On minimal two-letter propositional logic with disjunction

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1. Introduction. In this note we shall use the notation of [2]. In particular $E^*(\{a,b\}; \rightarrow, \wedge, \vee; \Omega)$ denotes the Lindenbaum algebra for propositional logic with connectives $\rightarrow, \wedge, \vee$ and natural deduction derivation rules, with alphabet $\{a,b\}$ and with $\Omega$ as a set of formulas which are taken as axioms. There is no negation, but if we wish, we can take $b$ to be "falsum" and define $\neg \lambda$ as $\lambda \rightarrow b$.

With this notation, we can describe intuitionistic one-letter propositional logic by $E^*(\{a,b\} ; \rightarrow, \wedge, \vee ; \{b \rightarrow a\})$; the axiom $b \rightarrow a$ is the so-called "falsum rule". (We note that from $\vdash b \rightarrow a$ it follows that $\vdash b \rightarrow p$ for all $p \in E(\{a,b\}; \rightarrow, \wedge, \vee)$ (this $E$ stands for the set of all formulas built from $a,b$ by means of the connectives $\rightarrow, \wedge, \vee$). The structure of this Lindenbaum algebra was described by Nishimura [4] in the form of a lattice. This description can be simplified a little by describing Nishimura's lattice as the lattice of all closed sets in a particular topological space. In this note we shall expose the material necessary for this purpose; our verification will be independent of Nishimura's paper.

We shall also try to get some further results of the same type. These might possibly help to get insight into the unsolved problem of finding the structure of the Lindenbaum algebra for the case without falsum rule, i.e. $E^*(\{a,b\}; \rightarrow, \wedge, \vee)$.

2. Topological description of Nishimura's lattice. By $N$ we shall denote the following topological space. The point set is the set $N = \{1,2,3,\ldots\}$, and the closure $\text{cl}(S)$ of a set $S$ (with $S \subseteq N$) is simply the union of all $\text{cl}([i])$ for all $i \in S$, and $\text{cl}([i])$ is the set of all $j \in N$ with $j = i$ or $j > i+1$.

The closed sets of $N$ are (i) the empty set, (ii) the sets $\text{cl}([i]) = \{i, i+2, i+3, \ldots\}$ ($i \in N$), (iii) the unions $\text{cl}([i]) \cup \text{cl}([i+1]) = \{i, i+1, i+2, \ldots\}$ ($i \in N$). The lattice of these closed sets (with set inclusion as lattice operation) is Nishimura's lattice.
We draw two pictures of this lattice. The first one is the one given by Nishimura (D.Scott referred to it as the "garden lattice"), the second one looks less attractive, but is better in showing the connection with \( \mathbb{N} \). In the pictures, we write \( i \) and \( i,j \) instead of \( c1({i}) \) and \( c1({i}) \cup c1({j}) \).

3. Exact valuations. In [1],[2] the notions of valuations and exact valuations of \( E^*(A; \rightarrow, \wedge, \vee; \Omega) \) are explained. In particular we mention that if there is an exact valuation by means of the finitely generated closed sets of a topologial space, then the points of that space represent the primes of the Lindenbaum algebra. (\( \alpha \in E^* \) is called a prime if \( \alpha \) is non-derivable and if for all dissections \( \alpha = \beta \wedge \gamma \) we have \( \alpha = \beta \) or \( \alpha = \gamma \)).

We shall not use any results on exactness of valuations in this note. We do use valuations, however, for showing non-derivability of formulas.
4. Abbreviations. The following abbreviations will be used, where $A$ is an alphabet and $\Omega$ a set of axioms.

(i) $E = E(A; \rightarrow, \wedge, \vee)$, $E^* = E^*(A; \rightarrow, \wedge, \vee; \Omega)$.

(ii) If $p, q \in E$, and if $\Omega \vdash p \rightarrow q$ and $\Omega \vdash q \rightarrow p$, we write $p \equiv q$.

(iii) If $p, q \in E$, and if both $p \rightarrow q \equiv q$ and $q \rightarrow p \equiv p$ we write $p \equiv q$.

(iv) If $p_1, p_2, \ldots \in E$ then $p_1 p_2$ denotes $p_1 \rightarrow p_2$, $p_1 p_2 p_3$ denotes $(p_1 \rightarrow p_2) \rightarrow p_3, \ldots, p_1 \ldots p_n$ denotes $(p_1 \ldots p_{n-1}) \rightarrow p_n$.

5. Some lemmas.

Lemma 1. Assume $p \in E$, $q \in E$, $r \in E$, with $\vdash pqq$ and $\vdash qrr$. Then, putting $s = r \rightarrow (p \lor q)$ we have $\vdash rss$, $\vdash sqs$ and $\vdash srs$.

Proof. For typographical reasons the proof is given in Appendix 1.

Lemma 2. Assume $f \in E$, $g \in E$, $h \in E$, $k \in E$, with $\vdash gff$, $\vdash fk$, $\vdash gk$. Then $\vdash kfk$.

Proof. Appendix 2.

Lemma 3. The binary relation $\vdash yxy$ is transitive, i.e. if $x \in E$, $y \in E$, $z \in E$, $\vdash yxy$ and $\vdash zyz$ then $\vdash zxz$.

Proof. Appendix 3.

Remark. Another non-trivial binary relation is $\vdash yxxxy$, which, however, plays no role in this note.

Lemma 4. Assume $u_1, u_2, u_3 \in E$, and $u_1 \vdash u_2$, $u_2 \vdash u_3$, $\vdash u_3 u_1 u_3$. We define $u_4, u_5, \ldots$ recursively by $u_{i+3} = u_{i+2} \rightarrow (u_i \lor u_{i+1})$ ($i = 1, 2, \ldots$). $P$ is the set $\{u_i \mid i \in \mathbb{N}\} \cup \{u_i \land u_{i+1} \mid i \in \mathbb{N}\}$. Then we have

(i) $u_i \vdash u_{i+1}$ ($i = 1, 2, 3, \ldots$),

(ii) $\vdash u_i u_j$ (whence $\vdash u_i u_j$) ($i = 1, 2, \ldots, j > i+1$),

(iii) for all $p, q \in P$ there exist $r, s, t \in P$ such that $p \rightarrow q \equiv r$, $p \land q \equiv s$, $p \lor q \equiv t$.

Proof. (i) By lemma 1, applied to $p = u_i$, $q = u_{i+1}$, $r = u_{i+2}$, $s = u_{i+3}$ we
deduce, by induction, that \( \vdash u_{i+2}u_{i+3} \) for \( i = 1, 2, \ldots \), and that, moreover \( \vdash u_{i+3}u_{i+2}u_{i+2} \) and \( \vdash u_{i+3}u_{i+1}u_{i+3} \) for these values of \( i \).

(ii) By lemma 2 (with \( f = u_i \), \( g = u_{i+1} \), \( h = u_{i+2} \), \( k = u_{i+3} \)) we have \( \vdash u_{i+3}u_{i+3} \) (\( i = 1, 2, \ldots \)). Together with \( \vdash u_{i+3}u_{3} \) and the \( \vdash u_{i+3}u_{i+1}u_{i+3} \) obtained under (i), this leads to \( \vdash u_{i}u_{i}u_{3} \) for all \( i, j \) with \( j-i = 2 \) or 3.

Now by lemma 3 we get it for all \( i, j \) with \( j > i + 1 \).

(iii) It suffices to take \( p = u_i \), \( q = u_j \); the other cases then follow by elementary relations between the connectives.

If \( j-i = 0 \) or \( j-i > 2 \) we have \( u_{i}u_{j} \equiv T \), and otherwise we get \( u_{i}u_{j} \equiv u_{j} \) (if \( i-j = 1 \) by (i), if \( i-j > 1 \) (by ii); note that \( pqp \) leads to \( pq \equiv q \).

If \( u_{i}u_{j} \equiv T \) we have \( u_{i} \land u_{j} \equiv u_{j} \) and \( u_{i} \lor u_{j} \equiv u_{i} \).

The cases with \( i = j \) being trivial, it remains to investigate \( j = i + 1 \).

Trivially \( u_{i} \land u_{i+1} \in P \). Finally we prove \( u_{i} \lor u_{i+1} = u_{i+3} \land u_{i+4} \in P \). We have \( \vdash u_{i}(u_{i+3} \land u_{i+4}) \) and \( \vdash u_{i+1}(u_{i+3} \land u_{i+4}) \) (since \( i+3 -(i+1) > 1 \)), so

\( \vdash (u_{i} \lor u_{i+1})(u_{i+3} \land u_{i+4}) \). On the other hand, if we start from \( \vdash u_{i+3} \land u_{i+4} \), we derive \( \vdash u_{i+1} \lor u_{i+2} \) (by the definition of \( u_{i+4} \)). Since both \( \vdash u_{i+1}(u_{i} \lor u_{i+1}) \) and \( \vdash u_{i+2}(u_{i} \lor u_{i+1}) \) (note that we have \( \vdash u_{i+3} \), and apply the definition of \( u_{i+3} \)), so by \( \lor \)-elimination we get \( \vdash (u_{i} \lor u_{i+1}) \).

**Lemma 6.** Let \( p, q, r \in E \), and \( p \perp q, q \perp r, \vdash rpr \). Then abbreviating \( x = q \land r, y = p \land q \) we have

\( \vdash yx, p \equiv xy, q \equiv xyy, r \equiv xyyx \).

**Proof.** For proofs of \( \vdash yx, \vdash p(xy), \vdash xyp, \vdash q(xyy) \) we refer to appendix 4.

That settles \( p \equiv xy \). Next we have \( xyy \equiv py = p(p \land q) \equiv pq \); by \( p \perp q \) we infer \( \vdash (xyy)q \). That settles \( q \equiv xyy \). Finally \( q \perp r \) leads to \( r \equiv q(q \land r) \), whence \( r \equiv xyyx \).

**Lemma 7.** Let \( x, y \in E \), and assume \( \vdash yx \). Then, abbreviating \( p = xy, q = xyy, r = xyyx \), we have \( p \perp q, q \perp r, \vdash rpr \).

**Proof.** Appendix 5.
6. The Lindenbaum algebra $E^*(\{a,b\}; \vdash, \wedge, \vee; \{ba\})$.

**Theorem.** Define $u_1, u_2, u_3$ by $u_1 = ab$, $u_2 = abb$, $u_3 = abba$, and $u_4, u_5, \ldots$ and $P$ as in lemma 4. Then every $e \in E$ is (under the axiom $ba$) equivalent to exactly one $p \in P$.

**Proof.** By lemma 7 we have $u_1 \not\vdash u_2$, $u_2 \not\vdash u_3$, $\vdash u_3 u_1 u_3$, whence lemma 4 can be applied. Furthermore we have (cf. lemma 6) $a \equiv u_2 \wedge u_3$, $b \equiv u_1 \wedge u_2$. The closure of $P$ with respect to $\vdash, \wedge, \vee$, as expressed by Lemma 4(iii), now shows that every $e \in E$ is equivalent to some $p \in P$.

The fact that the $u_i$ and the $u_i \wedge u_{i+1}$ are pairwise inequivalent, is shown by the valuation $v$ of $E$ into $N$, generated by $v(a) = \{2, 3, \ldots\}$, $v(b) = \{1, 2, 3, \ldots\}$. We obtain $v(ba) = \emptyset$ as it should for the value of an axiom, and $v(u_i) = \text{cl}(\{i\})$, $v(u_i \wedge u_{i+1}) = \text{cl}(\{i\}) \cup \text{cl}(\{i+1\})$. These values being all different, we observe the inequivalence of the $u_i$'s and $u_i \wedge u_{i+1}$'s.

7. Semiorthogonal systems. In [1], [3] the following situation plays a vital rôle. It involves a topological space $(X, \text{cl})$ ($\text{cl}$ is the closure operator) and a mapping $f$ of $X$ into $E$ such that

(i) if $x \in X$, $y \in Y$, $x \not\in \text{cl}(\{y\})$, $y \not\in \text{cl}(\{x\})$ then $f(x) \not\vdash f(y)$,

(ii) if $x \in \text{cl}(\{y\})$, $y \not\in \text{cl}(\{x\})$ then $f(y)f(x) \equiv f(x)$, and $\vdash f(x)f(y)$,

(iii) if both $x \in \text{cl}(\{y\})$ and $y \in \text{cl}(\{x\})$ then $f(x) \not\vdash f(y)$.

Let us call such an $f$ semiorthogonal.

We observe that this situation holds for our space $N$ (section 2) and the mapping $f$ defined by $f(i) = u_i$ ($i = 1, 2, \ldots$) where the $u_i$ are taken from section 6.

**Remark.** From lemma 1 we can derive a more general construction of such a system. We start from $p, q, r$ with $\vdash pqq$ and $q \vdash r$. Then with $u_1 = q$, $u_2 = r$, $u_3 = r \rightarrow (p \wedge q)$ we get $u_1, u_2, u_3$ with $u_1 \not\vdash u_2$, $u_2 \not\vdash u_3$, $\vdash u_3 u_1 u_3$, and lemma 4 can be applied.

Semiorthogonal $f$'s are particularly useful for showing that the structure of the Lindenbaum algebra is the lattice of finitely generated closed sets if we have

(i) For every $u \in \Delta$ we have a finite set $x_1, \ldots, x_n$ with $u \equiv f(x_1) \wedge \ldots \wedge f(x_n)$,

(ii) If $x, y \in X$ and $x \not\equiv y$ then $f(x) \not\equiv f(y)$.

These things can provide a more general setting for the verification of section 6.
8. The Lindenbaum algebra $E^*([a,b]; \to, \land, \lor; \{a,b\})$.

This one can be shown to have the same structure as the one of section 6. We take $u_1 = ab$, $u_2 = ba$, $u_3 = baa$, whence lemma 4 can be applied. Furthermore (cf. (i) of section 7) $a \equiv u_2 \land u_3$, $b \equiv u_1$. Finally, the valuation $v$, generated by

$$v(a) = cl({2}) \cup cl({3}), \quad v(b) = cl({1})$$

has the property that $v(u_i) = cl({i})$ ($i = 1,2,\ldots$) and $v(abb) = \emptyset$. These things guarantee, by the argument of section 6, that the Lindenbaum algebra has the same structure as the one of section 6.

Remark. Construction of systems like the one of section 8 help to throw light on the unsolved problem of the structure of the Lindenbaum algebra $E^*([a,b]; \to, \land, \lor; \{a,b\})$.

The following two lemmas have the same relation to the axiom $abb$ as lemmas 6 and 7 have to the axiom $abo$. Together they show how the Lindenbaum algebras of sections 6 and 8 can be mapped onto each other. That mapping follows from lemma 10.

Lemma 8. Let $p,q,r \in E$, $p \perp q$, $q \perp r$, $\vdash rpr$. Then abbreviating $u = q \land r$, $v = p$ we have

$$\vdash uvv, \quad p \equiv uv, \quad q \equiv vu, \quad r \equiv vuu.$$

Lemma 9. Let $u,v \in E$ and assume $\vdash uvv$. Then, abbreviating $p = uv$, $q = vu$, $r = vuu$ we have $p \perp q$, $q \perp r$, $\vdash rpr$.

Lemma 10. Let $x,y,u,v \in E$. Then we have

$$\vdash yx, \quad u \equiv x, \quad v \equiv xy$$

if and only if

$$\vdash uvv, \quad x \equiv u, \quad y \equiv uv \land vu$$

We omit the proofs.
Appendix I. Proof of lemma 1.

1. \( p \rightarrow q \) (Assumption)
2. \( q \rightarrow r \) (Assumption)
3. \( s = r \rightarrow (p \lor q) \) (Abbreviation)

Note: MP(i,j) means that Modus Ponens is applied to lines i and j; if i reads \( x \rightarrow y \) and j reads \( x \) then the conclusion is \( y \).

4. \( r \rightarrow s \)
5. \( s \)
6. \( p \lor q \) (from 4,5)
7. \( p \lor q \) (from 6,5)
8. \( p \lor q \) (from 4,5)
9. \( p \lor q \) (from 6,5)
10. \( p \lor q \) (from 5,6)
11. \( p \lor q \) (from 10,11)
12. \( p \lor q \) (from 11,12)
13. \( p \lor q \) (from 12,13)
14. \( p \lor q \) (from 13,14)
15. \( p \lor q \) (from 14,15)
16. \( p \lor q \) (from 15,16)
17. \( p \lor q \) (from 16,17)
18. \( p \lor q \) (from 17,18)
19. \( p \lor q \) (from 18,19)
20. \( p \lor q \) (from 19,20)
21. \( p \lor q \) (from 20,21)
22. \( p \lor q \) (from 21,22)
23. \( p \lor q \) (from 22,23)
24. \( p \lor q \) (from 23,24)
25. \( p \lor q \) (from 24,25)
26. \( p \lor q \) (from 25,26)
27. \( p \lor q \) (from 26,27)
28. \( p \lor q \) (from 27,28)
29. \( p \lor q \) (from 28,29)
30. \( p \lor q \) (from 29,30)
31. \( p \lor q \) (from 30,31)

Discharge 5 from 7
Discharge 4 from 8
Discharge 12 from 14
Discharge 16 from 18
Discharge 11 from 21
Discharge 10 from 22
Discharge 25 from 28
Discharge 24 from 30

Note: MP(i,j) means that Modus Ponens is applied to lines i and j; if i reads \( x \rightarrow y \) and j reads \( x \) then the conclusion is \( y \).
Appendix 2. Proof of Lemma 2.

1. gff (Assumption)
2. fk (Assumption)
3. gk (Assumption)
4. kf
5. g
6. k MP(3,5)
7. f MP(4,6)
8. gf Discharge 5 from 7
9. f MP(1,8)
10. k MP(2,9)
11. kfk Discharge 4 from 10


1. yxy (Assumption)
2. zyz (Assumption)
3. zx
4. z
5. x MP(3,4)
6. yx from 5
7. y MP(1,6)
8. zy Discharge 4 from 7
9. z MP(2,8)
10. zzx Discharge 3 from 9
Appendix 4. Details of proof of lemma 6. \((x = q \land r, \ y = p \land q)\).

1. \(pqq\)    
   Assumption
2. \(qpp\)    
3. \(qrr\)    
4. \(rrq\)    
5. \(rpr\)    
6. \(p \land q\) from 6
7. \(p\) from 7
8. \(rp\) from 7
9. \(r\) MP(5,8)
10. \(q \land r\) from 6,9
11. \(yx\) Discharge 6 from 10
12. \(q \land r\)
13. \(p \land q\) from 12,13
14. \(xy\) Discharge 13 from 14
15. \(q\)
16. \(x\) from 17,18
17. \(y\) MP(16,19)
18. \(p\) from 20
19. \(rp\) Discharge 18 from 21
20. \(r\) MP(5,22)
21. \(p\) MP(22,23)
22. \(qp\) Discharge 17 from 24
23. \(p\) MP(2,25)
24. \(q\)
25. \(xy\)
26. \(p\) same derivation as 21
27. \(p\) same derivation as 24
28. \(y\) from 27,31
29. \(x\) Discharge 28 from 32.
Appendix 5. Proof of lemma 7. \((p = xy, q = xyy, r = xyyx)\). We use composite MP's in an obvious manner.

1. \(yx\) Assumption
2. \((xy)(xyy)\)
3. \(xy\)
4. \(y\) MP(MP(2,3),3)
5. \(xyy\) Discharge 2 from 4
6. \((xyy)(xy)\)
7. \(x\)
8. \(xy\)
9. \(y\) MP(8,7)
10. \(xxy\) Discharge 8 from 9
11. \(y\) MP(MP(6,10),7)
12. \(xy\) Discharge 7 from 11
13. \((xyy)(xyyx)\)
14. \(xyy\)
15. \(x\) MP(MP(13,14),14)
16. \(xyyx\) Discharge 14 from 15
17. \((xyyx)(xy)\)
18. \(xy\)
19. \(xyyx\) MP(MP(1,MP(19,18)))
20. \(x\) Discharge 19 from 20
21. \(xyyx\)
22. \(y\) MP(18,MP(21,MP(17,21)))
23. \(xyyx\) Discharge 18 from 23
24. \((xyyx)(xy)\)
25. \(xyy\)
26. \(x\) from 26
27. \(xyyx\) MP(MP(24,27),26)
28. \(y\) Discharge 26 from 28
29. \(x\) MP(1,MP(25,29))
30. \(xyyx\) Discharge 25 from 30
References.


