix}
\]

The matrix entries \( a_{32}(\varphi, \rho) \) and \( g_{32}(\varphi, \dot{\varphi}) \) are linear functions of \( \rho \) and \( \dot{\rho} \). Secondly, it is assumed that the end-effector is and remains on the desired trajectory, so \( \varphi = \varphi_\rho = \varphi_d \) and \( \dot{\varphi} = \dot{\varphi}_d = \dot{\varphi}_\rho \). Then, equation (11) can be expanded into

\[
a_{33}(\varphi, \dot{\varphi}) + [a_{31}(\varphi, \rho)]\dot{\varphi}_\rho + [a_{32}(\varphi, \rho)]\dot{\varphi}_\rho + g_{33}(\varphi, \dot{\varphi}_\rho) \dot{\varphi}_\rho + [0 \ g_{32}(\varphi, \dot{\varphi}_\rho)] \dot{\varphi}_\rho + k_\rho = 0
\]  

(12)

Because \( \varphi_d \) and \( \dot{\varphi}_\rho \) are periodical functions, a linear second-order differential equation in \( \rho_r \) results with time-dependent, periodic coefficients. This enables us to investigate the stability of the reference equation (11) with Floquet's theory.
7 Application of Floquet's Theory

For stability analysis on the reference equation (9), Floquet's theory (lit. 1, 3, 6, 9) is used. A short summary of this theory can be found in Appendix C of this report. Floquet's theory has been developed for stability analysis on periodical solutions of linear differential equations with time-dependent coefficients such as the reference equation after fading out of the transient response.

As was mentioned before, equation (9) can be rewritten into

$$N^T A N \dot{\rho}_r + N^T A \dot{\theta}_r + N^T G N \dot{\rho}_r + N G \dot{\theta}_r + N^T k - N^T D N \ddot{e}^* - N^T P N e^* = 0$$  \hspace{1cm} (13)

In Appendix D, it shown that equation (13) can be approximated by a second order differential equation in \( \rho_r \) with time-dependent coefficients. This is the kind of equation that can be analyzed with the Floquet-theory when rewritten in state space notation. With \( \dot{x}(t) = [\rho_r, \dot{\rho}_r]^T \), this yields

$$\ddot{x}(t) = \mathbf{R}(t)\dot{x}(t) + \mathbf{S}(t)$$  \hspace{1cm} (14)

According to Appendix C, a conclusion on stability can be drawn on the eigenvalues of the matrix \( \Phi_A \)

$$\Phi_A = \Phi^{-1}(t)\Phi(t + T)$$

When starting at \( t = 0 \) [s], after one period (i.e. \( t = T = 2 \) [s]), the eigenvalues of \( \Phi_A = \Phi(T) \) are both equal to one. This implies that the reference equation (9) is on the edge of stability which means that an error will not asymptotically decay to zero. This behaviour can be seen in figure 3 and 5, where an existing tracking error stays present.
8 The Two-Point Boundary-Value Problem

8.1 Introduction

In the previous section, we saw rather bad tracking in the problems solved with conventional integration methods especially when an initial tracking error was present. After fading out of the transient response a periodic trajectory remains with period $T$: both the starting point $\mathbf{x}(t_0)$ and the final point $\mathbf{x}(t_0) = \mathbf{x}(t_0 + T)$ after one period are equal, and used to describe the trajectory. Therefore, it seems wise to use a two-point boundary-value problem (lit.2, 10) instead of the previous single-point boundary-value problem.

8.2 Solving the Two-Point Boundary-Value Problem

To solve the two-point boundary-value problem, a state-space notation is used like in section 6: $\mathbf{x} = [\rho_r \ \dot{\rho}_r]^T$. The problem is described by

$$\dot{\mathbf{x}}(t) = \mathbf{R}(t)\mathbf{x}(t) + \mathbf{S}(t)$$

In this equation, $\mathbf{R}(t)$ and $\mathbf{S}(t)$ are periodic with period $T$. The general solution of this differential equation is given by

$$\mathbf{x}(t) = \Phi(t) \int_{\tau=t_0}^{t} \Phi^{-1}(\tau) \mathbf{S}(\tau) \, d\tau + \Phi(t)\Phi^{-1}(t_0)\mathbf{x}(t_0)$$

In equation (15), $\Phi^{-1}(t_0)$ can be chosen equal to the identity matrix $I$. This yields

$$\mathbf{x}(t) = \Phi(t) \int_{\tau=t_0}^{t} \Phi^{-1}(\tau) \mathbf{S}(\tau) \, d\tau + \Phi(t)\Phi^{-1}(t_0)\mathbf{x}(t_0) = \mathbf{g}(t) + \Phi(t)\mathbf{x}(t_0) = \mathbf{x}(t + T)$$

(16)

$\Phi(t)$ is the so-called fundamental matrix or transition matrix. This transition matrix still has to be determined. This is done by setting $\mathbf{S}(\tau) = 0$ and sequentially solving equation (16) with $\mathbf{x}(t_0) = [1 \ 0]^T$ and $\mathbf{x}(t_0) = [0 \ 1]^T$. As soon as this is accomplished, the integral $\mathbf{g}(t)$ can be solved by setting $\mathbf{x}(t_0) = 0$.

In the next step, we use the periodicity of the solution to gain the initial position $\mathbf{x}(t_0)$ for the periodic part of the trajectory.

$$\mathbf{x}(t_0 + T) = \Phi(t_0 + T)\mathbf{x}(t_0) + \mathbf{g}(t_0 + T) = \mathbf{x}(t_0) \rightarrow \mathbf{g}(t_0 + T) = [I - \Phi(t_0 + T)]\mathbf{x}(t_0)$$

$$\mathbf{x}(t_0) = [I - \Phi(t_0 + T)]^{-1} \mathbf{g}(t_0 + T)$$

(17)

In equation (17), the initial position for the periodic part of the trajectory is derived. With equation (16), the position $\mathbf{x}(t)$ can be determined on any time past $t_0$.

The whole periodical part is now determined; the only remaining problem is to get the manipulator into the computed initial position for the periodical part.

8.3 Introduction to the Simulations

On basis of the previously described solution of the two-point boundary-value problem, the system is simulated in MATLAB. First, the desired trajectory for the third coordinate has to be determined. This is done with a fourth order Runge-Kutta algorithm with variable step-size. Because this is a rather time-intensive part of the simulation which is equal for all simulations with the same desired trajectory for the first two coordinates, this is done only once, and subsequently saved. The first simulation expresses the system behaviour while starting in the initial point for the
periodic part of the trajectory $\bar{x}(t_0)$ (section 8.2), so no initial tracking error was present. During simulation, all differential equations were solved using a fourth order Runge-Kutta integration scheme with variable step-size. This integration scheme is accurate, but rather slow, so in the second simulation, an Euler scheme was used. Earlier simulations (section 5) learned that, except for the system input, differences were small, but speed was considerably higher. In this second simulation, the situation with an initial tracking error was used.

### 8.4 Runge-Kutta Integration Schema Without Initial Tracking-Error

As was mentioned in the introduction, the initial position for this simulation was chosen exactly on the desired trajectory. This means that $\mathbf{e}^*(t_0) = \mathbf{e}^\prime(t_0) = 0$, so perfect tracking could be expected especially because the desired trajectory is purely periodical. In the previous simulations, $\rho_d$ didn't have to be periodical.

As can be seen in the previous figure, errors are small, but during this 20 second simulation it could not be proved that all errors are asymptotically decreasing to zero.
8.5 Euler Integration Scheme With Initial Tracking Error

In this simulation, the end-effector starts in the origin with velocity zero. A considerable initial tracking error is therefore present. When reminding the results of the simulation with two-point boundary-value problem and without initial tracking error, it should be expected that the initial tracking error will not disappear here either.

Indeed, the initial tracking error does not decrease, and actually increases in $x$-direction. The deformation remains small, but a consequence of this is that $x$ increases drastically. Again, the $y$-coordinate is tracked well.

8.6 Discussion of the Results

After simulating with straight-forward integration routines, it seemed wise to use an off-line two-point boundary-value problem to determine the desired trajectory instead of an on-line straight-forward integration. Simulations using this two-point boundary-value problem indicate that not all problems were solved by this step. Therefore, more research is needed to find out what the real problem is. This is done in the next subsection.
8.7 Further Analysis of CRCTC

As was mentioned in the introduction of this section, the idea behind the two-point boundary-value problem is to determine a bounded periodic solution \( \rho_d = \rho_d(t) \) for all \( t \) and use that solution to determine the input \( g \). This process is investigated using the error-equation for the controlled system (equation (5)):

\[
\begin{align*}
M^* \dot{\rho}^* + C^* \dot{\rho}^* + K_P \rho^* + K_P \dot{\rho}^* &= \mathbf{0} \\
\end{align*}
\]

When substituting \( \dot{\rho}^* = p^*_d - p^* \) and using the transformed equations of motion (4) to eliminate the term \( M^* \dot{\rho}^* \), this leads to

\[
\begin{align*}
M^* \ddot{p}^*_d + C^* \dot{p}^*_d + K_P \rho^* = M^* p^* + C^* \dot{p}^* = H^* y - k^* = \Phi^T (H_0 - k) \\
\end{align*}
\]

The partitioning \( p = H_0 + N \rho \) which was introduced in section 4 is used, and after premultiplication with \( \Phi^{-T} \), equation (18) can be written as

\[
\begin{align*}
\dot{\rho}_d = A N \dot{\rho}_d + s \\
\end{align*}
\]

which is the same equation as equation (7).

Because for the XY-table \( N^T H = 0^T \) is valid, premultiplication of equation (19) with \( N^T \) yields

\[
\begin{align*}
\dot{\rho}_d = -(N^T A N)^{-1} N^T s \\
\end{align*}
\]

if \( N^T A N \neq 0 \) for all relevant \( \rho^* \) (which is true for the XY-table). It should be mentioned that equation (20) equals equation (9), but is written in a different form.

With the off-line determination of the reference trajectory, it was assumed that \( p^*_r = p^*_d = p^* \), and therefore equation (20) becomes

\[
\begin{align*}
\dot{\rho}_d = -(N^T A_0 N)^{-1} N^T s_d \\
\end{align*}
\]

where \( A_0 = A(p^*_r) = A(H_0 \bar{\theta}_d + N \rho_d) = A(t, \rho_d) \) and \( s_d = A_0 \dot{\bar{\theta}}_d + G_0 \rho^*_d + k_d = s_d(t, \rho_d, \dot{\rho}_d) \). The reference trajectory \( \rho_d = \rho_d(t) \) is determined from equation (21), so the problem is that the solution \( \dot{\rho}_d = \rho_d(t) \) doesn’t satisfy equation (9) but only equation (21) while equation (21) is only valid if no tracking error occurs. In fact, the determined \( \rho_d = \rho_d(t) \) represents the deformation that will occur anyway if the end-effector tracks the desired trajectory perfectly.

All further analysis is based on the previous simulations: the desired trajectory \( \bar{\theta}_d = \bar{\theta}_d(t) \) is periodic with period \( T \), so \( \bar{\theta}_d(t + T) = \bar{\theta}_d(t) \quad \forall \ t \geq t_0 \). With the known solution of equation (21), the system input \( y \) can be determined. According to equation (7), \( y \) and \( \dot{\rho}_d \) must satisfy

\[
\begin{align*}
H_0 = AN \dot{\rho}_d + s \\
\end{align*}
\]

from which concludes that \( \dot{\rho}_d \) should satisfy \( \dot{\rho}_d = -(N^T A N)^{-1} N^T s \)

However, the deformation \( \rho_d \) as was determined in section 8.2 will generally not satisfy \( \dot{\rho}_d = -(N^T A N)^{-1} N^T s \). Therefore, equation (22) seems to be not very reliable. A possible solution is to substitute the previously determined \( \dot{\rho}_d \) in equation (22) and subsequently tread (22) as a set of three algebraic equations for the two unknown components of \( y \). Then, \( y \) can be determined as a least-squares solution of equation (22). This leads to

\[
\begin{align*}
y = (H^T H)^{-1} H^T (AN \dot{\rho}_d + s) ; \ \ \dot{\rho}_d \ \ \text{periodic} \\
\end{align*}
\]

Using equation (23) and the equations of motion yields

\[
\begin{align*}
\ddot{p}^* + \dot{p}^* + k = H(H^T H)^{-1} H^T (AN \dot{\rho}_d + s) \\
\end{align*}
\]

and when it is realised that for the studied XY-table \( H^T H = I \) and \( N H = 0^T \), it follows that

\[
\begin{align*}
H^T [\ddot{p}^* + \dot{p}^* + k - AN \dot{\rho}_d - s] = 0 \\
\end{align*}
\]
\[ N^T [\Delta \ddot{p}^* + G \dot{p}^* + k] = 0 \]

With the substitution of \( s = A H \ddot{\theta}_d + G \dot{p}_d + k + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) \), this leads to

\[ H^T \left[ A(\ddot{\theta}^* - \ddot{\theta}_d^*) + G(\dot{p}^* - \dot{p}_d^*) - \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) \right] = 0 \]
\[ N^T \left[ A(\ddot{\theta}^* - \ddot{\theta}_d^*) + G(\dot{p}^* - \dot{p}_d^*) + \Delta \ddot{p}_d^* + G \dot{p}_d^* + k \right] = 0 \]

and because \( [H, N] = I \) also to

\[ \Delta \ddot{p}_d^* + G \dot{p}_d^* + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) = \left[ \begin{array}{c} 0 \\ N^T(\Delta N \tilde{\rho}_d + s) \end{array} \right] \] (24)

Because the right-hand side of equation (24) can't be guaranteed to be equal to zero, it also can't be guaranteed that the tracking error will decrease to zero!

Another possible solution is using the control law (8), so

\[ y = H^T \left[ I - AN (N^T AN)^{-1} N^T \right] s \]
\[ s = A H \ddot{\theta}_d + G \dot{p}_d^* + k + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) = \Delta \ddot{p}_d^* + G \dot{p}_d^* + k + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) - \Delta N \tilde{\rho}_d \]

Substitution of equation (25) in the equation of motion leads to

\[ \Delta \ddot{p}_d^* + G \dot{p}_d^* + k = H H^T \left[ I - AN (N^T AN)^{-1} N^T \right] s \]

With \( \Delta \ddot{p}_d^* + G \dot{p}_d^* + k = \Delta \ddot{p}_d^* + G \dot{p}_d^* + k - \Delta \dot{\epsilon}^* - G \dot{\epsilon}^* = s + \Delta N \tilde{\rho}_d - \Delta \dot{\epsilon}^* - G \dot{\epsilon}^* - \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) \), this yields

\[ \Delta \ddot{p}_d^* + G \dot{p}_d^* + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*) = s + \Delta N \tilde{\rho}_d - H H^T \left[ I - AN (N^T AN)^{-1} N^T \right] s = \]

\[ [N (N^T AN) + H H^T AN] (N^T AN)^{-1} N^T s + \Delta N \tilde{\rho}_d = \]

\[ \Delta N \left[ (N^T AN)^{-1} N^T s + \tilde{\rho}_d \right] \]

As in the rest of this section, \( \tilde{\rho}_d \) in this equation is periodical. The term \( (N^T AN)^{-1} N^T s + \tilde{\rho}_d \) represents the error that was made in determining \( \tilde{\rho}_d \).

The error equation then becomes

\[ M^* \ddot{\epsilon} + \zeta^* \dot{\epsilon}^* + K_D \dot{\epsilon}^* + K_P \epsilon^* = M^* N \left[ (N^T AN)^{-1} N^T s + \tilde{\rho}_d \right] = \]
\[ M^* N (N^T AN)^{-1} N^T [\Delta \ddot{p}_d^* + G \dot{p}_d^* + k + \Psi^{-T}(K_D \dot{\epsilon}^* + K_P \epsilon^*)] \] (26)

The right-hand side of equation (26) in general will not equal zero, and therefore it can't be guaranteed that the tracking error will disappear!

### 8.8 Conclusion

In section 8.7, it was proved that it can't be guaranteed that the tracking error will disappear. Therefore, the CRCTC-theory doesn't reach it's expectations of controlling flexible systems. Neither tuning nor application of Adaptive Computed Reference Computed Torque Control will generally enable good tracking because they don't influence the form of the error equation.
9 Conclusions and Recommendations

9.1 Conclusions

- Computed Reference Computed Torque Control does not work well with an off-line determination of the reference trajectory. The problem is that in order to produce off-line a reference trajectory, some assumptions had to be made that proved to be invalid later. With systems without an initial tracking error, performance seems to be acceptable, but the deformation is large.

- Adaptive CRCTC is not a solution for the encountered problems because this algorithm has no effect on the error equation.

9.2 Recommendations

- Earlier experiments showed that CRCTC works well when the reference trajectory can be determined on-line with straight-forward integration. It should be investigated whether it is possible to determine the reference trajectory for the XY-table on-line. Multiple shooting techniques (lit.10) might enable this.
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A Model of the XY-Table in Motor Coordinates

The model of the XY-table that is used to derive a mathematical model is given in the next figure.

Figure 9: Model of the XY-table

v.d. Molengraft (lit.8) and Willems (lit.11) derived a model for this flexible TT-manipulator in terms of the motor coordinates \( q = [\varphi_1 \ \varphi_2 \ \varphi_3]^T \):

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + Kq + w = Hu
\]  
\[ (27) \]

The model matrices have the following structure:

\[
M = \begin{bmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad K = \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix}
\]

\[
w = \begin{bmatrix} w_1 \text{sign}(\varphi_1) \\ w_2 \text{sign}(\varphi_2) \\ w_3 \text{sign}(\varphi_3) \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]
where $m_{ij}$ and $c_{ij}$ are given by

\[
\begin{align*}
m_{11} &= J_1 + \left( m_s + \frac{1}{3} m_t \left( \frac{1}{d} \right)^2 + m_e \left( \frac{r}{d} \right)^2 \right) r^2 \\
m_{13} &= \left( \frac{1}{6} m_t \frac{1}{l} - \frac{1}{3} m_t \left( \frac{1}{d} \right)^2 + m_e \phi_2 \frac{r}{d} - m_e \left( \frac{r}{d} \right)^2 \right) r^2 \\
m_{22} &= J_2 + m_r r^2 \\
m_{23} &= m_e \left( \phi_1 - \phi_3 \right) \frac{r}{d} \frac{r^2}{d} \\
m_{31} &= m_{13} \\
m_{32} &= m_{23} \\
m_{33} &= \left( m_e + m_r - m_t \left( \frac{1}{d} \right) + \frac{1}{3} m_t \left( \frac{1}{d} \right)^2 + m_e - 2 m_e \left( \phi_2 \frac{r}{d} \right) + m_e \left( \frac{r}{d} \right)^2 \right) r^2 \\
c_{11} &= m_e \phi_2 \frac{r}{d} \phi_2 \\
c_{12} &= m_e \phi_2 \frac{r}{d} \phi_2 \left( \phi_1 - \phi_2 \right) \\
c_{13} &= -c_{11} \\
c_{21} &= -c_{12} \\
c_{22} &= c_{12} \\
c_{31} &= m_e \phi_2 \frac{r}{d} \left( d - \phi_2 r \right) \phi_2 \\
c_{32} &= m_e \phi_2 \frac{r}{d} \left( d - \phi_2 r \right) \left( \phi_1 - \phi_3 \right) \\
c_{33} &= -c_{31}
\end{align*}
\]

The physical parameters of the XY-table are given in Table 1.

<table>
<thead>
<tr>
<th>$m_s$</th>
<th>$m_t$</th>
<th>$m_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3 [kg]</td>
<td>8.5 [kg]</td>
<td>2.3 [kg]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$l$</th>
<th>$d$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25 [m]</td>
<td>1.00 [m]</td>
<td>0.01 [m]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 [Nm]</td>
<td>9 [Nm]</td>
<td>20 [Nm]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J_1$</th>
<th>$J_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0022 [kgm$^2$]</td>
<td>0.000158 [kgm$^2$]</td>
</tr>
</tbody>
</table>

Table 1: Physical parameters as estimated by v.d. Molengraft (lit.8)
B Model of the XY-Table in End-Effector Coordinates

The previously given model in terms of the motor coordinates $q$ is transformed into a model in terms of the end-effector coordinates $p^*$. An overview of the coordinate systems is given in the next figure.

Figure 10: The coordinate systems

$\dot{p}^* = \begin{bmatrix} \frac{x}{\sqrt{D(\varphi_1 - \varphi_3)}} \\ \frac{y}{\sqrt{D(\varphi_1 - \varphi_3)}} \end{bmatrix}$

$q = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \frac{1}{r} \begin{bmatrix} p_1^* + Dp_3^*(1 + p_2^*) \\ Dp_3^*\sqrt{1 + (p_2^*)^2} \\ p_1^* - Dp_3^*(1 - p_2^*) \end{bmatrix} = \psi(p^*)$

$\dot{q} = \Psi \dot{p}^*$

$\Psi = \frac{1}{r} \begin{bmatrix} 1 & 0 & D(1 + p_2^*) \\ 0 & D\sqrt{1 + (p_2^*)^2} & D\sqrt{1 + (p_2^*)^2} \\ 0 & -D(1 - p_2^*) & 1 \end{bmatrix}$

$\dot{\Psi} = \frac{1}{r} \begin{bmatrix} 0 & Dp_3^* & Dp_3^* \\ 0 & Dp_3^* & Dp_3^* \\ 0 & Dp_3^* & Dp_3^* \end{bmatrix}$

$s^* = \sqrt{1 + (p_2^*)^2}$

The original equations of motion (Willems (lit.11)) with $k = Kq + \psi$,

$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + k(q, \dot{q}) = H\dot{u}, \quad (28)$

now can be rewritten as

$M^*(p^*)\ddot{p}^* + C^*(p^*, \dot{p}^*)\dot{p}^* + k^*(p^*, \dot{p}^*) = H^*\dot{u} \quad (29)$

where, according to Lammerts (lit.7),

$M^* = \Psi^T M \Psi$ ; $C^* = \Psi^T \left( M \dot{\Psi} + C \Psi \right)$ ; $k^* = \Psi^T k$ ; $H^* = \Psi^T H$
Due to the choice for the end-effector coordinates \( \mathbf{p}^* \), the mass matrix \( \mathbf{M}^* \) and the Coriolis matrix \( \mathbf{C}^* \) can be simplified considerably.

\[
\mathbf{M}^*(q, \dot{q}, \ddot{q}) = r^2 \begin{bmatrix}
    m_{11}^* & 0 & m_{13}^* \\
    0 & m_{22}^* & m_{23}^* \\
    m_{13}^* & m_{23}^* & m_{33}^* 
\end{bmatrix}
\]

where \( m_{ij}^* \) is given by

\[
\begin{align*}
    m_{11}^* &= \frac{r^2}{r^2} + m_s + \frac{1}{2} m_t \left( \frac{r^2}{d} \right)^2 + m_e \left( \frac{r^2}{d} \right)^2 \\
    m_{13}^* &= \frac{m_s}{d} \left( 1 - \frac{2 r^2}{3 d} \right) + m_e \frac{(r^2 q_2)}{d} \left( 1 - \frac{r^2 q_2}{d} \right) \\
    m_{22}^* &= \frac{r^2}{r^2} + m_e \\
    m_{23}^* &= m_e p_3^2 \\
    m_{33}^* &= m_s + m_t \left( 1 - \frac{1}{d} + \frac{1}{3} \frac{r^2}{d^2} \right) + m_e \left( 1 - \frac{r^2 q_2}{d} \right)^2
\end{align*}
\]

\[
\mathbf{C}^*(q, \dot{q}, \ddot{q}) = m_e \left( \frac{r}{d} \right)^2 \begin{bmatrix}
    r^2 q_2 \dot{q}_2 & 2r D q_2 \ddot{p}_3^* & -r^2 q_2 \dot{q}_2 \\
    -2r D q_2 \ddot{p}_3^* & 0 & 2r D q_2 \ddot{p}_3^* \\
    r(d - r q_2) \dot{q}_2 & (d - r q_2) \cdot 2D \ddot{p}_3^* & r(r q_2 - d) \ddot{q}_2
\end{bmatrix}
\]
C Summary of Floquet’s Theory

Floquet’s theory has been developed for stability analysis of periodical solutions of linear differential equations with time-dependent coefficients, which are written in the form

\[ \dot{x}(t) = A(t)x(t) + B(t) \]  

(30)

In this equation, \( A \) is periodical with period \( T \). The solution is given by

\[ x(t) = \Phi(t, t_0)x(t_0) + \Phi(t, t_0) \int_{t_0}^{t} \Phi^{-1}(\tau, t_0)B(\tau) \, d\tau \]  

(31)

Obviously, \( \Phi(t, t_0) \) should satisfy \( \Phi(t_0, t_0) = I \). Furthermore, \( \Phi(t, t_0) \) satisfies the linear matrix differential equation

\[ \Phi(t, t_0) = A(t)\Phi(t, t_0) \]  

(32)

Because \( A(t) \) is periodical with period \( T \), it accounts that \( A(t + T) = A(t) \), which also concludes that if \( \Phi(t, t_0) \) is a solution, so is \( \Phi(t + T, t_0) \). According to Floquet it follows that

\[ \Phi(t + T, t_0) = \Phi(t, t_0)\Phi_\lambda \]  

(33)

In this equation, \( \Phi_\lambda \) is a non-singular constant matrix. By repeated use of equation (33), it can be found that

\[ \Phi(t + \kappa T, t_0) = \Phi(t, t_0)\Phi_\lambda^\kappa \]  

(34)

With a bounded value of \( \Phi(t, t_0) \), it will now be clear that for a bounded \( \Phi(t + \kappa T, t_0) \), the eigenvalues of the matrix \( \Phi_\lambda \) have to be equal to or smaller than 1.

It can be concluded that, with \( \lambda_1 > \lambda_2 > \lambda_3 \),

- \( |\lambda_1| < 1 \) \( \forall \, i \) \( \rightarrow \) (asymptotically) stable
- \( |\lambda_1| = 1 \) \( \rightarrow \) stable
- \( |\lambda_3| > 1 \) \( \rightarrow \) non-stable
D Matrix Entries for Periodicity Investigation

In this appendix, the following abbreviations are used for simplicity of notation:

\[ \lambda = \frac{1}{2} \]

\[ s^* = \sqrt{1 + (p_3^*)^2} \]

The matrix \( A \) is defined by \( A = M \Psi \):

\[
A = \begin{bmatrix}
H^T A H & H^T A N \\
N^T A H & N^T A N
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

where \( a_{ij} \) is given by

\[
a_{11} = r \left( \frac{1}{2} + m_s + \frac{1}{2} m_l \lambda \right) + \frac{1}{2} m_e s^* p_3^* + \frac{1}{2} m_e s^* (p_3^*)^2
\]

\[
a_{12} = \frac{1}{2} \left( m_{11} + m_{13} \right) D p_3^*
\]

\[
a_{13} = Dr \left( \frac{1}{2} s^* + m_e \left( s^* - p_3^* \right) \right)
\]

\[
a_{21} = \frac{1}{2} m_e s^* (1 + s^*) (p_3^*)^2
\]

\[
a_{22} = Dr \left( \frac{1}{2} s^* + m_e \left( s^* - p_3^* \right) \right)
\]

\[
a_{23} = Dr \left( -m_e p_3^* \right) + \left( m_e p_3^* + \left( \frac{1}{2} + m_e \frac{p_3^*}{s} \right) \right) p_3^* + \frac{1}{2} m_e s^* (p_3^*)^2
\]

\[
a_{31} = r \left( m_s + m_t - \frac{1}{2} m_t \lambda + m_e \right) - \frac{1}{2} m_e s^* p_3^*
\]

\[
a_{32} = Dr \left( \frac{1}{2} s^* + m_e \left( s^* - p_3^* \right) \right)
\]

\[
a_{33} = Dr \left( m_s + m_t - \frac{1}{2} m_t \lambda + m_e \left( s^* - p_3^* \right) \right) + \left( m_s + m_t - \frac{1}{2} m_t \lambda + m_e \left( s^* - p_3^* \right) \right) p_3^* + \frac{1}{2} m_e s^* (p_3^*)^2
\]

The matrix \( G = M \Psi + C \Psi \) has the following structure:

\[
G = \begin{bmatrix}
H^T G H & H^T G N \\
N^T G H & N^T G N
\end{bmatrix}
= \begin{bmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{bmatrix}
\]

where \( g_{ij} \) is given by

\[
g_{11} = 0
\]

\[
g_{12} = Dr \left( \frac{1}{2} + m_s + \frac{1}{2} m_l \lambda \right) + \frac{1}{2} m_e \left( 1 + (p_3^*)^2 + s^* \right) \frac{p_3^*}{s}
\]

\[
g_{13} = Dr \left( \frac{1}{2} + m_s + \frac{1}{2} m_l \lambda \right) + \frac{1}{2} m_e \left( s^* + 1 + (p_3^*)^2 \right) \frac{p_3^*}{s} + Dm_e \frac{p_3^*}{s} + \frac{1}{2} m_e s^* \frac{p_3^*}{s}
\]

\[
g_{21} = 0
\]

\[
g_{22} = D \left( \frac{1}{2} + m_e \left( p_3^* + \frac{p_3^*}{s} \right) \right) \frac{p_3^*}{s} + Dr \left( \frac{1}{2} + m_e \lambda \right) \frac{1}{s} - m_e \frac{1}{s}
\]

\[
g_{23} = 0
\]

\[
g_{31} = 0
\]

\[
g_{32} = Dr \left( m_s + m_t - \frac{1}{2} m_t \lambda + m_e \left( 1 + \frac{(p_3^*)^2}{s} \right) \right) + \frac{1}{2} m_e \left( s^* + 1 + (p_3^*)^2 \right) \frac{p_3^*}{s} + Dm_e \frac{p_3^*}{s} + \frac{1}{2} m_e s^* \frac{p_3^*}{s} + \frac{1}{2} m_e \frac{p_3^*}{s}
\]

\[
g_{33} = Dr \left( m_s + m_t - \frac{1}{2} m_t \lambda + m_e \left( s^* + \frac{(p_3^*)^2}{s} \right) \right) + \frac{1}{2} m_e \left( s^* + 1 + (p_3^*)^2 \right) \frac{p_3^*}{s} + Dm_e \frac{1}{s} + \frac{1}{2} m_e \frac{p_3^*}{s} - \frac{1}{2} m_e \frac{p_3^*}{s}
\]

Because \( p_3^* \) is small with respect to 1 (remember that \( p_3^* \) is the flexible coordinate: \( p_3^* = \frac{\varphi_1 - \varphi_2}{2D} \)), the matrix entries of \( A \) and \( G \) can be simplified because this also means that \( s^* \approx 1 \). From equation (11) follows that only \( N^T A N = a_{33} \), \( N^T A H = [a_{31} \ a_{32}] \), \( N^T G N = g_{33} \) and \( N^T G H = [g_{31} \ g_{32}] \) are significant for stability analysis.
These entries can be simplified into

\[a_{31} = r \left\{ m_s + m_t - \frac{1}{2} m_t \lambda + m_e \right\} - \frac{1}{2} m_e p^2_2\]

\[a_{32} = Dr \left\{ m_s + m_t - \frac{1}{2} m_t \lambda + 2 m_e \right\} p^2_2 - \frac{1}{2} m_e p^2_2^2\]

\[a_{33} = Dr \left\{ -m_s - m_t + \frac{3}{2} m_t \lambda - \frac{3}{2} m_t \lambda^2 - m_e \right\} + \left\{ m_s + m_t - \frac{1}{2} m_t \lambda + \frac{5}{2} m_e \right\} p^2_2 - m_e (p^2_2)^2\]

\[g_{31} = 0\]

\[g_{32} = Dr \left\{ m_s + m_t - \frac{1}{2} m_t \lambda \right\} - m_e p^2_2\]

\[g_{33} = Dr \left\{ m_s + m_t - \frac{1}{2} m_t \lambda + 2 m_e \right\} - m_e p^2_2\]

In these matrix entries, \(p^2_2\) and \(p^2_2\) are prescribed functions of the time, so all these entries are a function of the time and \(p^2_2\) or \(p^2_2\). Because \(p^2_1\) and \(p^2_2\) are periodical functions of the time, equation (11) turns into a time-dependent second order differential equation.