Identification for Robust Control Using an H-infinity Norm

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Identification for Robust Control Using an $H_\infty$ norm

Abstract
Conventional identification techniques are based on stochastic assumptions and yield minimum variance estimates. A major drawback of these techniques is that disturbances as well as model-uncertainties are treated as additive noise and are thus minimized together. Consequently, hardly any indication of model-error bounds can be given, whereas robust control theory is based on a nominal model together with both norm-bounded model-uncertainties and noise characterized in the frequency domain.

It is our goal to present a new identification method using $H_\infty$ norms which yields a nominal model and information about the model-uncertainties applicable to $H_\infty$ robust control design. In the proposed method, disturbances and noise signals at the input and output of the system are characterized as norm-bounded.

The first step in the identification method is to compute uncertainty regions for the process dynamics in the complex plane. The a priori knowledge about the norm-bounds on the disturbances and properties of conformal mappings are used to compute these regions. The second step is to find a model in a model-set that minimizes the $H_\infty$ norm of the model errors, given the uncertainty regions. The first elementary implementation of the method is presented with two simulation studies.

Keywords: Identification, parameter estimation, robust control

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Notation:

\( P_t \) = True process-description  
\( P \) = Model  
\( \Delta_t \) = True model error ( = \( P_t - P \) )  
\( u_t \) = True input signal  
\( y_t \) = True output signal  
\( d_t \) = True input noise signal  
\( e_t \) = True output noise signal  
\( P_c \) = Centerpoint of uncertainty set \( P_c \)  
\( r_c \) = Radius of uncertainty set \( P_c \)  
\( P \) = Set of possible process-descriptions \( \tilde{P} \)  
\( P_c \) = Enclosing set for \( P ( P \subseteq P_c ) \) with elements \( \tilde{P}_c \)  
\( S \) = Model set with elements \( P \)  
\( U \) = Set with all possible input signals \( \tilde{u} \)  
\( Y \) = Set with all possible output signals \( \tilde{y} \)  
\( D \) = Set of possible input noise signals \( \tilde{d} \)  
\( E \) = Set of possible output noise signals \( \tilde{e} \)  

(The subscript 't' denotes 'true')


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**Section 1 : Introduction**

In conventional identification techniques a model set is proposed which is supposed to be capable of representing the process behaviour under study. Parameters are then tuned such that the model outputs correspond according to some criterion to the dominant part of a measured data set. Deviations are thought to be concentrated in some error source in the model such as output error, prediction error, equation error and the kind. This artificial error source is to explain for all disturbances acting on the process as well as for all model deviations from the real dynamic behaviour of the process. Furthermore stochastic assumptions have to be proposed concerning the errors leading to the criterion and as a result a "best" model is produced together with some stochastically based range for the parameters and/or dynamic behaviour.

For robust control design it is required to obtain a nominal model which is best in the sense that a known model error bound can be guaranteed (Doyle & Stein [2], Francis [3]). The disturbances should preferably be characterised by filters with bounded normed inputs. We seek new ways to cope directly with these requirements.

![Diagram](image)

**Fig.1 : Detailed setup**

A clear and detailed distinction between modelling errors and various disturbances is therefore indispensable. As a consequence an increased amount of detailed information should be acquired about the process in order to arrive at acceptable error bounds and just input/output data will be far from sufficient.

**Fig.1** superficially indicates our ideas about details. Extensive preliminary measurements and data processing should provide us with information about the actuator dynamics \( P_a \), bounds for the actuator modelling error \( \Delta_a \) (e.g. \( \| W_a \Delta_a \|_\infty \leq 1 \)) and disturbances, the disturbances acting on the process as detailed as possible, the sensor dynamics and measurement errors and finally the structure of the process as well...
as the structure of the modelling errors to be expected. For the sake of simplicity we indicated a nonstructured additive model error source (\( P_t = P + \Delta \)) to begin with, but of course other types will also appear relevant. In first instance we think of estimations and constraints in frequency domain so that \( P_s, W_s, W_t, W_o, W_w \) and \( W_m \) can be thought of as filters in frequency domain obtained from a priori studies. Also bounds in some norm on error sources \( \xi_s, \xi_t, \xi_i \) and \( \xi_o \) should be available as well as an appropriate input sequence \( r \) which constitutes the dataset \( \{r, v, y\} \). This dataset will generally consist of time samples and estimation in frequency domain requires a Fourier transform causing additive errors to be combined with the measurement noise. We mainly think of industrial processes where analysis directly in the frequency domain is either impossible or too costly. The estimation phase then consists of choosing a model \( P \) from a modelset \( S \) such that the model error \( \Delta \) (preliminary unstructured) is as small as possible in the sense that \( \|W_m\Delta\|_\infty \) is minimized with \( W_m \) is an appropriate weighting filter (\( W_m \) and \( W_m^{-1} \in \text{RH}_\infty \)).

Related papers as Helmicki et al. [2] and Zhu [3] also concern model error bounds in their identification-methods, the approaches however are different from the one in the present paper.

As an example we can depart from \( H_\infty \) norms on all disturbances: \( \|\xi_s\|_\infty \leq 1 \), \( \|\xi_t\|_\infty \leq 1 \), \( \|\xi_i\|_\infty \leq 1 \), \( \|\xi_o\|_\infty \leq 1 \). The additive errors due to Fourier transforms can be captured in above bounds. Since all analysis is in the frequency domain we may write for the true process \( P_t \) and the nominal model \( P \) to be estimated:

\[
\min_{P \in S} \| W_m (P_t - P) \|_\infty = \min_{P \in S} \| W_m \frac{y_t}{u_t} - P \|_\infty = \delta_m \tag{1}
\]

where \( y_t \) is in the allowed region \( y - W_s\xi_o - W_t\xi_s \) and \( u_t \) is in the cross section of \( (P_s + \Delta_s) \cdot r + W_s\xi_s \) and \( v - W_t\xi_i \).

The problem definition, therefore, turns out to be very simple for \( H_\infty \) domains. The same principle can be followed for e.g. \( \ell_1 \) norms or stochastic constraints. In a latter stage one can even think of combinations. The actual found bound \( \delta_m \) can be loosened in order to obtain a nominal model which fits better expectations based on stochastical knowledge. The control design constraints either allow this bound \( \delta_m \) or require a smaller bound. In the latter case a smaller bound can then be obtained by improving actuators and sensors and/or extending the modelset and dataset.

In this paper we will consider a stripped version of above concept, because we will use only information from either the measured input signal \( v \) or control signal \( r \). The other signal is not taken in consideration or is unknown.
We consider the situation of Fig. 2

\[
\begin{array}{c}
u \\
\downarrow d_t \\
\hline
P \\
\hline
\Delta \\
\hline
y_t \\
\downarrow y \\
\end{array}
\]

Fig. 2: Basic setup

True input and output signals \( u_t, y_t \) are measured in \( u \) and \( y \) where noise signals \( d \) and \( e \) are involved such that \( u = u_t - d \) and \( y = y_t + e \). The signal \( u \) can stand for either \( v \) or \( r \), the other signal is supposed to be unknown. We can think of a situation where the control signal \( r \) is unknown and so we choose \( u = v, d = -W_i \xi_1 \) and \( e = W_i \xi_1 + W_e \xi_2 \) to get from Fig. 1 to Fig. 2. We can also think of doing no measurements (\( v \) is unknown) and we choose \( u = P_r r, d = \xi_1 r + W_e \xi_2 \) and again \( e = W_e \xi_1 + W_e \xi_2 \). In both cases we arrive at the setup of Fig. 2 and we can give bounds in the frequency domain for the noise signals:

\[
\| W_d d \|_\infty \leq 1 \quad \text{and} \quad \| W_e e \|_\infty \leq 1
\]

For the filters \( W_d \) and \( W_e \) we get:

\[
\begin{array}{l}
| W_e | = | W_e | + | W_e | \\
| W_d | = | W_i | \\
| W_d | = | W_w | \cdot | r | + | W_e | \quad \text{for} \quad u = v \\
| W_d | = | W_w | \cdot | r | + | W_e | \quad \text{for} \quad u = P_r r
\end{array}
\]

The proposed identification method is done completely in the frequency-domain. The given data set \{ \( u(kT), y(kT) \) \} (see Fig. 2), with \( k \) an integer and \( T \) the sampling interval, consists of samples in the time-domain. In order to apply our identification method, the data set has to be transferred to the frequency-domain. However, only a limited number of data-samples are available, so the Fourier transformation will introduce an error on the frequency functions.

In Section 2 the error of the Fourier transformation of the measured finite-length time-domain signals to the frequency-domain will be studied. In Sections 3 and 4 we will consider the identification in the frequency-domain as was introduced by Van den Boom, Klompstra and Damen [5]. Section 3 gives some basic ideas how to derive uncertainty regions for the system dynamics in the frequency complex plane. In Section 4 it will be shown how to find a model in a model set that minimizes the upper bound of the \( H_\infty \)-norm of the model-error, given the uncertainty regions. In Section 5 two simulation studies are presented and finally Section 6 consists of a discussion and some concluding remarks.
Section 2 : Time-frequency domain transformation

Let us assume that the noise-free infinite-length data set \( \{ u(kT), y(kT) \} \) (see Figure 2) with \( k = 0,1,\ldots,\infty \) is available. The input-output relation of a linear time-invariant discrete system can be written as:

\[
y_i(kT) = h(kT) \ast u_i(kT)
\]

where \( h(kT) \) denotes the sampled impulse response and \( \ast \) stands for convolution. The discrete time Fourier transform is defined as:

\[
F(e^{j\omega}) = \sum_{k=0}^{\infty} f(kT)e^{-j\omega k}
\]

Applying the discrete time Fourier transform, \( (2) \), to \( (1) \) gives:

\[
Y_i(e^{j\omega}) = H(e^{j\omega})U_i(e^{j\omega})
\]

From \( (3) \) follows that the quotient \( Y_i(e^{j\omega})/U_i(e^{j\omega}) \) will be an exact description of the transfer function of the process, however, this is an ideal situation. The major problems we have to deal with in practice are:

- the length of the time series are finite
- the data will be corrupted by noise.

First, the effect of the finite length of the time series on relation \( (3) \) will be considered. Thus, given the noise-free finite-length data set \( \{ u_i(kT), y_i(kT) \} \) for \( k = 0,\ldots,N-1 \). Define the N-point discrete Fourier transform (DFT):

\[
F^N(e^{j\Omega m}) = \sum_{k=0}^{N-1} f(kT)e^{-j\Omega mk} \quad \Omega = \frac{2\pi}{NT} \quad m = 0,\ldots,N-1
\]

For most industrial processes the impulse response \( h(kT) \) is causal and of finite length, i.e \( h(kT) = 0 \) for \( k < 0 \) and \( k > L \). The length \( L \) of the impulse response can be obtained by preliminary process measurements (Backx & Damen [1]). Note that due to the finite length of \( h(kT) \) the discrete time Fourier transform \( (2) \) and the DFT \( (4) \) of \( h(kT) \) are equal at \( e^{j\Omega m} \) thus \( H(e^{j\Omega m}) = H^N(e^{j\Omega m}) \), where the length of the data set \( N-1 \) is chosen such that \( L \ll N-1 \). The DFTs of \( y_i(kT) \) and \( u_i(kT) \) are denoted by \( Y_i^N(e^{j\Omega m}) \) and \( U_i^N(e^{j\Omega m}) \) respectively. Assuming \( u_i(kT) = y_i(kT) = 0 \) for \( k < 0 \) and \( k > N-1 \) and that \( h(kT) = 0 \) for \( k < 0 \) and \( k > L \) the following formulas can be derived:
combining (5) and (6) gives the error in the frequency domain between the DFT estimate $Y_{N}^{N}(e^{j\omega n})/U_{N}^{N}(e^{j\omega n})$ and the true transfer function:

$$Y_{N}^{N}(e^{j\omega n}) - H(e^{j\omega n}) U_{N}^{N}(e^{j\omega n}) = \sum_{k=0}^{L} h(kT) e^{-j\omega nk} \sum_{s=0}^{N-1-k} u_{l}(sT) e^{-j\omega ms}$$

From equation (7) it is clear that the transformation error can be made zero by choosing $u_{l}(sT) = 0$ for $N-1-L \leq s \leq N-1$.

As already mentioned, the data set will be corrupted by noise. Consider the data set $\{u(kT), y(kT)\}$ with $k = 0, 1, ..., N-1$. Then, as is shown in Figure 2, there exists true input and output noise signals $d_{l}$ and $e_{l}$ such that $u_{l}(kT) = u(kT) + d_{l}(kT)$ and $y_{l}(kT) = y(kT) - e_{l}(kT)$. The noise signal $d_{l}$ is e.g. input sensor and/or actuator noise, and $e_{l}$ is e.g. output sensor noise. Assuming that the impulse response is of finite length, $L$ with $L \ll N-1$, and that $u_{l}(kT) = 0$ for $N-1-L \leq k \leq N-1$, hence the transformation error (7) will be zero. Then the DFT leads to the following estimate of $H(e^{j\omega n})$:

$$H(e^{j\omega n}) = \frac{Y_{N}^{N}(e^{j\omega n}) - E_{l}(e^{j\omega n})}{U_{N}^{N}(e^{j\omega n}) + D_{l}(e^{j\omega n})} \quad m = 0, 1, ..., N-1$$

Of course, in practice the true noise signals $d_{l}(kT)$ and $e_{l}(kT)$ are unknown, the only information we have is that they belong to certain classes of norm bounded signals.

In the simulation studies of Section 5 $u(kT) = v(kT)$, see Figure 1, there $u$ is chosen such that $u(kT) = 0$ for $N-1-L \leq k \leq N-1$. This means that due to the input noise $u_{l}(kT)$ is not exactly zero for $N-1-L \leq k \leq N-1$. Consequently there will be a small transformation error, which can be seen as disturbance on the output and can thus be easily included in signal class of $e$. How the estimate (8) can be used to identify the process will become clear in the next sections.
Section 3: Derivation of uncertainty regions

Consider the setup of Fig. 2 where a SISO linear time-invariant 'true' process $P_t(z)$ is excited by an unknown true input signal $u_t(z)$ that results in an unknown true output $y_t(z)$. The true input and output signals are corrupted by additive noise, $d_t(z)$ and $e_t(z)$ resp.

**A priori knowledge:** We have a measured input signal $u(z)$ and measured output signal $y(z)$. This data set $\{u(z), y(z)\}$ consists of data that is sampled in the frequency-domain on a sufficient number of frequencies from the set $\mathcal{O} = \{z_1, ..., z_n\} = \{e^{i\omega}, ..., e^{i\omega_n}\}$. Sufficient means that there are enough samples to describe the relevant frequency-behaviour of the system. The frequencies should cover the whole frequency-range (bandwidth) of the system, and the distance between the frequency-samples should be small enough to 'follow' the changes of the frequency-response. Further we have descriptions of two sets for the noise-signals in the frequency domain:

$$D = \{ \tilde{d}(z) | \| W_P^{-1}(z) \tilde{d}(z) \|_\omega \leq 1 \}$$
$$E = \{ \tilde{e}(z) | \| W_{P_t}^{-1}(z) \tilde{e}(z) \|_\omega \leq 1 \}$$

and we assume that $d_t(z) \in D$ and $e_t(z) \in E$.

**Purpose:** Find a model $P$ in a given model set $S$ such that the upper bound for the $H_\omega$-norm of the weighted additive model error $\Delta$ is minimized, so:

$$\min_{P \in S} \{ \max_{\tilde{d} \in D, \tilde{e} \in E} \| W_m \Delta \|_\omega \}$$

The first step in the identification procedure is to derive uncertainty regions for the system dynamics in the frequency complex plane.

From Fig. 2 we can easily see that

$$u_t(z) = u(z) + d_t(z) \quad \text{and} \quad y_t(z) = y(z) - e_t(z).$$

However, we do not know the signals $d_t(z)$ and $e_t(z)$, we only know that they belong to $D$ and $E$ respectively. Therefore we define the following sets:

$U$ is the set of signals $\tilde{u}(z) = u(z) + \tilde{d}(z)$ for all $\tilde{d} \in D$

$Y$ is the set of signals $\tilde{y}(z) = y(z) - \tilde{e}(z)$ for all $\tilde{e} \in E$

so

$$U = \{ \tilde{u}(z) \mid \| \tilde{u}(z) - u(z) \| \leq \| W_d(z) \| \}$$
$$Y = \{ \tilde{y}(z) \mid \| \tilde{y}(z) - y(z) \| \leq \| W_e(z) \| \}$$

Note that $u_t(z) \in U$ and $y_t(z) \in Y$. 

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Now we like to have an estimate for the true process: \( P_t(z) = y_t(z) / u_t(z) \).

Therefore we define the following set:
\( P \) is the set of functions \( \tilde{P}(z) = \tilde{y}(z) / \tilde{u}(z) \) for all \( \tilde{u}(z) \in U \) and \( \tilde{y}(z) \in Y \) (where we assume that \( \tilde{u}(z) \neq 0 \), \( \forall z \in O \)). This means that we deal with a persistently exciting input \( u_t(z) \) and a sufficiently small input noise-signal \( d \).

Note that the true process \( P_t \) is an element of set \( P \).

Remark: From now on in this section we will consider all signals and functions for a specific frequency. Therefore we will use a simplified notation (i.e. \( \tilde{u} \) in stead of \( u_t(z) \), \( W_d \) in stead of \( W_d(z) \)).

To derive the set \( P \) we will need an auxiliary set \( X \) (see Fig.3) with signals \( \tilde{x} = \tilde{u}^{-1} \) for all \( \tilde{u} \in U \).

**Theorem 1:**
Suppose \( U \) is the set of signals \( \tilde{u} \) satifying \( |\tilde{u} - u| \leq |W_d| \).
Assume that \( |\tilde{u}| > |W_d| > 0 \).
The set \( X \) with signals \( \tilde{x} = \tilde{u}^{-1} \) for all \( \tilde{u} \in U \) is given by the disk:
\[
X = \left\{ \tilde{x} \mid \left| \tilde{x} - \frac{u^*}{uu^* - W_d W_d^*} \right| \leq \frac{|W_d|}{uu^* - W_d W_d^*} \right\}
\]
The proof is in appendix 1.

This means that the set \( X \) is represented by a disk in the complex plane, like the sets \( Y \) and \( U \). For an easier notation we use:
\[
x \sim \frac{u^*}{uu^* - W_d W_d^*}
\]
and we get:
\[
|\tilde{x} - x| \leq \left| \frac{W_d}{u} \right|
\]

In Fig. 3a/d the sets \( U, Y \) and \( X \) are shown in the complex plane for one frequency sample. For every plot there is a different noise-to-signal ratio. As an example we can take \( d \) and \( e \) as white gaussian noises and let the bound be given by the \( 3\sigma \) bound. A thousand realizations of these disturbances have been presented by points in Fig.3a/d.
That way the point densities indicate the expected signal values. Certainly other distributions are possible and we do not have to look at it from a statistical point of view at all. Note that the point density in the set \( X \) is not concentrated around the center \( x \) but around \( u^{-1} \).
Fig. 3a: The sets $U$, $Y$, $X$ and $P$ in the complex plane.

Fig. 4a: The sets $P$, $P_d$ and $P_{c2}$ in the complex plane.
Fig. 3b: The sets $U, Y, X$ and $P$ in the complex plane.

Fig. 4b: The sets $P, P_{c1}$ and $P_{c2}$ in the complex plane.
Fig. 3c: The sets $U$, $Y$ and $P$ in the complex plane.

Fig. 4c: The sets $P$, $P_c$ and $P_c'$ in the complex plane.
Fig. 3d: The sets $U$, $Y$ and $P$ in the complex plane.

Fig. 4d: The sets $P$, $P_{C_1}$ and $P_{C_2}$ in the complex plane.
So we have the sets $Y$ and $X$ as

$$Y = \{ \tilde{y} \mid \tilde{y} = y(1 + \alpha_y \cdot e^{i\phi}), \ 0 \leq \phi \leq 2\pi, \ 0 \leq \alpha_y \leq \frac{|W_e|}{|y|} \}$$

$$X = \{ \tilde{x} \mid \tilde{x} = x(1 + \alpha_u \cdot e^{i\psi}), \ 0 \leq \psi \leq 2\pi, \ 0 \leq \alpha_u \leq \frac{|W_d|}{|u|} \}$$

We can derive the set $P$ with functions $\tilde{P}$ by multiplying all elements $\tilde{y}$ of $Y$ with all elements $\tilde{x}$ of $X$. This results in

$$P = \{ \tilde{P} \mid \tilde{P} = yx(1 + \alpha_y \cdot e^{i\phi})(1 + \alpha_u \cdot e^{i\psi})$$

$$, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \psi \leq 2\pi, \ 0 \leq \alpha_y \leq \frac{|W_e|}{|y|}, \ 0 \leq \alpha_u \leq \frac{|W_d|}{|u|} \}$$

This is a region with a typical bean-form. The next step will be to calculate a boundary function for the region.

First note that $\tilde{P}(\alpha_y, \alpha_u, \phi, \psi)$ for certain values $\alpha_y$, $\alpha_u$, $\phi$ and $\psi$ will only be a boundary point of the region if $\tilde{y}(\alpha_y, \phi)$, $\tilde{u}(\alpha_u, \psi)$ are also both boundary points. So we have to fix $\alpha_y = \frac{|W_e|}{|y|} \Delta r$ and $\alpha_u = \frac{|W_d|}{|u|} \Delta s$, and we obtain

$$\tilde{P}(r, s, \phi, \psi) = yx(1 + r e^{i\phi})(1 + s e^{i\psi}) \Delta yx h(\phi, \psi)$$

The problem reduces into finding the boundary function of the function $h(\phi, \psi) = (1 + r e^{i\phi})(1 + s e^{i\psi})$. 

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Theorem 2:
Let \( r \) and \( s \) be real constants with \( 0 < r < 1 \) and \( 0 < s < 1 \). The bounding curve for the region with points \( h(\phi, \psi) = (1 + re^{i\phi})(1 + se^{i\psi}) \), where \( 0 \leq \phi \leq \pi \) and \( 0 \leq \psi \leq \pi \), is given by the function:

\[
h_b(\psi) = 1 + se^{i\psi} + \frac{r^2}{2}(e^{i2\psi} - 1) + \]
\[+ re^{i\psi} \sqrt{-r^2 \sin^2 \psi + 2s \cos \psi + 1 + s^2}
\]

The proof of this theorem is in appendix 2.

The boundary of the region becomes \( \tilde{P}_b(\psi) = y \cdot x \cdot h_b(\psi) \).

In Fig.3a/d the regions \( P \) are given with its boundary functions. Also shown in the figure is the result of the original 1000 random realizations of \( d \) and \( e \).

The boundary function \( \tilde{P}_b(\psi) \) is not an nice function to work with. Therefore we will try to find a disk in the complex plane that encloses the region \( P \).

A simple circular boundary for this set \( P \) can be derived very easily. Define:

\[
P_{c1} \triangleq y \cdot x \quad \text{and} \quad r_{c1} \triangleq |P_{c1}| \cdot \left( \frac{W_e + W_d}{|y|} \cdot \frac{W_e \cdot W_d}{|y \cdot u|} \right) = |P_{c1}| (r + s + rs)
\]

Then it will hold for all \( \psi, \phi, \alpha_y \) and \( \alpha_u \) that:

\[
|\tilde{P} - P_{c1}| = |yx (1 + \alpha_y \cdot e^{i\phi}) (1 + \alpha_u \cdot e^{i\psi}) - yx| =
\]
\[= |yx \left(\alpha_y \cdot e^{i\phi} + \alpha_u \cdot e^{i\psi} + \alpha_y \cdot \alpha_u \cdot e^{i(\phi + \psi)}\right)| \leq
\]
\[\leq |P_{c1}| (r + s + rs) = r_{c1}
\]

So an enclosing set \( P_{c1} \supseteq P \) can be given as:

\[
P_{c1} = \{ \tilde{P}_{c1} = P_{c1} + \alpha_c \cdot e^{i\theta}, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \alpha_c \leq r_{c1} \}
\]
The set $P_{ci}$ will enclose the set $P$ very tightly as long as

$$r = \frac{|W_x|}{|y|} < 1 \quad \text{and} \quad s = \frac{|W_d|}{|u|} < 1 .$$

If these values will increase the enclosing will be less tight.

This 'simple' enclosing set is easy to calculate and will satisfy in most cases.

If this enclosing set does not satisfy, we will have to find the smallest enclosing set $P_{ci}$ for the set $P$. This can be done by calculating a smaller enclosing circle for the set $P$.

Before we derive this smallest enclosing set we will consider a special point on the boundary, namely the points where $\text{Im}(h(\phi,\psi))$ reaches extrema.

In all points on the boundary it hold:

$$\frac{\partial h(\phi,\psi)}{\partial \phi} = \frac{\partial h(\phi,\psi)}{\partial \psi} \cdot C_0 \quad \text{with} \quad C_0 \in \mathbb{R}$$

and on top of that, in the extrema it will hold:

$$\text{Im} \left( \frac{\partial h(\phi,\psi)}{\partial \phi} \right) = r \cos(\phi) + rs \cos(\phi + \psi) = 0$$

and

$$\text{Im} \left( \frac{\partial h(\phi,\psi)}{\partial \psi} \right) = s \cos(\psi) + rs \cos(\phi + \psi) = 0$$

resulting in the condition

$$r \cos(\phi) = -rs \cos(\phi + \psi) .$$

Suppose that $(\phi_0,\psi_0)$ is the solution corresponding to the maximum, and because of the symmetry with respect to the real axis $(-\phi_0,-\psi_0)$ the solution corresponding to the minimum, then define: $c_0 \triangleq \text{Re}(h(\phi_0,\psi_0)) \in \mathbb{R}$, $\rho_0 \triangleq \text{Im}(h(\phi_0,\psi_0)) \in \mathbb{R}$.

**Theorem 3:**

Let $\phi_0$, $\psi_0$, $c_0$ and $\rho_0$ be defined as above.

Then we can define the circle $C$ with centerpoint $c_0$ and radius $\rho_0$.
This circle $C$ is the smallest enclosing circle of $h(\phi,\psi)$.

The proof of this theorem is in appendix 3.
So the smallest enclosing set $P_{c2} \supseteq P$ can be given as:

$$P_{c2} = \{ \tilde{P}_{c2} = P_{c2} + \alpha_c \cdot e^{i\phi}, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \alpha_c \leq r_{c2} \}$$

where $P_{c2} = yx \mathcal{C}_0 \triangleq \text{Re}(h(\phi_0, \psi_0))$ and $r_{c2} = |yx| \rho_0 \triangleq \text{Im}(h(\phi_0, \psi_0))$.

In Fig. 4a/d the sets $P$ from Fig. 3a/d are given with the enclosing sets $P_{cl}$ and $P_{c2}$. It is clear that the 1000 original random realizations of $d$ and $e$ do not lead to a clustering around $P_{cl}$ or $P_{c2}$. Consequently in expectation $P_{cl}$ and $P_{c2}$ are not necessarily close to the real system transfer.
Section 4: Optimal $H_{\infty}$-model fitting

In this Section we will consider the parametrization of an optimal model in the sense that the upper bound of the $H_{\infty}$-norm of the model error $\Delta$ is minimized.

We define a model set $S$ with the models $P(\theta,z)$ where $\theta \in \Theta$ is a vector with the model-parameters. Of course we can choose many different types of models like ARMAX, Matrix fraction description, State-space models, etc. For we are interested in model-errors we define a set $F$ with functions

$$\Delta(z) = \hat{P}(z) - P(z) \text{ for all } \hat{P}(z) \in P \text{ and } P(z) \in S.$$  

Because $P(z) \in P$ we note that the true model error, defined as $\Delta(z) = P(z) - P(z)$, will be in this set $F$.

In fact we would like to minimize the $H_{\infty}$-norm of the model-error ($\|\Delta\|_{\infty}$) over all admissible models in the modelset $S$. However in the presence of input and output noise we can only give an upper bound:

$$\min_{P \in S} \|\Delta\|_{\infty} \leq \min_{P \in S} \max_{\hat{P} \in P} \|\hat{\Delta}\|_{\infty} = \min_{P \in S} \max_{\hat{P} \in P} \|\hat{P} - P\|_{\infty}.$$  

To emphasize specific frequency-ranges we can introduce a (stable and minimum phase) weighting filter $W_m(z)$, and we can minimize the $H_{\infty}$-norm of the weighted model-error:

$$\min_{P \in S} \|W_m \Delta\|_{\infty} \leq \min_{P \in S} \max_{\hat{P} \in P} \|W_m \hat{\Delta}\|_{\infty} = \min_{P \in S} \max_{\hat{P} \in P} \|W_m (\hat{P} - P)\|_{\infty}.$$  

The problem is now reduced to a min-max problem, but with the use of an approximate set $P_c \supseteq P$ ($P_c$ is either $P_{CI}$ or $P_{CO}$) we can write:

$$\min_{P \in S} \max_{\hat{P} \in P} \|W_m \cdot (\hat{P} - P)\|_{\infty} \leq $$

$$\min_{P \in S} \max_{\hat{P} \in P} \|W_m \cdot (\bar{P}_c - P)\|_{\infty} =$$
\[
\min_{\theta \in \Theta} \max_{\bar{P} \in P} \max_{z \in \mathcal{O}} | W_m(z) \cdot (\bar{P}_c(z) - P(\theta, z)) | =
\]

\[
\min_{\theta \in \Theta} \max_{z \in \mathcal{O}} | W_m(z) | \cdot | \bar{P}_c(z) - P(\theta, z) | =
\]

\[
\min_{\theta \in \Theta} \max_{z \in \mathcal{O}} | W_m(z) | \cdot \left( | P_c(z) - P(\theta, z) | + r_c \right)
\]

Now we define the upper bound of the model error for a model with parameter-vector \( \theta \) and for a frequency \( z \in \mathcal{O} \) as:

\[
\epsilon_{\max}(\theta, z) = | P_c(z) - P(\theta, z) | + r_c(z).
\]

The final problem we like to solve now becomes:

\[
\min_{\bar{P} \in S} \| W_m A \|_\infty \leq \min_{\theta \in \Theta} \max_{z \in \mathcal{O}} \left\{ | W_m(z) | \cdot | P(\theta, z) - \bar{P}_c(z) | \right\} =
\]

\[
= \min_{\theta \in \Theta} \| W_m(z) \epsilon_{\max}(\theta, z) \|_\infty
\]

The problem turns out to be the minimization of the \( H_\infty \)-norm of a function \( W_m \epsilon_{\max} \) over all admissible \( \theta \). Here we find the major drawback of using an \( \infty \)-norm, namely that the cost-criterion \( W_m \epsilon_{\max} \) is not differentiable. This means that we can not directly use a gradient method search for the minimum. One way to solve the problem is using methods which do not need gradients e.g. simplex-methods. The problem however with these methods is that convergence is not guaranteed if the initial value of \( \theta \) is far from the optimal value. During the simulations in section 5 this problem will not present because we know the actual true parameters. In a practical situation we can use the results from an earlier identification.
Section 5 : Simulation studies

In this Section we present two simulation studies. First we study a second order simulation model

\[
P_1 = \frac{z^2 - 1.1z + 0.24}{z^2 - 1.6z + 0.68}
\]

that is excited by a control signal \( u(k) = r(k) \) ( \( P_s = 1 \), \( \Delta_s = 0 \) ), and where we measure the output \( y(k) \). \( r(k) \) is generated for 1024 samples and has approximately a flat spectrum \( |r(z)| > 29 \) and care have been taken that \(-4 \leq r(k) \leq 4\) and that \( r(k) = 0 \) for \( k = 1,\ldots,100 \) and \( k = 924,\ldots,1024 \). The control input signal and measured output signal are corrupted by additive white gaussian noise \( d \) resp. \( e \), with \( W_d \) and \( W_e \) are the \( 3\sigma \) bounds in the frequency domain. This results in the following values for the noise to signal ratios:

\[
\frac{|W_d(z)|}{|y(z)|} \leq 0.16 \quad \text{and} \quad \frac{|W_e(z)|}{|u(z)|} \leq 0.11 .
\]

We now obtained an artificial data-set \( \{ u(k), y(k) \} \). For this choise of control signal \( u(k) = r(k) \) the errors due to the Fourier transformation of \( u(k) \) and \( y(k) \) are negligible. For the frequency set \( \Theta \) we choose \( z_i = \exp(j\omega) \) with \( \omega_i = i\pi/512 \) for \( i = 1,\ldots,512 \). For all frequencies \( z_i \) we calculate \( P_c(z_i) \) and \( r_c(z_i) \) and we get the regions as in Fig.5.

![Fig.5 : Uncertainty regions](image1.png)
Now we must optimize the function

$$\gamma_{opt} = \min_{\theta \in \Theta} \max_{\theta \in \Theta} | W_m(z) | \cdot (| P_C - P(\theta, z) | + r_c )$$

where we choose a low pass filter $W_m(z) = 1$.

In a first run we choose as a model set $S$ all first order transfer functions

$$P(z) = \frac{\theta_2 z + \theta_2}{z + \theta_1}$$

so $P_i$ is not in the model set $S$.

We find an optimal

$$\theta_{opt} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.2752 \\ 0.8948 \\ 0.7574 \end{bmatrix} \quad \text{where } \gamma_{opt} = 0.9363.$$

In Fig.6 $P_i(z)$, optimal model $P(z)$ and region centerpoints $P_i(z)$ are given in the complex plane.

Now we define three functions:

$$\epsilon_{\max}(\theta,z) = | P_C(z) - P(z) | + r \quad \text{(see section 4)}$$
$$\epsilon_{\med}(\theta,z) = | P_C(z) - P(z) |$$
$$\epsilon_{\min}(\theta,z) = \max \{ ( | P_C(z) - P(z) | - r_c ) , 0 \}$$

The function $\epsilon_{\max}$ gives an upper bound for the model error, the function $\epsilon_{\min}$ gives a lower bound, and the function $\epsilon_{\med}$ is centered between these bounds. In Fig.7a the functions $\epsilon_{\max}$, $\epsilon_{\med}$, $\epsilon_{\min}$ and the true model error $| \Delta_i |$ are plotted for the estimated model. The lower bound $\epsilon_{\min}$ in this example is for nearly all frequencies larger than zero, which indicates, that the found nominal model cannot describe the system accurately. Note that these 'estimates' $\epsilon_{\max}$, $\epsilon_{\med}$ and $\epsilon_{\min}$ can always be calculated and be used for defining a weighting filter $W_m$ in a next iteration. If we want a better model for higher frequencies we choose a filter that will emphasize the error in the higher frequencies, so $W_m(z)$ is large for higher frequencies, and will be small for the lower frequencies. Therefore we define a highpass filter as a weighting filter.
\[ W_m(z) = \frac{z + 0.16}{z + 0.7} . \]

For this choice of weighting filter we find \[ \theta_{opt} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.4016 \\ 1.2403 \\ 0.1895 \end{bmatrix} \]

where \( \gamma_{opt} = 1.0307 . \)

In Fig. 6 \( P(z) \), optimal model \( P(z) \) and region centerpoints \( P_c(z) \) are given in the complex plane.

In Fig. 7b the functions \( e_{max} \), \( e_{med} \), \( e_{min} \) and the true model error \( | \Delta_t | \) are plotted for the estimated model. Comparing the curves of the unweighted and the weighted case in Fig. 6 we can see that the model error decreased very much for the higher frequencies, with the cost of a small increase for the lower frequencies.

If we want a better model for lower frequencies we choose a lowpass filter as a weighting filter

\[ W_m(z) = \frac{z + 0.7}{z + 0.16} . \]

For this choice of weighting filter we find \[ \theta_{opt} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.2264 \\ 0.8096 \\ 0.9622 \end{bmatrix} \]

where \( \gamma_{opt} = 1.1645 . \)

In Fig. 6 \( P(z) \), optimal model \( P(z) \) and region centerpoints \( P_c(z) \) are given in the complex plane.

In Fig. 7c the functions \( e_{max} \), \( e_{med} \), \( e_{min} \) and the true model error \( | \Delta_t | \) are plotted for the estimated model. It is clear the model did not improve very much for lower frequencies, on the other hand it got much worse for the higher frequencies. It is clear that approximating the true process by a low order model is much more difficult for the lower frequencies than for the higher frequencies.

In a second run we choose as a model set \( S \) all second order transfer functions

\[ P = \frac{\theta_3 z^2 + \theta_4 z + \theta_5}{z^2 + \theta_1 z + \theta_2} \]
so \( P_t \) is in the model set \( S \) and as a weighting filter \( W_m(z) = 1 \).

We find an optimal
\[
\theta_{\text{opt}} = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\end{bmatrix} = \begin{bmatrix}
-1.5890 \\
0.6785 \\
1.0207 \\
-1.1161 \\
0.2591 \\
\end{bmatrix}
\]
where \( \gamma_{\text{opt}} = 0.5360 \).

In Fig. 6 \( P_t(z) \), optimal model \( P(z) \) and region centerpoints \( P_r(z) \) are given in the complex plane.

In Fig. 7d the functions \( \epsilon_{\text{max}} \), \( \epsilon_{\text{med}} \), \( \epsilon_{\text{min}} \) and the true model error \( |\Delta_t| \) are plotted for the estimated model. The lower bound \( \epsilon_{\text{min}} \) in this example is exactly zero, which indicates that the found nominal model might indeed describe the system exactly.

In a third run we choose as a model set \( S \) all third order transfer functions
\[
P = \frac{\theta_3 z^2 + \theta_4 z + \theta_5}{z^2 + \theta_1 z + \theta_2}
\]
so \( P_t \) is in the model set \( S \).

We find an optimal
\[
\theta_{\text{opt}} = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\theta_7 \\
\end{bmatrix} = \begin{bmatrix}
-1.6101 \\
0.7029 \\
-0.0101 \\
1.0382 \\
-1.1565 \\
0.2612 \\
0.0063 \\
\end{bmatrix}
\]
where \( \gamma_{\text{opt}} = 0.5347 \).

In Fig. 6 \( P_t(z) \), optimal model \( P(z) \) and region centerpoints \( P_r(z) \) are given in the complex plane.

In Fig. 7e the functions \( \epsilon_{\text{max}} \), \( \epsilon_{\text{med}} \), \( \epsilon_{\text{min}} \) and the true model error \( |\Delta_t| \) are plotted for the estimated model. Also in this case the lower bound \( \epsilon_{\text{min}} \) in this example is equal to zero. However the upper bound for the model error is not decreased very much, so it looks as if a second order model will satisfy in this case (as could be expected).
Fig. 6: $P(z)$, $P_c(z)$ and various models $P(z)$ in the complex plane:

(a) : 1st order model without weighting.
(b) : 1st order model with weighting on higher frequencies.
(c) : 1st order model with weighting on lower frequencies.
(d) : 2nd order model without weighting.
(e) : 3rd order model without weighting.

Fig. 7a: True model error with bounds (1st order, no weighting).
Fig. 7b: True model error with bounds (1st order, high frequency weighting).

Fig. 7c: True model error with bounds (1st order, low frequency weighting).

Fig. 7d: True model error with bounds (2nd order).

Fig. 7e: True model error with bounds (3rd order).
In a second simulation study we replace the simulation model of the first example by a first order process with a delay of 2 samples

\[ P_1 = \frac{z - 0.1}{z - 0.8} \cdot z^{-2} \]

that is excited by a control signal \( u(k) = r(k) \) (\( P_s = 1 \), \( \Delta_s = 0 \)), and where we measure the output \( y(k) \). \( r(k) \) is generated for 2048 samples and has approximately a flat spectrum |\( r(z) \)| > 20 and care have been taken that \(-2 \leq r(k) \leq 2 \) and that \( r(k) = 0 \) for \( k = 1, \ldots, 100 \) and \( k = 1949, \ldots, 2048 \). The control input signal and measured output signal are corrupted by additive white gaussian noise \( d \) resp. \( e \), with \( W_d \) and \( W_e \) are the 3\( \sigma \) bounds in the frequency domain. This results in the following values for the noise to signal ratios:

\[
\frac{|W_e(z)|}{|y(z)|} \leq 0.13 \quad \text{and} \quad \frac{|W_d(z)|}{|u(z)|} \leq 0.11 .
\]

We choose \( W_m(z) = 1 \). In the same way as the first example we obtain a artificial data-set \{\( v(k), y(k) \}\}. For the frequency set \( O \) we choose \( z_i = \exp(i\omega_i) \) with \( \omega_i = i \cdot \pi/512 \) for \( i = 1, \ldots, 200 \) (So we are only interested in low frequency behaviour), and we start the identification. In the first run we choose as a model set \( S \) all first order transfer functions

Clearly the system is not in the modelset. We find an optimal \( \theta_{\text{opt}} \):

\[
\theta_{\text{opt}} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.916 \\ 0.600 \\ -0.109 \end{bmatrix}
\]

In a second run we choose as a model set \( S \) all second order transfer functions

\[
P = \frac{\theta_3 z^2 + \theta_4 z + \theta_5}{z^2 + \theta_1 z + \theta_2}
\]
Again the system is not in the modelset. We find an optimal \( \theta_{opt} \):

\[
\theta_{opt} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} -0.272 \\ -0.444 \\ 0.039 \\ -0.108 \\ 1.163 \end{bmatrix}
\]

In Fig. 8 and Fig. 9 the three functions \( \epsilon_{max}, \epsilon_{med} \) and \( \epsilon_{min} \) are plotted for the first and second order estimation. In both experiments we evidently find \( \epsilon_{min} > 0 \) and we see that the upper bound \( \epsilon_{max} \) and the lower bound \( \epsilon_{min} \) are relatively very close to each other, indicating that the model error is mainly determined by undermodelling and not by the noise. \( \epsilon_{med} \) gives a good indication for the true model error.

![Fig. 8: True model error with bounds (1st order).](image)

![Fig. 9: True model error with bounds (2nd order).](image)
Section 6: Discussion and conclusions

General view:
In this report we concern an identification method that uses detailed information from a priori knowledge and experiments. Various error sources need to be studied intensively, and characteristics of the actuators and sensors are separated. The identification method makes use of error sources with a known bound in the frequency domain. A nominal model is found with a minimal model error in H∞-norm.

Input design:
The results of the identification procedure can be improved by choosing appropriate input signals. If we have an indication about the length of the impulse response of the true process, we can reduce the error due to the discrete Fourier transformation to zero. Further we have to make sure that the designed input signal is persistently exciting with a sufficiently large signal-to-noise ratio.

Particular stripped implementation & simulation studies:
The identification method in the frequency domain is a clear straightforward procedure in a classical appealing way. Minimum, maximum and medium errors give an indication about the adaptation of weighting filter Wm in the next iteration step and whether the model could represent the system.

Drawbacks and remedies:
One of the main drawbacks of the identification method is its conservatism. As could be seen in Fig.4a/d a symmetric distribution in u and y does not lead to a symmetric distribution in P around PC1 or PC2. Therefore a minimal distance to PC truely leads to a minimum of the maximal error. However the resulting nominal model will expect not to be close to the true system. One can think of putting the expectation of the nominal model closer to the true process at the expense of a larger upper bound for the model error. In the case of Gaussian noise a combination of H2 and H∞ norms has to be minimized. If the noise to signal ratio is bad the proposed method will fail or the model error bounds will be very large. More a priori knowledge about the true process have to be used to improve the results.
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Appendix 1

Proof of theorem 1:

We can give the sets $U$ and $Y$ for one specific frequency as:

$$
U = \left\{ \tilde{u} = u\left(1 + \alpha_u e^{i\phi}\right), \ 0 \leq \theta < 2\pi, \ 0 \leq \alpha_u \leq \frac{|W_d|}{|u|} \right\}
$$

$$
Y = \left\{ \tilde{y} = y\left(1 + \alpha_y e^{i\phi}\right), \ 0 \leq \phi < 2\pi, \ 0 \leq \alpha_y \leq \frac{|W_c|}{|y|} \right\}
$$

To derive the set $P$ we will need an auxiliary set $X$ (see Fig.2.2) with signals $\tilde{z} = \tilde{u}^{-1}$ for all $\tilde{u} \in U$.

Under the assumption that $|u| > |W_d| > 0$ so $(uu^* - W_dW_d^*) > 0$ it holds:

$$
|\tilde{u} - u| \leq |W_d| \Leftrightarrow (\tilde{u} - u)(\tilde{u} - u)^* \leq W_dW_d^* \Leftrightarrow
$$

$$
\frac{(\tilde{u}\tilde{u}^* - \tilde{u}u^* - u\tilde{u}^* + uu^* - W_dW_d^*)}{\tilde{x}\tilde{x}^*} \leq 0 \Leftrightarrow
$$

$$
\frac{1}{\tilde{x}\tilde{x}^*} - \frac{u^*}{\tilde{x}} - \frac{u}{\tilde{x}} + (uu^* - W_dW_d^*) \leq 0 \Leftrightarrow
$$

$$
1 - u\tilde{x}^* - u^*\tilde{x} + (uu^* - W_dW_d^*)\tilde{x}\tilde{x}^* \leq 0 \Leftrightarrow
$$

$$
\tilde{x}\tilde{x}^* - \frac{u\tilde{x}^* + u^*\tilde{x}}{uu^* - W_dW_d^*} + \frac{uu^*}{(uu^* - W_dW_d^*)^2} - \frac{W_dW_d^*}{(uu^* - W_dW_d^*)^2} \leq 0 \Leftrightarrow
$$

$$
\left[ \tilde{x} - \frac{u^*}{(uu^* - W_dW_d^*)} \right] \left[ \tilde{x} - \frac{u}{(uu^* - W_dW_d^*)} \right]^* \leq \frac{W_dW_d^*}{(uu^* - W_dW_d^*)^2} \Leftrightarrow
$$

$$
\left| \tilde{x} - \frac{u^*}{(uu^* - W_dW_d^*)} \right| \leq \frac{|W_d|}{(uu^* - W_dW_d^*)}
$$

So the set $X$ is given by the ball:

$$
X = \left\{ \tilde{x} \mid \left| \tilde{x} - \frac{u^*}{(uu^* - W_dW_d^*)} \right| \leq \frac{|W_d|}{(uu^* - W_dW_d^*)} \right\} = \left\{ \tilde{x} = \frac{u^*}{(uu^* - W_dW_d^*)}(1 + \alpha_u e^{i\psi}), \ 0 \leq \psi < 2\pi, \ 0 \leq \alpha_u \leq \frac{|W_d|}{|u|} \right\}
$$
Appendix 2

Proof of theorem 2:

We have the region 

$$ h(q, \psi) = (1 + re^{i\phi})(1 + se^{j\psi}) $$

where \(0 \leq \phi < 2\pi\) and \(0 \leq \psi < 2\pi\). If \(h(q, \psi)\) is a boundary point of the region for some \(\phi\) and \(\psi\), then it holds for the derivatives (see Fig.A1):

$$ \frac{\partial h(\phi, \psi)}{\partial \phi} = \frac{\partial h(\phi, \psi)}{\partial \psi} C_0 \quad \text{with} \quad C_0 \in \mathbb{R} $$

This leads to

$$ \text{Im} \left( \frac{\partial h(\phi, \psi)}{\partial \phi} \right) = 0 $$

and therefore:

$$ \text{Im} \left( \frac{\partial h(\phi, \psi)}{\partial \psi} \right) = \text{Im} \left( \frac{jre^{i\phi}(1 + se^{j\psi})}{jse^{j\psi}(1 + re^{i\phi})} \right) = $$

$$ = \text{Im} \left( \frac{(r e^{i\phi} + rs e^{j(\phi+\psi)})(e^{-j\psi} + r e^{-j(\phi+\psi)})}{(s e^{j\psi} + rs e^{j(\phi+\psi)})(e^{-j\psi} + r e^{-j(\phi+\psi)})} \right) = $$

$$ = \text{Im} \left( \frac{r(e^{j(\phi-\psi)} + s e^{i\phi} + r e^{-j\psi} + rs)}{s(1 + r e^{i\psi} + r e^{-j\psi} + r^2)} \right) = $$

$$ = \frac{r(s \sin(\phi - \psi) + s \sin\phi - r \sin\psi)}{s(1 + 2r \cos\phi + r^2)} = 0 $$

So we find the condition

\[
\sin(\phi - \psi) + s \sin\phi - r \sin\psi = 0
\]

or

\[
\frac{e^{i(\phi-\psi)} - e^{-i(\phi-\psi)}}{2j} + s e^{i\phi} - s e^{-j\phi} - \frac{re^{j\psi} - re^{-j\psi}}{2j} = 0
\]
Now we define $a = e^{i\psi}$ and $b = e^{i\phi}$ and we get the condition

$$\frac{a}{b} - \frac{b}{a} + sa - \frac{s}{a} + \frac{r}{b} - rb = 0$$

and because $a \neq 0$ and $b \neq 0$ we can write:

$$a^2 - b^2 + sa^2b - sb + ra - r^2b =$$

$$= a^2(1 + sb) + a(r - rb^2) + (-b - sb) = 0$$

We get the solutions

$$a_{1,2} = \frac{r (b^2 - 1) \pm \sqrt{r^2(1 - b^2)^2 + 4(1 + sb)(b^2 + sb)}}{2(1 + sb)} =$$

$$= \frac{r (e^{2i\psi} - 1) \pm e^{i\psi} \sqrt{r^2(e^{-i\psi} - e^{i\psi})^2 + 4(1 + se\psi)(1 + se^{-i\psi})}}{2(1 + se^{i\psi})} =$$

$$= \frac{r(e^{2i\psi} - 1) \pm 2e^{i\psi}\sqrt{-r^2\sin^2\psi + 2s\cos\psi + 1 + s^2}}{2(1 + se^{i\psi})}$$

We substitute this solution for $a$ in $h(\phi, \psi)$ and we find two functions $h_{b1}(\psi)$ and $h_{b2}(\psi)$ one corresponding to a '+' sign and one corresponding to a '−' sign:

$$h_{b1}(\psi) = (1 + ra)(1 + sb) = 1 + sb + (1 + sb)ra =$$

$$= 1 + se^{i\psi} + \frac{r^2}{2}(e^{2i\psi} - 1) + r e^{i\psi}\sqrt{-r^2\sin^2\psi + 2s\cos\psi + 1 + s^2}$$

likewise

$$h_{b2}(\psi) = 1 + se^{i\psi} + \frac{r^2}{2}(e^{2i\psi} - 1) - r e^{i\psi}\sqrt{-r^2\sin^2\psi + 2s\cos\psi + 1 + s^2}$$

Now we substitute $\psi = 0$ and $\psi = \pi$, resulting in:

$$h_{b1}(0) = 1 + s + \frac{r^2}{2}(1 - 1) + r\sqrt{1 + 2s + s^2} =$$

$$= 1 + s + r(1 + s) = 1 + s + r + rs$$

$$h_{b1}(\pi) = 1 - s + \frac{r^2}{2}(1 - 1) - r\sqrt{1 - 2s + s^2} =$$

$$= 1 - s - r(1 - s) = 1 - s - r + rs$$
\begin{align*}
  h_{b2}(0) &= 1 + s + \frac{r^2}{2}(1 - 1) - r \sqrt{1 + 2s + s^2} = \\
  &= 1 + s - r(1 + s) = 1 + s - r - rs \\
  h_{b2}(\pi) &= 1 - s + \frac{r^2}{2}(1 - 1) + r \sqrt{1 - 2s + s^2} = \\
  &= 1 - s + r(1 - s) = 1 - s + r - rs
\end{align*}

Fig. A2: The two boundary functions $h_{b1}(\psi)$ and $h_{b2}(\psi)$.

So the function $h_{b1}(\psi)$ is the desired bound of the region $k(\phi, \psi)$. The function $h_{b2}(\psi)$ is an inner boundary that is caused by fixing the values $\alpha_u = s$ and $\alpha_v = r$. If we consider again $0 \leq \alpha_u \leq s$ and $0 \leq \alpha_v \leq r$, then all inner points will be allowed, and the inner boundary vanishes.
Appendix 3

Proof of theorem 3:

The function
\[ h(\phi, \psi) = (1 + r e^{i\phi})(1 + s e^{i\psi}) \]
where \( \phi = f(\psi) \), gives the boundary \( h(f(\psi), \psi)) = h_b(\psi) \) (see Appendix 2) of the uncertainty region \( \hat{p}(\alpha_1, \alpha_2, \phi, \psi) \) with \( \alpha_1, \alpha_2, \phi \) and \( \psi \) are in the allowed regions. We seek for a disk with a minimal radius, that encloses the region \( \hat{p}(\alpha_1, \alpha_2, \phi, \psi) \).

We will consider the distance \( \rho \) between the point \( c_0 > 1 \) and the function \( h_b(\psi) \), and determine the maxima of this function \( |h_b(\psi) - c_0| \). Then we prove that maxima are reached in the points \( h(\phi_0, \psi_0) = h_b(\psi_0) = c_0 + j \rho_0 \) and \( h(-\phi_0, -\psi_0) = h_b(-\psi_0) = c_0 - j \rho_0 \). This means that the circle with center point \( c_0 \) and radius \( \rho_0 \) encloses the region \( h(\phi, \psi) \). Because the points \( c_0 + j \rho_0 \) and \( c_0 - j \rho_0 \) are both in the region, the mentioned circle will be the smallest enclosing circle, see Fig.A3.

First we will take a closer look at the centre-point \( c_0 \) and radius \( \rho_0 \):
\[ \rho_0 = \text{Im}\left(h(\phi_0, \psi_0)\right) = r \sin \phi_0 + s \sin \psi_0 + rs \sin(\phi_0 + \psi_0) \]
it is clear that \( \rho_0 > r + s = \text{Im}\left(h(\frac{\pi}{2}, \frac{\pi}{2})\right) \) and thus it must hold:
\[ \sin \phi_0 > 0, \quad \sin \psi_0 > 0 \quad \text{and} \quad \sin(\phi_0 + \psi_0) > 0 \]
Further we know from the conditions for \( \phi_0 \) and \( \psi_0 \) that:
\[ r \cos \phi_0 = s \cos \psi_0 = -rs \cos(\phi_0 + \psi_0) \overset{\text{df}}{=} y_0 \]
Therefore we find: \( 0 < \phi_0 < \frac{\pi}{2} \), \( 0 < \psi_0 < \frac{\pi}{2} \) and \( \frac{\pi}{2} < (\phi_0 + \psi_0) < \pi \).
We can derive: \( \cos \phi_0 = y_0/r \), \( \cos \psi_0 = y_0/s \), \( \cos(\phi_0 + \psi_0) = -y_0/rs \),
\[ \sin \phi_0 = \sqrt{1 - r^2 y_0^2 - y_0^2}, \quad \sin \psi_0 = \sqrt{1 - s^2 y_0^2 - y_0^2} \]
and so:
\[ y_0 = -rs \cos(\phi_0 + \psi_0) = rs \sin \phi_0 \sin \psi_0 = -rs \cos \phi_0 \cos \psi_0 = \]
\[ = \sqrt{r^2 - y_0^2} \sqrt{s^2 - y_0^2} - y_0^2 \]
this leads to
\[ y_0^2 + y_0 = \sqrt{r^2 - y_0^2} \sqrt{s^2 - y_0^2} \]
\[ (y_0^2 + y_0)^2 = (r^2 - y_0^2)(s^2 - y_0^2) \]
\[ y_0^4 + 2y_0^3 + y_0^2 = y_0^4 + r^2 y_0^2 + s^2 y_0^2 - r^2 s^2 = 0 \]
\[ 2y_0^2 + y_0^2(1 + r^2 + s^2) - r^2s^2 = 0 \]  \hfill (A1)

where we choose the solution \(0 < y_0 < rs < 1\), which is unique for \(0 < r < 1\) and \(0 < s < 1\). From this \(y_0\) we can easily determine 
\[ \phi_0 = \arccos(y_0/r) \] 
and 
\[ \psi_0 = \arccos(y_0/s). \]

We can now compute:
\[ c_0 = 1 + r \cos \phi_0 + s \cos \psi_0 + rs \cos(\phi_0 + \psi_0) = \]
\[ = 1 + y_0 + y_0 - y_0 = 1 + y_0 > 1 \]
\[ \rho_0 = r \sin \phi_0 + s \sin \psi_0 + rs \sin(\phi_0 + \psi_0) = \]
\[ = \sqrt{r^2 - y_0^2} + \sqrt{s^2 - y_0^2} + \sqrt{r^2s^2 - y_0^2} \]

So finally we can give two important results:
\[ 2y_0^2 + y_0^2(1 + r^2 + s^2) - r^2s^2 = 0 \]
and
\[ 1 < c_0 = 1 + y_0 < 1 + rs \]

We will use these results later.

Fig. A3: The boundary function \(h_b(\psi)\) and distance \(\rho\) to \(c_0\).

Now we are interested in the maximum distance
\[ \rho_{\text{max}} = \max_{\phi, \psi} \rho(\phi, \psi) = \max_{\phi, \psi} |h(\phi, \psi) - c_0| = \]
\[ = \max_{\phi, \psi} |(1 - c_0 + r e^{i\phi}) + s(1 + r e^{i\phi}) e^{i\psi}| \]  \hfill (A2)
For every choice of \( \phi \) we can choose a \( \psi \) such that:

\[
\rho_{\text{max}} = \max_{\phi, \psi} \rho(\phi, \psi) = 
\]

\[
= \max_{\phi, \psi} \left( |1 - c_0 + re^{i\phi}| + s(1 + re^{i\phi}) e^{i\psi} \right) = 
\]

\[
= \max_{\phi} \left( \left| r e^{i\phi} - (c_0 - 1) \right| + \left| s(1 + re^{i\phi}) \right| \right) = 
\]

\[
= \max_{\phi} \left( \sqrt{r^2 + (c_0 - 1)^2} - 2r(c_0 - 1) \cos \phi + s \sqrt{r^2 + 1 + 2r \cos \phi} \right)
\]

Note that the maximization problem over two variables \( \phi \) and \( \psi \) has reduced to a maximization problem over only one variable \( \phi \).

To compute the maximum we put the derivative of \( \rho(\phi) \) equal to zero:

\[
\frac{d\rho(\phi)}{d\phi} = \frac{r(c_0 - 1) \sin \phi}{\sqrt{r^2 + (c_0 - 1)^2} - 2r(c_0 - 1) \cos \phi} + \frac{-rs \sin \phi}{\sqrt{r^2 + 1 + 2r \cos \phi}} = 0
\]

so either

\[
\sin \phi = 0 \quad \text{(A3)}
\]

or

\[
(c_0 - 1)\sqrt{r^2 + 1 + 2r \cos \phi} - s\sqrt{r^2 + (c_0 - 1)^2} - 2r(c_0 - 1) \cos \phi = 0 \quad \text{(A4)}
\]

From equation (A3) we get the solutions

\[
\phi_1 = 0 \quad \text{and} \quad \phi_2 = \pi
\]

(We restrict ourselves to \( 0 \leq \phi < 2\pi \)).

\( \phi_1 = 0 \) and \( \phi_2 = \pi \) correspond to the points \( x_1 \) and \( x_2 \) in Fig.A3.

We know that \( c_0 \geq 1 \) and from equation (A4) we derive

\[
(c_0 - 1)^2(r^2 + 1 + 2r \cos \phi) - s^2(r^2 + (c_0 - 1)^2 - 2r(c_0 - 1) \cos \phi) = 0 \quad \text{(A5)}
\]

and so

\[
\cos \phi = \frac{(c_0 - 1)^2(r^2 + s^2 + 1) + r^2s^2}{2r(c_0 - 1)(s^2 + 1 - c_0)} \equiv b_0
\]

If \( |b_0| \leq 1 \) we obtain solutions

\[
\phi_3 = \arccos(b_0) \quad \text{and} \quad \phi_4 = -\arccos(b_0) = -\phi_3
\]
To see which solutions are maxima we calculate the second derivative of the function $\rho(\phi)$:

$$\frac{d^2 \rho(\phi)}{d \phi^2} =$$

$$= r \cos \phi \left( \frac{(c_0 - 1)}{\sqrt{r^2 + (c_0 - 1)^2 - 2r(c_0 - 1) \cos \phi}} + \frac{-s}{\sqrt{r^2 + 1 + 2r \cos \phi}} \right) +$$

$$+ r \sin \phi \left( \frac{-r(c_0 - 1)^2 \sin \phi}{\sqrt{r^2 + (c_0 - 1)^2 - 2r(c_0 - 1) \cos \phi}} + \frac{-rs \sin \phi}{\sqrt{r^2 + 1 + 2r \cos \phi}} \right)$$

If we fill in the solutions $\phi_3$ or $\phi_4$ we find the first term to be zero because

$$\frac{(c_0 - 1)}{\sqrt{r^2 + (c_0 - 1)^2 - 2r(c_0 - 1) \cos \phi}} + \frac{-s}{\sqrt{r^2 + 1 + 2r \cos \phi}} = 0$$

and the second term is negative because $(c_0 - 1) > 0$, $r > 0$ and $s > 0$. This means that

$$\frac{d^2 \rho(\phi)}{d \phi^2} < 0$$

for $\phi_3$ and $\phi_4$ and so $\rho(\phi_3)$ and $\rho(\phi_4)$ are maxima. Consequently $\rho(\phi_1)$ and $\rho(\phi_2)$ must be minima.

We fill in the solutions $\phi_3$ in equation (A2) and we find the corresponding solution $\psi_3$ by making:

$$\psi_3 = \arg \left[ \frac{r e^{j\phi_3} + 1 - c_0}{r e^{j\phi_3} + 1} \right]$$

In the same way we can find for $\phi_4 = -\phi_3$ the corresponding $\psi_4 = -\psi_3$. These solutions $(\phi_3, \psi_3)$ and $(\phi_4, \psi_4)$ will lead to to the points $h(\phi_3, \psi_3)$ and $h(\phi_4, \psi_4)$ which are a complex conjugated pair, symmetric with respect to the real axis. This points are the maxima of the function $|h(\phi, \psi) - c_0|$ for $c_0 > 1$.

Now we will prove that the points

$$h(\phi_3, \psi_3) = c_0 + j \rho_0$$

and

$$h(\phi_4, \psi_4) = c_0 - j \rho_0$$

We can prove this by showing that $\phi_3 = \phi_0$: If we choose $\cos \phi = \cos \phi_0 = y_0/r$ and we fill in $c_0 - 1 = y_0$ in equation (A5) we get:

$$(c_0 - 1)^2 \left( r^2 + 1 + 2r \cos \phi \right) - s^2 \left( r^2 + (c_0 - 1)^2 - 2r(c_0 - 1) \cos \phi \right) =$$
\[ y_0^2 (r^2 + 1 + 2y_0) - s^2 (r^2 + y_0^2 - 2y_0 y_0) = \]
\[ = 2y_0^3 + y_0^2 (1 + r^2 + s^2) - r^2 s^2 \]

From equation (A1) we know that this is zero, so we know that \( \phi_3 = \phi_0 \) and thus \( \psi_3 = \psi_0 \).

We proved that the points \( c_0 \pm j \rho_0 \) are maxima of the function \( |h(\phi, \psi) - c_0| \) and so the circle \( c_0 + \rho_0 e^{j\alpha} \) is the smallest enclosing circle of \( h_t(\psi) \).
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