Control concept for a semi-active suspension with preview using a continuously variable damper

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Control Concept for a Semi-Active Suspension with Preview Using a Continuously Variable Damper

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Report on a period of practical training

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Abstract

New developments in valve technology make it possible to construct continuously variable dampers. In this report, a control strategy is derived for a semi-active suspension system with an adjustable damper. Preview information about the approaching road is used in the control of the suspension. A new approach is to take the damping coefficient of this damper as an input to be determined. This leads to a bilinear system in the input and the state. The control law, which minimizes a quadratic performance index under passivity constraints, is derived using optimal control theory. The closed loop system varies piecewise linear between two passive systems and a semi-active one.

The semi-active suspension with preview is applied to a two DOF model of the rear side of a vehicle. The performance of the suspension is evaluated on a deterministic (range of rounded pulses) and a stochastic (brick paved) road surface, and compared with those of a semi-active system without preview and a passive system. On the brick paved road, the suspension shows the potential of preview in improving all aspects of system performance compared to the nonpreview case. However, on the rounded pulses hardly any advantage can be taken from the preview information.
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Notation

\( a, A, \alpha \) scalars (italic characters)
\( [a, \alpha] \) columns (small bold characters)
\( A \) matrix (bold uppercase character)
\( 0 \) zero column
\( 0 \) zero matrix
\( a^T, A^T \) transposition
\( A^{-1} \) inversion
\( \dot{a}, \dot{a}, \dot{A} \) first order time derivative of \( a, A \)
\( \ddot{a}, \ddot{a}, \ddot{A} \) second order time derivative of \( a, A \)
1 Introduction

The design of vehicle suspensions is a trade-off between conflicting requirements (Sharp and Crolla [12]). In order to obtain a high ride comfort, it is desired to isolate the driver from the road unevenness as good as possible. However, ride comfort is rather difficult to quantify. For simple models, comfort is usually assumed to be related to vertical and pitch accelerations of the vehicle. Another important aspect is road holding: variations of the dynamic tire force should be small in connection with the handling properties of the vehicle. Furthermore, the suspension working space is limited because all suspension elements have a finite length, and because a low centre of gravity is desirable, for instance, for vehicle safety.

Unlike a passive suspension system comprised of springs and dampers, active systems require energy sources. However, the improved performance of the active suspension is coupled with a high energy consumption, high costs, and low reliability. As a less expensive alternative to active suspensions, a semi-active suspension was proposed by Karnopp et al. [11]. A semi-active suspension is characterized by a adjustable damper which can vary the rate of dissipation in response to instantaneous conditions of motion. There are two types of semi-active dampers:

1. switchable dampers. These dampers have two or more states.

2. continuously variable dampers. The damping coefficient can be adjusted continuously between certain limits.

An important factor limiting the performance of active and semi-active suspensions is a lack of information about the road input. The idea of preview control of active suspensions was first proposed by Bender [2]. Preview information can be obtained by using sonar sensors which look ahead of the front wheels of the vehicle (Frühauf et al. [6], Tomizuka [13]). A more reliable and cheaper strategy to obtain this information is to reconstruct the road surface from simple measurements at the front wheels of the vehicle (Huisman et al. [9], Crolla and Abdel-Hady [5]). Assuming that the excitation of the rear suspension is a time delayed version of that at the front, this knowledge allows the rear suspension to react to road profile before it reaches them. In this report the latter, more practical method is used, which implies that the road elevation is known over the wheelbase of the vehicle.

Here, a semi-active suspension is applied at the rear wheels of a vehicle. The continuously variable damper rate of this suspension is the input to be determined. The suspension has to reduce the maximum acceleration of the body of the vehicle and the required suspension working space, without increase of the dynamic tire force variation. A preview control strategy is derived using optimal control theory. The control problem is formulated in the next chapter, and analytically solved in Chapter 3. In Chapter 4, simulations results of the semi-active suspension with preview are described for a deterministic and a stochastic road input (a range of rounded pulses and a brick paved road, respectively). Conclusions and recommendations for further investigation are presented in Chapter 5.
2 Formulation of the problem

In this section, a model of the rear side of a vehicle with a semi-active suspension is introduced. It is followed by the formulation of the suspension preview control problem.

2.1 Description of the model

The rear side of the vehicle is modeled as a two DOF model. The model is shown in Figure 1, where \( m_1 \) denotes a quarter of the body mass (sprung mass), \( m_2 \) the mass of the wheel with semi-axis (unsprung mass), \( k_1 \) the suspension stiffness, \( b \) the viscous damping constant, \( u(t) \) the damping coefficient of a variable shock absorber, and \( k_2 \) the tire stiffness. The damping coefficient \( u(t) \) can be varied in a given range by changing the size of the orifice in the shock absorber. We assume that this damping coefficient can be adjusted without time delay (no damper dynamics). Furthermore, \( z_1 \) and \( z_2 \) are the absolute vertical displacements of the body and the wheels measured with respect to the position of equilibrium, while \( z_0 \) describes the road elevation.

![Figure 1: Vehicle system model.](image)

Then, the system equations are given by

\[
\begin{align*}
    m_1 \ddot{z}_1 + k_1 (z_1 - z_2) + [b + u(t)](\dot{z}_1 - \dot{z}_2) &= 0, \\
    m_2 \ddot{z}_2 + k_2 (z_2 - z_0) + [b + u(t)](\dot{z}_2 - \dot{z}_1) + k_1 (z_1 - z_2) &= 0.
\end{align*}
\]

(2.1) (2.2)

It is assumed that the road elevation \( z_0 \) is known over the time interval \([t, t + t_p]\) by reconstruction at the front wheels of the vehicle. The preview time \( t_p \) is equal to the ratio \( L/V \), where \( L \) is the wheelbase and \( V \) is the forward vehicle speed.
2.2 The semi-active suspension preview control problem

The semi-active suspension must be optimized with respect to ride comfort, suspension working space, and road holding. To take this into account, we introduce a performance index which includes the mean square of the following variables: the vehicle body acceleration $\ddot{z}_1$, the relative displacement $z_1 - z_2$ between the body and the wheel, and the dynamic tire force which is proportional to the wheel-to-road displacement $z_2 - z_0$. Thus the performance index to be minimized can be written as

$$ J = \frac{1}{2} \int_{t}^{t+t_p} \left[ \ddot{z}_1^2 + \rho_1(z_1 - z_2)^2 + \rho_2(z_2 - z_0)^2 \right] \, dt. \quad (2.3) $$

where $\rho_1$ and $\rho_2$ are weighting constants reflecting the preferences of the designer. The integral is defined over the time interval $[t, t+t_p]$ in which the road elevation is supposed to be known. Furthermore, $u(t)$ has to satisfy the condition

$$ u_{\text{min}} \leq u(t) \leq u_{\text{max}}, \quad u_{\text{min}} \geq 0. \quad (2.4) $$

Using the state vector $x$

$$ x = [z_1 - z_2, \dot{z}_1, z_2, \dot{z}_2]^T \quad \text{and} \quad z_0 = w, \quad (2.5) $$

the system of Eqs. (2.1) and (2.2) can be represented by the state equation

$$ \dot{x} = Ax - uBTx + Dw, \quad x(t_0) = x_0, \quad (2.6) $$

where

$$ A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -k_1/m_1 & -b/m_1 & 0 & b/m_1 \\ 0 & 0 & 0 & 1 \\ k_1/m_2 & b/m_2 & -k_2/m_2 & -b/m_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix}, $$

$$ D = \begin{bmatrix} 0 & 0 & 0 & k_2/m_2 \end{bmatrix}^T, \quad T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.7) $$

Because the state equation is linear in both $x$ and $u$ and also involves the product $xu$, the system is called bilinear. Using Eqs. (2.3), (2.5), (2.6), and (2.7) the performance index can be rewritten as

$$ J = \frac{1}{2} \int_{t}^{t+t_p} \left[ x^T Q_1 x - 2ux^T Ux + u^2 x^T V x + 2x^T Q_{12} w + Q_2 w^2 \right] \, dt, \quad (2.8) $$
Formulation of the problem

where

\[ Q_1 = \frac{1}{m_1^2} \begin{bmatrix} k_1^2 + \rho_1 m_1^2 & k_1 b & 0 & -k_1 b \\ k_1 b & b^2 & 0 & -b^3 \\ 0 & 0 & \rho_2 m_1^2 & 0 \\ -k_1 b & -b^3 & 0 & b^2 \end{bmatrix}, \quad N = \frac{1}{m_1^2} \begin{bmatrix} -k_1 \\ -b \\ 0 \\ b \end{bmatrix}, \]

and furthermore

\[ R = \frac{1}{m_1^2}, \quad Q_{12} = \begin{bmatrix} 0 & 0 & -\rho_2 & 0 \end{bmatrix}^T, \quad Q_2 = \rho_2, \]

\[ U = NT, \quad V = T^T R T. \] (2.9)

Under the assumption that \( w(\tau), \tau \in [t, t + t_p] \) is known, we want to find the optimal control \( u(t) \), that minimizes performance index (2.8) subject to the inequality and dynamic constraints Eqs. (2.4) and (2.6).
3 Preview control concept

In this chapter, the control strategy is derived for a semi-active suspension with preview by applying calculus of variations (Bryson and Ho [3]). First, we have to obtain the necessary conditions for minimization of the performance index $J$ under dynamic and passivity constraints.

3.1 Necessary conditions for optimality

The input $u(t)$ has to minimize

$$J = \frac{1}{2} \int_t^{t+T_p} \left[ x^T Q_1 x - 2ux^T U x + u^2 x^T V x + 2x^T Q_{12} w + Q_2 w^2 \right] dt,$$

subject to the constraints (2.4) and (2.6), where Eq. (2.4) can be written as a pair of inequality constraints

$$-u(t) + u_{min} \leq 0 \quad \text{and} \quad u(t) - u_{max} \leq 0.$$  

In Appendix A, it is shown that these constraints can be incorporated in a similar way as the dynamic constraints, i.e. by using Lagrange multipliers [3]. Then, the Hamiltonian for this problem is written as

$$H = \frac{1}{2} x^T Q_1 x - 2ux^T U x + \frac{1}{2} u^2 x^T V x + x^T Q_{12} w + \frac{1}{2} w^2 Q_2 +$$

$$+ \lambda^T [Ax - uBx + Dw] + \mu_1 (-u + u_{min}) + \mu_2 (u - u_{max}),$$

where the vector $\lambda(t)$ and the scalars $\mu_1(t), \mu_2(t)$ are Lagrange multipliers for the dynamic and the inequality constraints, respectively. Necessary conditions for optimality are ([3])

$$\begin{align*}
a) \frac{\partial H}{\partial u} &= 0, \\
b) \dot{\lambda} &= -\frac{\partial H}{\partial x}, \\
c) \lambda(t + T_p) &= 0, \\
\end{align*} \tag{3.4}$$

and, depending on whether the first, none, or the second of the inequality constraints is active, we have the following cases and necessary conditions belonging to them:

Case 1. $\mu_1 > 0$, $\mu_2 = 0$, then $\frac{\partial H}{\partial \mu_1} = 0.$  

Case 2. $\mu_1 = 0$, $\mu_2 = 0$, no additional condition.  

Case 3. $\mu_1 = 0$, $\mu_2 > 0$, then $\frac{\partial H}{\partial \mu_2} = 0.$  

(3.5) (3.6) (3.7)
3.2 Analytical solution

Elaboration of the conditions (3.4) yields

\[ a) \frac{\partial H}{\partial u} = -x^T U x + u x^T V x - \lambda^T B T x - \mu_1 + \mu_2 = 0, \]  
(3.8)

\[ b) \dot{\lambda} = -Q_1 x + 2 u U x - u^2 V x - Q_{12} w - A^T \lambda + u B T \lambda, \]  
(3.9)

\[ c) \lambda(t + t_p) = 0. \]  
(3.10)

If \( x_2 - x_4 \neq 0 \) then \( x^T V x \neq 0 \), and Eq. (3.8) gives

\[ u = (x^T V x)^{-1} \left[ x^T U x + \lambda^T B T x + \mu_1 - \mu_2 \right], \]  
(3.11)

and in combination with Eq. (2.6) and Eqs. (3.8)-(3.10), one gets

\[ \dot{x} = A_n x - B R^{-1} B^T \lambda + D w - (\mu_1 - \mu_2)(T x)^{-1} R^{-1} B, \]  
(3.12)

\[ x(t_0) = x_0; \]

\[ \dot{\lambda} = -Q_n x - A_n^T \lambda - (x^T V x)^{-2} \left[ (\mu_1 - \mu_2) \lambda^T B T x + (\mu_1 - \mu_2)^2 \right] V x, \]  
(3.13)

\[ \lambda(t + t_p) = 0, \]

where \( A_n = A - B R^{-1} N^T \) and \( Q_n = Q_1 - N R^{-1} N^T \).

Now, we can elaborate the cases (3.5)-(3.7).

**Case 1**
The necessary condition \( \partial H/\partial \mu_1 = 0 \) yields \( u(t) = u_{\text{min}} \). This together with Eq. (3.8) gives

\[ \mu_1 = -x^T (U - u_{\text{min}} V) x - \lambda^T B T x. \]  
(3.14)

Inserting this into Eq. (3.12), invoking (3.5), gives

\[ \dot{x} = A(u_{\text{min}}) x + D w, \quad x(t_0) = x_0, \]  
(3.15)

with

\[ A(u_{\text{min}}) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -k_1/m_1 & -(b + u_{\text{min}})/m_1 & 0 & (b + u_{\text{min}})/m_1 \\ 0 & 0 & -k_2/m_2 & -(b + u_{\text{min}})/m_2 \\ k_1/m_2 & (b + u_{\text{min}})/m_2 & 0 & 1 \end{bmatrix} \]  
(3.16)
Case 2
Due to (3.6), Eqs. (3.12) and (3.13) become

\begin{align*}
\dot{x} &= A_n x - B R^{-1} B^T \lambda + D w, \quad x(t_0) = x_0, \\
\dot{\lambda} &= -Q_n x - A_n^T \lambda - Q_{12} w, \quad \lambda(t + t_p) = 0.
\end{align*}

Equations (3.17)-(3.18) form a two point boundary value problem of order $2n$. To solve this problem a new vector is introduced, defined by

\begin{equation}
\mathbf{r}(t) = \lambda(t) - P x(t),
\end{equation}

where $P \in \mathbb{R}^{n \times n}$ is a constant matrix, $\mathbf{r}(t + t_p) = -P x(t + t_p)$, and $x(t) \in \mathbb{R}^n$ is a vector dependent on the road elevation $w(t)$. Inserting solution (3.19) in Eqs. (3.17)-(3.18), yields

\begin{equation}
\begin{bmatrix} P A_n - A_n^T P - P B R^{-1} B^T P + Q_n \end{bmatrix} x = -\dot{r} + \begin{bmatrix} P B R^{-1} B^T - A_n^T \end{bmatrix} \mathbf{r} + - (PD + Q_{12}) w.
\end{equation}

If $P$ is chosen as the symmetric positive definite solution of the algebraic Riccati equation

\begin{equation}
PA_n - A_n^T P - P B R^{-1} B^T P + Q_n = 0,
\end{equation}

then we can write for vector $\mathbf{r}$

\begin{equation}
\dot{\mathbf{r}} = \begin{bmatrix} -A_n^T + P B R^{-1} B^T \end{bmatrix} \mathbf{r} - (PD + Q_{12}) w, \quad \mathbf{r}(t + t_p) = -P x(t + t_p).
\end{equation}

Thus, in this case the system is described by

\begin{equation}
\dot{x} = A_c x + D w - B R^{-1} B^T \mathbf{r}, \quad x(t_0) = x_0,
\end{equation}

with $A_c = A_n - B R^{-1} B^T P$, and the solutions $P$ and $\mathbf{r}$ from Eqs. (3.21)-(3.22).

Case 3
Condition $\partial H/\partial \mu_2 = 0$ yields $u(t) = u_{\text{max}}$. Taking this and (3.7) into account, Eq. (3.8) gives

\begin{equation}
\mu_2 = -x^T (U - u_{\text{max}} V) x - \lambda^T B T x.
\end{equation}

In this case, Eq. (3.12) becomes

\begin{equation}
\dot{x} = A(u_{\text{max}}) x + D w, \quad x(t_0) = x_0,
\end{equation}
with

$$A(u_{\text{max}}) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -k_1/m_1 & -(b + u_{\text{max}})/m_1 & 0 & (b + u_{\text{max}})/m_1 \\ 0 & 0 & 0 & 1 \\ k_1/m_2 & (b + u_{\text{max}})/m_2 & -k_2/m_2 & -(b + u_{\text{max}})/m_2 \end{bmatrix}. \quad (3.26)$$

Now, we can state the main result of this section. Using Eqs. (3.11), (3.14), (3.19), and (3.24), the optimal preview control $u(t)$ is given by

$$u(t) = \begin{cases} u_{\text{min}} & \text{if } R^{-1} \left[ (N^T + B^TP)x + B^Tr \right] Tx \leq u_{\text{min}}(Tx)^2 \\ u_{\text{max}} & \text{if } R^{-1} \left[ (N^T + B^TP)x + B^Tr \right] Tx \geq u_{\text{max}}(Tx)^2 \\ R^{-1} \left[ (N^T + B^TP)x + B^Tr \right] (Tx)^{-1} & \text{otherwise} \end{cases} \quad (3.27)$$

The problem has a singularity at $Tx = x_2 - x_4$. At this point, therefore, switching should not occur, so the control retains its most recent value. Note that control law (3.27) consists of a feedback part and a feedforward part that uses preview information about the road input.

Furthermore, the closed loop system is described by

$$\dot{x} = A(u_{\text{min}})x + Dw \quad \text{if } R^{-1} \left[ (N^T + B^TP)x + B^Tr \right] Tx \leq u_{\text{min}}(Tx)^2$$

$$\dot{x} = A(u_{\text{max}})x + Dw \quad \text{if } R^{-1} \left[ (N^T + B^TP)x + B^Tr \right] Tx \geq u_{\text{max}}(Tx)^2 \quad (3.28)$$

The solution to this constrained preview control problem yields the so-called “clipped optimal control” (Butsuen and Hedrick [4], Hac and Youn [7]): the force generated by the damper follows the one required by an active system (see for derivation Appendix B) when dissipation of energy is required. Otherwise, the damping coefficient is set to one of the limit values.

### 3.3 Practical aspects of the optimal control

To calculate the optimal control $u(t)$, the current state $x(t)$ and the solution $r(t)$ are needed. To solve $r(t)$, the state $x(t + t_p)$ must be known. However, this state is not known a priori. In this section, two approaches are treated to take this boundary value into account.

A practical approach is the following: under the assumption that the preview time is large enough and that the real parts of the eigenvalues of $A_c$ are negative and large
enough, the influence of \( r(t+t_p) \) on \( r(t) \) is negligible. Then, the vector \( r(t) \) is calculated by backward integration of Eq. (3.22)

\[
r(t) = - \int_{t+t_p}^{t} e^{-A_c^T (t - \tau)} (PD + Q_{12}) w(\tau) \, d\tau,
\]

(3.29)

with \( A_c \) as given in Eq. (3.23). Switching the limits of integration and using the transformation of variables \( \sigma = \tau - t \), gives

\[
r(t) = \int_{0}^{t_p} e^{A_c^T \sigma} (PD + Q_{12}) w(t + \sigma) \, d\sigma,
\]

(3.30)

Note that \( r(t) \) is calculated for the unconstrained situation (case 2). This approach which actually ignores the boundary value, will be called 'semi-active with preview I'.

Huisman et al. [9] have determined the state \( x(t+t_p) \) from the analytical solution of Eqs. (3.22)-(3.23) (see Appendix D). With state \( x(t+t_p) \), the current state \( x(t) \), and the solution of Eq. (3.22), the actual input to be supplied \( u(t) \) is computed from control law (3.27). This approach to handle the boundary value is referred to as 'semi-active with preview II'.
4 Performance of the semi-active suspension with preview

To test the performance of the control law derived in Chapter 3, simulations are done for a deterministic and a stochastic road surface. Here, we use rounded pulses as deterministic road irregularities. Furthermore, the dynamic behaviour of the semi-active suspension is tested on a stochastic brick paved road.

4.1 Rounded pulses

Alanoly and Sankar [1] use rounded pulses as road input. These road elevations are described by (see also Figure 2)

$$z_0(t) = z_{0\text{max}} \cdot \frac{e^2}{4} \cdot \left(\frac{t}{\pi t_d}\right)^2 \cdot e^{-2\pi \frac{t}{t_d}},$$

where $z_0$ is the vertical position of the road surface at the front wheels of the vehicle (see Figure 1). The suspension behaviour is determined by calculating the response of the vehicle for a range of pulse-widths $t_d$ and pulse-heights $z_{0\text{max}}$. Huisman et al. [10] choose this range such that for a two DOF vehicle model with a passive suspension, either the available suspension working space or the maximum allowable tire deflection is reached. The parameter values of the passive system, the critical limits regarding the dynamic behaviour of the suspension system, and the range of rounded pulses for which the passive system reaches these limits, are given in Appendix C.

![Figure 2: Rounded pulse.](image)

4.2 Simulations on rounded pulses

In simulations, the pulse-height $z_{0\text{max}}$ is chosen equal to half the critical heights for the passive system, because otherwise no improvement in the performance of the semi-active
4 Performance of the semi-active suspension with preview

suspension can be achieved in comparison with the passive system: a reduction of the body acceleration is obtained at the expense of 'extra' tire deflection and suspension deflection. Therefore, the weighting factors $\rho_1$ and $\rho_2$ are determined such that a maximum reduction of the body acceleration is achieved on a range of 'half-critical' pulses, keeping the tire deflection and the suspension deflection within their limits. The combination of weighting factors $\rho_1 = 1.5 \cdot 10^3$ and $\rho_2 = 8 \cdot 10^3$ has been selected from a large number of combinations for which the response of the semi-active system is determined. Furthermore, a preview time of $1/8 \, [s]$ is available (wheelbase $L = 3.5 \, [m]$ and vehicle speed $V = 28 \, [m/s]$). In Figure 3 the dynamic behaviour of the tire deflection, the suspension deflection, the body acceleration and the variable damper rate is shown for one pulse.

![Figure 3: Tire deflection, suspension deflection, body acceleration, and the variable damper of the semi-active suspension for the pulse $t_d = 0.756 \, [s]$, $\sigma_{0\text{max}} = 0.088 \, [m]$ as road input. (---) passive; (---) semi-active without preview; (---) semi-active with preview I. Preview time $t_p = 1/8 \, [s]$.](image-url)
This figure shows that:

- the semi-active systems (with and without preview I) reduce the body acceleration remarkably at the expense of an increase of the negative suspension deflection.

- the semi-active suspension with preview I is able to reduce the (negative) acceleration even further compared with the nonpreview case. This reduction is accompanied by an increase of the positive suspension deflection.

- no reduction in the positive acceleration can be achieved with preview: an active system with preview prepares itself for an approaching pulse by pumping energy in the suspension system. The suspension acts to lift the wheels over the pulse. This reduces the forces transmitted to the body and improves tracking of the road input by the wheel. Since a semi-active suspension is only able to dissipate energy, the information of the road input can't be utilized and the damper switches to the lowest possible damping rate.

- the semi-active systems damp out very fast: after $t = 0.7 \, [s]$ the damping settings are higher than the one in the passive system, so oscillations of the wheels and the body of the vehicle are much better damped.

The performance quantities for the whole range of pulses are the maxima and minima of the tire deflection, the suspension deflection, and the body acceleration. Note that important comfort features such as the frequency dependence of the human body (ISO 2631) and the rate of damping after an excitation, are not taken into account. Figure 4 illustrates the performance of the semi-active suspension for the series of rounded pulses as road inputs:

- For the semi-active suspension without preview a reduction in the body acceleration is coupled with an increase in the maximum positive tire deflection for $t_d < 0.45 \, [s]$, and an increase in the maximum negative suspension deflection for longer pulses.

- The suspension systems with preview can hardly take any advantage of the available road information: the semi-active system with preview I is only able to reduce (negative) accelerations over the pulse-widths $0.34 \leq t_d \leq 2.5 \, [s]$ compared with the nonpreview case. None of the preview systems can reduce the maximum positive acceleration because of their dissipative character mentioned above.

- For low-frequent excitations ($t_d \geq 1.8 \, [s]$) the semi-active systems with preview even behave worse than the passive system with regard to the maximum positive accelerations: a larger preview time is required to imitate the passive acceleration behaviour.
4 Performance of the semi-active suspension with preview

Figure 4: Performance of the semi-active suspension systems (without, with I and II) compared with the passive suspension for rounded pulses as road input. (--) passive; (···) semi-active without preview; (--) semi-active with preview I; (·--) semi-active with preview II. Preview time $t_p = 1/8$ [s].
4.3 Stochastic road surface

In this section, a more realistic type of road input is described: a stochastic road profile. This road is represented by its power spectral density (see Sharp and Crolla [12]):

$$\text{PSD}(f) = \begin{cases} R_c \cdot V^{\kappa-1} \cdot f_0^{1-\kappa} & \text{if } f \leq f_0, \\ R_c \cdot V^{\kappa-1} \cdot f^{-\kappa} & \text{if } f > f_0, \end{cases}$$

(4.2)

where $R_c$ is the roughness of the road surface, $\kappa$ is the slope of the power density function (see Figure 4) and $V$ the forward vehicle speed. The description of a brick paved road measured by DAF Trucks is used in the simulations.

A realisation of this stochastic road is generated by adding a large number of sinusoidals (e.g. 1000) with uniformly distributed phase and amplitude. The frequencies of these sinusoidals are uniformly distributed in a frequency interval $[f_{\text{min}}, f_{\text{max}}]$.

![Figure 4: Power spectral density function of a stochastic road surface.](image)

4.4 Simulations on a stochastic road surface

The performance of the suspension is tested on 100 [s] brick paved road for a vehicle speed $V = 14$ [m/s], invoking a preview time $t_p = 1/4$ [s]. The performance of the semi-active suspension is evaluated on basis of the root mean square values of the tire deflection, the suspension deflection, and the body acceleration (not frequency weighted). These RMS values, the percentage of time in which road contact is lost, and the percentage of time in which collisions between axle and body occur, are shown in Table 1.

In Table 1, the following attracts the attention:

- Again, the semi-active systems (without, with I and II) achieve a substantial acceleration reduction, which is accompanied by a strong increase in the tire
deflection and a moderate increase in the suspension deflection. Particularly the increase in tire deflection of the semi-active suspension system without preview is inadmissible (tire lift-off occurs 2.5% of the time).

- The two semi-active suspensions with preview reduce this increase in the tire deflection: preview appears to be useful in limiting the tire deflection.

- On this road surface, the semi-active suspension systems differ significantly: the semi-active suspension with preview II improves all performance quantities compared with the semi-active nonpreview suspension. On a stochastic road surface, the approach to handle the boundary value $r(t + t_p)$ appears to be reasonable.

Table 1: Performance of the semi-active suspensions with preview I and II compared with the passive suspension and the semi-active suspension.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>tire defl. [cm]</th>
<th>lift-off [-]</th>
<th>susp. defl. [cm]</th>
<th>collision [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive</td>
<td>0.48</td>
<td>0.1%</td>
<td>1.30</td>
<td>0%</td>
</tr>
<tr>
<td>semi active</td>
<td>0.76 +58%</td>
<td>2.5%</td>
<td>1.43 +10%</td>
<td>0%</td>
</tr>
<tr>
<td>semi active with preview I</td>
<td>0.66 +38%</td>
<td>1.4%</td>
<td>1.53 +18%</td>
<td>0%</td>
</tr>
<tr>
<td>semi active with preview II</td>
<td>0.69 +44%</td>
<td>1.6%</td>
<td>1.41 +8%</td>
<td>0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>acceleration [m/s²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive</td>
<td>1.76</td>
</tr>
<tr>
<td>semi active</td>
<td>1.15 -35%</td>
</tr>
<tr>
<td>semi active with preview I</td>
<td>1.30 -26%</td>
</tr>
<tr>
<td>semi active with preview II</td>
<td>1.08 -39%</td>
</tr>
</tbody>
</table>
5 Conclusions and recommendations

In this report, a control strategy for a semi-active suspension system with a continuously variable damper was derived and tested on a deterministic and a stochastic road surface.

5.1 Conclusions

1. In general, a considerable reduction in the body accelerations can be achieved. However, this reduction is always accompanied with an increase in the tire deflection and the suspension working space.

2. For the rounded pulses as road input, preview can hardly improve the performance of the semi-active suspension because a supply of energy is required before an approaching pulse is traversed.

3. On the stochastic road surface examined, preview is able to reduce the tire deflection in comparison with the semi-active suspension without preview.

4. The boundary value $r(t + t_p)$ should not be ignored merely: the semi-active system with preview II results in the lowest accelerations reducing the tire deflection as well as the suspension deflection compared with the nonpreview case.

5. Here, the weighting factors $p_1$ and $p_2$ are chosen such that a maximum reduction in the body acceleration is achieved without exceeding critical limits in the tire deflection and the suspension working space for a range of half-critical rounded pulses. However, the simulations on the stochastic road surface show that for the chosen $p_1$ and $p_2$ still tire lift-offs occur: only an indirect influence can be exerted on the performance of the suspension system, or stated otherwise, the chosen quadratic performance index doesn't satisfy the actual requirements on the suspension.

5.2 Recommendations

- In fact, the tire deflection and the required suspension working space need not to be minimized, but we want them not to exceed certain limit values. A more direct approach to the suspension control problem is the following: minimize the maximum absolute body acceleration under constraints on (1) the system equations, (2) the input $u(t)$ and (3) the tire deflection and the suspension deflection. Since information about the approaching road is available, computations with damping settings to be optimized can be carried out to check whether the constraints are satisfied.

- As stated in Section 4.2, the settling time of the suspension system after an excitation is not taken into account as a performance quantity. It is recommendable to investigate the possibility to use this settling time in the performance index to be minimized.
• Here, the application of a continuously variable damper is examined. Potential benefits brought about by having springs with variable stiffness should be investigated.

• In this report, only two types of road input are considered: the dynamic behaviour of the semi-active suspension should be tested on other road surfaces (e.g. Belgium pavé and German Autobahn) and other road irregularities (e.g. branches, missing bricks and holes), which are available at the moment.

• The behaviour of an articulated vehicle can’t be modelled properly by a two DOF model as used here. A logical, subsequent step is the extension of this model to a half vehicle model.
References


A Problems with inequality constraints

An approach to handle inequality constraints in minimization problems is given by Bryson and Ho [3]. Consider the problem of minimizing $L(y)$ subject to $f(y) \leq 0$, where $y$ and $f$ are scalars. Then, there are two possibilities for the optimal value of $y$, $y_0$: $f(y_0) < 0$ or $f(y_0) = 0$. In the former case, the constraint is not effective and can be ignored. In the latter case, consider small perturbations about $y_0$; if $L(y_0)$ is a minimum, then we have

$$dL = \frac{\partial L}{\partial y} \bigg|_{y_0} \, dy \geq 0,$$

for all admissible values of $dy$, which must satisfy

$$df = \frac{\partial f}{\partial y} \, dy \leq 0.$$  \hspace{1cm} (A.2)

In order that Eqs. (A.1) and (A.2) be consistent, it is necessary that either

$$\text{sign} \left( \frac{\partial L}{\partial y} \right) = -\text{sign} \left( \frac{\partial f}{\partial y} \right) \quad \vee \quad \frac{\partial L}{\partial y} = 0.$$  \hspace{1cm} (A.3)

These two possibilities can be expressed in one relation as

$$\frac{\partial L}{\partial y} + \mu \frac{\partial f}{\partial y} = 0, \quad \mu \geq 0.$$  \hspace{1cm} (A.4)

The two situations are shown geometrically in Figures 5 (a) and (b).

![Figure 5](image_url)

Figure 5: The two possible types of minimum with inequality constraints.
The two cases may be treated analytically by augmenting the performance index $L(y)$
with the constraint relation $f(y) \leq 0$:

$$H(y, \mu) = L(y) + \mu f(y).$$

(A.5)

The necessary conditions become

$$\frac{\partial H}{\partial y} = 0 \quad \land \quad f(y) \leq 0,$$

(A.6)

where

$$\mu \geq 0, \quad f(y) = 0,$$

(A.7)

$$\mu = 0, \quad f(y) < 0.$$

The positivity of the multiplier $\mu$ when we have $f(y) = 0$ can be interpreted as the
requirement that the gradient $\partial L/\partial y$ be such that decrease of $L$ would violate the
constraint.

In the case that $L(y)$ is subject to a set of inequality constraints $f$, the necessary
conditions for optimality become

$$\frac{\partial L}{\partial y} + \mu^T \frac{\partial f}{\partial y}, \quad \mu = [\mu_1, \ldots, \mu_p]^T,$$

(A.8)

where

$$\mu_i \geq 0, \quad f(y) = 0, \quad i = 1, \ldots, p$$

(A.9)

$$\mu_i = 0, \quad f(y) < 0.$$
B Solution for the active suspension system

Consider the system given in Figure 1 with the variable damper replaced by an active force generator. The formulation of the optimal control problem remains the same as in Chapter 2 except that \( u(t) \) is now an actuator force and the inequality constraints (2.4) are removed. Then, the system to be controlled is described by

\[
\dot{x} = Ax - Bu + Dw, \quad x(t_0) = x_0. \tag{B.1}
\]

Performance index (2.3) can be written as

\[
J = \frac{1}{2} \int_{t}^{t+t_p} \left[ x^T Q_1 x + 2ux^T N + u^2 R + 2x^T Q_{12} w + Q_2 w^2 \right] \, dt. \tag{B.2}
\]

Introducing the notation \( A_n = A - BR^{-1}N^T \) and \( Q_n = Q_1 - NR^{-1}N^T \), it can be shown (see Hač [8]) that, if the pair \((A_n, B)\) is stabilizable and \((A_n, Q_n)\) is detectable, the control law that minimizes \( J \) is given by

\[
u_o = -R^{-1} \left[ (N^T + B^TP)x(t) + B^T r(t) \right], \quad R \neq 0, \tag{B.3}
\]

where \( P \) is the positive definite solution of the algebraic Riccati equation (3.21). The vector \( r(t) \) satisfies

\[
\dot{r}(t) = -A_r^T r(t) - [PD + Q_{12}] w, \quad r(t + t_p) = -Px(t + t_p), \tag{B.4}
\]

where \( A_r = A_n - BR^{-1}B^TP \) is the closed loop system matrix. Then, the closed loop system is described by

\[
\dot{x} = A_r x + Dw - BR^{-1}B^T r, \tag{B.5}
\]

which is equal to the semi-active system for case 2.
C  Model data

C.1  Model parameters

The parameter values of the vehicle model shown in Figure 1 are written down in Table 2. For the model with a passive suspension, the constant damper $b$ and the continuously variable damper $u$ are replaced by a linear damper with damping constant $b_p = 4.31 \cdot 10^4 \text{ [Ns/m]}$.

Table 2: Model parameters for the semi-active suspension system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>sprung mass $m_1$</td>
<td>8,650</td>
<td>constant damper $b$</td>
<td>$1.0 \cdot 10^4 \text{ [Ns/m]}$</td>
</tr>
<tr>
<td>unsprung mass $m_2$</td>
<td>1,350</td>
<td>lower limit $u: u_{\text{min}}$</td>
<td>0</td>
</tr>
<tr>
<td>suspension stiffness $k_1$</td>
<td>$4.4 \cdot 10^5 \text{ [N/m]}$</td>
<td>upper limit $u: u_{\text{max}}$</td>
<td>$8.0 \cdot 10^4 \text{ [Ns/m]}$</td>
</tr>
<tr>
<td>tire stiffness $k_2$</td>
<td>$6.5 \cdot 10^6 \text{ [N/m]}$</td>
<td>wheelbase $L$</td>
<td>3.5</td>
</tr>
</tbody>
</table>

C.2  Limits on the suspension system

The following properties should not be exceeded:

Maximum positive tire deflection: 0.015 [m]
Maximum negative suspension deflection: -0.090 [m]
Maximum positive suspension deflection: 0.140 [m]

The limit on the tire deflection is set equal to the static tire deflection: exceeding this constraint causes a tire lift-off. In order to avoid discomforting collisions between axle and chassis, there are two restrictions on available suspension working space.

C.3  Rounded pulses

The values $t_d$ and $z_{0\text{max}}$, as used by Huisman et al. [9], are shown in the Table 3.

Table 3: Combinations of $t_d$ and $z_{0\text{max}}$.

<table>
<thead>
<tr>
<th>$t_d$</th>
<th>$z_{0\text{max}}$</th>
<th>$t_d$</th>
<th>$z_{0\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[s]</td>
<td>[m]</td>
<td>[s]</td>
<td>[m]</td>
</tr>
<tr>
<td>1.00 \cdot 10^{-2}</td>
<td>5.20 \cdot 10^{-2}</td>
<td>3.06 \cdot 10^{-1}</td>
<td>1.02 \cdot 10^{-1}</td>
</tr>
<tr>
<td>2.20 \cdot 10^{-2}</td>
<td>2.64 \cdot 10^{-2}</td>
<td>3.44 \cdot 10^{-1}</td>
<td>1.09 \cdot 10^{-1}</td>
</tr>
<tr>
<td>4.82 \cdot 10^{-2}</td>
<td>1.87 \cdot 10^{-2}</td>
<td>3.87 \cdot 10^{-1}</td>
<td>1.17 \cdot 10^{-1}</td>
</tr>
<tr>
<td>7.14 \cdot 10^{-2}</td>
<td>1.93 \cdot 10^{-2}</td>
<td>4.36 \cdot 10^{-1}</td>
<td>1.26 \cdot 10^{-1}</td>
</tr>
<tr>
<td>1.06 \cdot 10^{-1}</td>
<td>2.36 \cdot 10^{-2}</td>
<td>5.10 \cdot 10^{-1}</td>
<td>1.40 \cdot 10^{-1}</td>
</tr>
<tr>
<td>1.29 \cdot 10^{-1}</td>
<td>2.82 \cdot 10^{-2}</td>
<td>7.56 \cdot 10^{-1}</td>
<td>1.76 \cdot 10^{-1}</td>
</tr>
<tr>
<td>1.57 \cdot 10^{-1}</td>
<td>3.56 \cdot 10^{-2}</td>
<td>1.12 \cdot 10^{0}</td>
<td>2.29 \cdot 10^{-1}</td>
</tr>
<tr>
<td>1.91 \cdot 10^{-1}</td>
<td>4.82 \cdot 10^{-2}</td>
<td>2.46 \cdot 10^{0}</td>
<td>4.85 \cdot 10^{-1}</td>
</tr>
<tr>
<td>2.32 \cdot 10^{-1}</td>
<td>7.08 \cdot 10^{-2}</td>
<td>5.40 \cdot 10^{0}</td>
<td>1.34 \cdot 10^{0}</td>
</tr>
<tr>
<td>2.72 \cdot 10^{-1}</td>
<td>9.60 \cdot 10^{-2}</td>
<td>1.00 \cdot 10^{1}</td>
<td>3.51 \cdot 10^{0}</td>
</tr>
</tbody>
</table>
D Calculation of the state \( x(t + t_p) \)

The solution of the 'active' part of Eq. (3.28) is

\[
x(t + t_p) = \Phi(t_p) x(t) + \int_t^{t+t_p} \left[ \Phi(t + t_p - \tau) \left[ -BR^{-1}B^T r(\tau) + Dw(\tau) \right] \right] d\tau,
\]

(D.1)

where \( \Phi \) is defined by

\[
\Phi(\tau) = e^{A_c \tau}.
\]

(D.2)

The solution of Eq. (3.22) is

\[
r(\tau) = -\Phi^T(t + t_p - \tau)Px(t + t_p) + r_0(\tau),
\]

(D.3)

where \( r_0 \) is the solution of

\[
\dot{r}_0(\tau) = -A_c^T r_0(\tau) - (PD + Q_{12})w(\tau), \quad r_0(t + t_p) = 0.
\]

(D.4)

Combination of (D.1) and (D.3) yields

\[
x(t + t_p) = \Phi(t_p) x(t) + GPx(t + t_p) + q_0(t + t_p),
\]

(D.5)

in which \( G \) and \( q_0(t + t_p) \) are defined by

\[
G = - \int_t^{t+t_p} \Phi(t + t_p - \tau)BR^{-1}B^T \Phi^T(t + t_p - \tau) d\tau,
\]

(D.6)

\[
q_0(\tau) = A_c q_0(\tau) - BR^{-1}B^T r_0(\tau) + Dw(\tau), \quad q_0(t) = 0.
\]

(D.7)

Matrix \( G \) satisfies

\[
G = Z - \Phi(t_p)Z \Phi^T(t_p),
\]

(D.8)

where \( Z \) is the solution of the Lyapunov equation

\[
A_c Z + Z A_c^T = -BR^{-1}B^T, \quad Z = Z^T.
\]

(D.9)

With Eq. (D.5), an explicit expression of the boundary value \( x(t + t_p) \) can be calculated:

\[
x(t + t_p) = (I - GP)^{-1} \left[ \Phi^T(t_p) x(t) + q_0(t + t_p) \right]
\]

(D.10)