Note on a Wiener-Hopf integral equation arising in some inference and queueing problems

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Recently, Atkinson (1974) considered the Wiener–Hopf integral equations

\[ a(x) = \int_0^{\infty} a(y)k(x-y)dy \quad (1) \]

and

\[ a(x) = \int_0^{\infty} a(y)k(x-y)dy + Pk(x) \quad , \quad (2) \]

where

\[ P = \int_{-\infty}^{0} a(x)dx \quad (3) \]

and \( k(x) \) stands for the standard normal density function shifted over a distance \( \Delta > 0 \), viz.

\[ k(x) = (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x+\Delta)^2\right] \quad . \quad (4) \]

For simplicity the dependence of \( a(x) \) on \( \Delta \) has been suppressed in the notation. The equation (1) is connected with the problem of inference about the change-point in a sequence of random variables. The second equation arises from the problem of the stationary distribution of queueing time in a single server queue for which the difference between the service time and the inter-arrival time is normally distributed.

Atkinson (1974) derived explicit exact solutions for (1) and (2) by means of the Wiener–Hopf technique. In his analysis the key step of factorization is performed in terms of infinite products. In this note we propose a slightly different approach that avoids the use of infinite products. Moreover, it is felt that the present approach can easily be extended to the case of more general density functions \( k(x) \).
Following Atkinson (1974), we introduce the Fourier transforms

\[ F_+ (\lambda) = (2\pi)^{-\frac{1}{2}} \int_0^\infty a(x)e^{i\lambda x}dx, \quad F_- (\lambda) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^0 a(x)e^{i\lambda x}dx, \]  

then equations (1) and (2) can be reduced to

\[ F_+(\lambda)H(\lambda) + F_-(\lambda) = 0, \]  

\[ F_+(\lambda)H(\lambda) + F_-(\lambda) = \frac{P}{(2\pi)^{\frac{1}{2}}} \{1 - H(\lambda)\} \]  

resp., where

\[ H(\lambda) = 1 - \exp[-\Delta i\lambda - \frac{1}{2}\lambda^2]. \]  

Application of the Wiener-Hopf technique now requires the factorization of \( H(\lambda) \) into a product \( H_+(\lambda)H_- (\lambda) \), such that \( H_+(\lambda) \) is regular and nonzero in \( \text{Im} \lambda > 0 \), and \( H_- (\lambda) \) is regular and nonzero in some lower half-plane \( \text{Im} \lambda < \mu \) with \( \mu > 0 \). Rather than using infinite products, this factorization is effected by the method described in Noble (1958, Sec. 1.3). It is observed that

\[ |\exp[-\Delta i\lambda - \frac{1}{2}\lambda^2]| < 1 \]

in the strip \(-2\Delta < \text{Im} \lambda < 0\). Hence, \( \log H(\lambda) \) is regular and \( \log H(\lambda) \to 0 \) as \( \text{Re} \lambda \to \infty \), in that strip. We now present the actual factorization:

\[ H_+(\lambda) = \lambda L_+(\lambda), \quad H_- (\lambda) = \lambda^{-1} L_- (\lambda), \]  

\[ L_+(\lambda) = \exp\left[\frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{\log(1 - e^{-\Delta it} - \frac{1}{2}t^2)}{t - \lambda} dt\right], \quad \text{Im} \lambda > c, \]  

\[ L_- (\lambda) = \exp\left[-\frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{\log(1 - e^{-\Delta it} - \frac{1}{2}t^2)}{t - \lambda} dt\right], \quad \text{Im} \lambda < d, \]

where \( c, d \) are arbitrary subject to \(-2\Delta < c < d < 0\) and the logarithm stands for its principal value. It can easily be verified that the factors \( H_\pm (\lambda) \) do indeed meet the requirements stated above. Furthermore, it is seen by inspection that \( H_+(\lambda) \to \lambda \) and \( H_- (\lambda) \to \lambda^{-1} \) as \( \lambda \to \infty \) in their respective half-
planes of regularity. Therefore, the present factors $H_\pm(\lambda)$ are identical (although of a different form) to those derived in Atkinson (1974) except for a multiplicative constant.

The further solution of equations (6) and (7) runs along the same lines as in Atkinson (1974), leading to the final results

$$F_+(\lambda) = A_0/H_+(\lambda),$$

$$F_+(\lambda) = \frac{p}{(2\pi)^{\frac{1}{2}}} \frac{\lambda - H_+(\lambda)}{H_+(\lambda)}$$

resp., where $A_0$ is a constant to be determined from the condition $\alpha(x) + 1$ as $x \to \infty$.

As for equation (1), the main interest is in the coefficient $a_1(\Delta)$ appearing in the asymptotic expansion

$$1 - \alpha(x) \sim a_1(\Delta) e^{-2\Delta x}$$

as $x \to \infty$. Proceeding as in Atkinson (1974), we now find

$$a_1(\Delta) = \frac{L_+(0)L_-(2\Delta i)}{2\Delta^2}.$$  

The actual values of $L_+(0), L_-(2\Delta i)$ are quoted from (10) where both $c$ and $d$ are set equal to $-\Delta$ and $t$ is replaced by $\Delta(t-i)$. Thus we obtain

$$a_1(\Delta) = \frac{1}{2\Delta^2} \exp \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left( 1 - e^{-\frac{1}{2} \Delta^2 (1+t^2)} \right) \frac{dt}{1+t^2} \right].$$

The latter result is to be compared with the representation in terms of an infinite product as derived by Atkinson (1974, equation (13)):

$$a_1(\Delta) = \exp \left[ \frac{2\Delta^2}{(2\pi)^{\frac{1}{2}}} \prod_{n=1}^{\infty} \frac{\left( r_n + \Delta^2 \right)^{\frac{1}{2}} + 2^\frac{1}{2} \Delta}{\left( r_n + \Delta^2 \right)^{\frac{1}{2}} - 2^\frac{1}{2} \Delta} \exp \left[ \frac{-2\Delta}{(2\pi)^{\frac{1}{2}}} \right] \right]$$

where $r_n = (\Delta^4 + 16\pi^2 n^2)^{\frac{1}{2}}$ and

$$A = \lim_{M \to \infty} \left( \sum_{N=1}^{M-1} N^{-\frac{1}{4}} - 2M^4 \right).$$
It is remarked that $A = \zeta(\frac{1}{2}) = -1.4603545...$ where $\zeta(s)$ stands for Riemann's zeta-function; see Abramovitz & Stegun (1964, form. 23.2.9), Powell (1952). Notice that Atkinson's result $A = -1.46055$ is slightly in error.

The probability $P$ as given by (3), can be determined in the same manner as in Atkinson (1974). Thus we find

$$P = \exp \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 - e^{-\frac{1}{4}\Delta^2(1 + t^2)}}{1 + t^2} \, dt \right] = 2^{\frac{1}{2}}(a_1(\Delta))^\frac{1}{4}$$

(17)

in agreement with Atkinson (1974, equation (24)). From (17) it is obvious that $P \to 1$ as $\Delta \to \infty$.

Both representations (15) and (16) are well-suited for the numerical evaluation of $a_1(\Delta)$, although it seems that (15) is superior at moderate and large values of $\Delta$. For small values of $\Delta$, $a_1(\Delta)$ is most suitably determined by means of a rapidly convergent series expansion to be derived hereafter.

Let the exponent in (15) be denoted by $E(\Delta)$, then differentiation with respect to $\Delta$ yields

$$E'(\Delta) = \frac{\Delta}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}\Delta^2(1 + t^2)}}{1 - e^{-\frac{1}{4}\Delta^2(1 + t^2)}} \, dt =$$

$$= \frac{\Delta}{\pi} \sum_{n=1}^{\infty} e^{-\frac{1}{4}n\Delta^2} \int_{-\infty}^{\infty} e^{-\frac{1}{4}n\Delta^2 t^2} \, dt = \left( \frac{2}{\pi} \right)^\frac{1}{2} \sum_{n=1}^{\infty} e^{-\frac{1}{4}n\Delta^2} \frac{1}{n^\frac{1}{2}}.$$  

(18)

Referring to Magnus, Oberhettinger & Soni (1966, Sec. 1.6), the latter sum can be expressed in terms of Lerch's transcendent and we deduce the expansion

$$E'(\Delta) = \frac{2}{\Delta} + \frac{2^\frac{1}{2}}{\pi} \sum_{r=0}^{\infty} (-1)^{\frac{1}{4}r} \frac{\Gamma(\frac{1}{2}) \Gamma(r+\frac{1}{2})}{r!} \frac{\Delta^{2r}}{4\pi}$$

(19)

valid for $\Delta^2 < 4\pi$, where $[\frac{1}{4}r]$ denotes the largest integer $\leq \frac{1}{4}r$. In addition it is found that

$$E(\Delta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log\left[\frac{1}{4}\Delta^2(1 + t^2)\right]}{1 + t^2} \, dt + o(1) = \log(2\Delta^2) + o(1)$$

(20)

as $\Delta \to 0$. Therefore, by integration of (19) we obtain
\[ E(\Delta) = \log(2\Delta^2) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} (-1)^{r+1} \frac{\Gamma(r+\frac{1}{2}) \Gamma(r+\frac{1}{2})}{r!(r+\frac{1}{2})} \frac{\Delta^{2r+1}}{(4\pi)^{r+\frac{1}{2}}}, \quad (21) \]

and consequently

\[ a_1(\Delta) = \exp\left[ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} (-1)^{r+1} \frac{\Gamma(r+\frac{1}{2}) \Gamma(r+\frac{1}{2})}{r!(r+\frac{1}{2})} \frac{\Delta^{2r+1}}{(4\pi)^{r+\frac{1}{2}}} \right], \quad (22) \]

both expansions being valid for \( \Delta^2 < 4\pi \). The leading term of (22) agrees with the approximation by Atkinson (1974):

\[ a_1(\Delta) = \exp\left[ 2\Delta A/(2\pi)^{\frac{1}{2}} \right] \]

where \( A = \zeta(\frac{1}{2}) \). Atkinson found that the latter approximation suffices over the range \( 0 \leq \Delta \leq 1 \), the error being about \( 2\frac{1}{2}\% \) at \( \Delta = 1 \).

A second expansion, useful for large values of \( \Delta \), is obtained by integration of (18) over the interval \([\Delta, \infty)\). In view of \( E(\infty) = 0 \), we find

\[ E(\Delta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\Delta}^{\infty} e^{-\frac{1}{2}nt^2} dt = -2 \sum_{n=1}^{\infty} \frac{1 - \phi(n^{\frac{1}{2}})}{n}, \quad (23) \]

where \( \phi \) denotes the standard normal distribution function

\[ \phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt. \quad (24) \]

Corresponding expansions for \( a_1(\Delta) \) and \( P \) are obtained by substitution of (23) into (15) and (17). In particular, it is found that

\[ P = \exp\left[ \{-1 - \phi(\Delta) + O(\Delta^{-1} e^{-\Delta^2}) \} = \phi(\Delta) + O(\Delta^{-1} e^{-\Delta^2}) \right] \]

as \( \Delta \to \infty \). The latter approximation yields \( P = 0.999999971 \ldots \) at \( \Delta = 5 \), which is at variance with the entry \( P = 0.9988 \) in Atkinson (1974, Table 2).
REFERENCES


