A method of optimal system identification with applications in control
Weiland, S.

Published: 01/01/1994

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author’s version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
A Method of Optimal System Identification with Applications in Control

S. Weiland

Internal report, 94I/02

Eindhoven, September 1994
A Method of Optimal System Identification with Applications in Control

Siep Weiland

Department of Electrical Engineering
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
s.weiland@ele.tue.nl

Contributed paper for the 33rd IEEE Conference on Decision and Control

Abstract

In this paper an optimal deterministic identification problem is solved in which a new measure for the misfit between data and system is minimized. It is shown that the misfit can be expressed as the Hankel norm of a specific operator. Optimal autonomous models are obtained by factorizing an optimal Hankel norm approximant of the Laplace transformed data matrix. An upperbound on the misfit between model and data is derived for a class of non-autonomous models of prescribed complexity. The identified autonomous systems are viewed as closed-loop behaviors of a feedback interconnection of two systems. Stability of these feedback interconnections is discussed.

Key Words

optimal system identification, stabilization, linear systems, behaviors

1 Introduction

In the usual methods of system identification uncertainty of models is expressed as uncertainty in the parameters defining the model. In traditional axiomatic frameworks this uncertainty is given a stochastic interpretation in the sense that deviations of nominal parameter values are modeled by prescribed probability distributions. The recent interest in deterministic techniques to quantify model uncertainty stems mainly from the present inability of robust control theory to cope with probabilistic assumptions on estimated model parameters.

A mathematical description of model uncertainty requires a quantification of a distance measure between models. In the context of deterministic system identification a distance measure needs to be specified between observed data and models belonging to an a priori specified set of candidate models. It is important that this misfit criterion is chosen independent of the parametrization of the model class and independent of the way individual models are represented. As such, parametrization and representation issues should be clearly separated from the model identification problem.

In the present paper we consider a deterministic identification problem using model sets of prescribed complexity. We restrict attention to the class of linear time-invariant systems and introduce a misfit criteria for the distance between data and elements in this model class. The proposed misfit function is independent of model representations and is characterized as the Hankel norm of an operator which is determined by the data and a specific (co-inner) kernel representation of the model. Optimal approximate models with prescribed complexity are characterized in this
way.

In the last section of this paper we will view these optimal models as the closed-loop behavior of a plant-controller feedback interconnection. Internal stability of these interconnections is characterized in an input-output independent context and it is shown how the class of all stabilizing controllers for a given plant (or the class of all plants stabilized by a given controller) can be parametrized.

\section*{2 Models and data}

Consider a finite set of observed time series

\[ \tilde{w}_i : Z_+ \rightarrow W, \quad i = 1, \ldots, n \]  

(2.1)

where \( W \) denotes the signal space which is assumed to be a \( q \) dimensional real vector space, i.e., \( W = \mathbb{R}^q \). We address the problem to identify linear time-invariant systems that model these time series.

Let \( l_2(Z_+, \mathbb{R}^q) \) (or \( l_2^f \) for short) denote the set of time series \( w : Z_+ \rightarrow \mathbb{R}^q \) for which \( ||w||_{l_2}^2 := \sum_{t \in \mathbb{Z}_+} w^T(t)w(t) < \infty \). It is assumed throughout that \( \tilde{w}_i \in l_2^f \) for all \( i = 1, \ldots, n \).

\begin{definition}
An \( l_2 \) system is a system \( \Sigma = (Z_+, \mathbb{R}^q, B) \) with time set \( Z_+ \), signal space \( \mathbb{R}^q \) and behavior \( B \subseteq l_2(Z_+, \mathbb{R}^q) \).
\end{definition}

Denote by \( S_2^f \) the class of all \( l_2 \) systems \( \Sigma = (Z_+, \mathbb{R}^q, B) \) defined on the time set \( Z_+ \) whose behavior \( B \subseteq l_2^f \) is a linear shift-invariant and closed subset of \( l_2(Z_+, \mathbb{R}^q) \). This class of linear time-invariant systems has been extensively studied in the work of Willems \([8]\) and Heij \([5]\) and corresponds to the class of all plants stabilized by a given controller.

\begin{definition}
A system \( \Sigma \in S_2^f \) is said to be autonomous if there exists \( t \in Z_+ \) such that the mapping \( \pi : B \rightarrow B_{[1, t]} \) defined by the restriction \( \pi_i(w) := w_{[1, i]} \) is injective.
\end{definition}

\begin{definition}
A system \( \Sigma = (Z_+, \mathbb{R}^q, B) \) is unfalsified by the data \( (2.1) \) if \( \tilde{w}_i \in B \) for \( i = 1, \ldots, n \). We call a system \( \Sigma_1 = (Z_+, \mathbb{R}^q, B_1) \) more powerful than \( \Sigma_2 = (Z_+, \mathbb{R}^q, B_2) \) if \( B_1 \subseteq B_2 \).
\end{definition}

This defines a partial order \( \preceq \) on systems in \( S_2^f \) by defining \( \Sigma_1 \preceq \Sigma_2 \) if \( \Sigma_1 \) is more powerful than \( \Sigma_2 \). Equivalently, \( \Sigma_i \) will be called a subsystem of \( \Sigma_2 \) if \( \Sigma_1 \preceq \Sigma_2 \). The most powerful unfalsified system (for the data \( (2.1) \)) is that system \( \Sigma_{MPUM} \in S_2^f \) which is unfalsified by \( (2.1) \) and which is more powerful than any other \( \Sigma \in S_2^f \) which is unfalsified by \( (2.1) \). In the following we use the fact that \( \Sigma_{MPUM} \in S_2^f \) exists and is unique for any set of time series \( (2.1) \) which belong to \( l_2^f \). See \([5, 8, 9]\).

\section*{3 Models for data}

In this section we address the question to characterize all systems \( \Sigma \in S_2^f \) which are unfalsified by the data \( (2.1) \).

Define the Laplace transform of the data sequences \( \tilde{w}_i \) by putting

\[ W(z) := \tilde{W}(1)z^{-1} + \tilde{W}(2)z^{-2} + \ldots \]  

(3.1)

where \( z \in \mathbb{C} \) and \( \tilde{W}(t) := [\tilde{w}_1(t) \ldots \tilde{w}_n(t)] \), \( t \in Z_+ \) denotes the data matrix. We will assume the following.

\begin{assumption}
Assumption 3.1 \( W(z) \) is rational and analytic for all \( z \in \mathbb{C} \) with \( |z| < 1 \).
\end{assumption}

Specific examples of data sets that satisfy assumption 3.1 include finite sets of frequency response measurements, spectral data, or polynomial-exponential data series. See \([6]\) for a methodology to approximate data sets by polynomial-exponential time series which satisfy assumption 3.1 using risk minimization techniques.

Let \( \Sigma_{MPUM} = (Z_+, \mathbb{R}^q, B_{MPUM}) \) denote the most powerful unfalsified model for this data and let \( \sigma \) denote the shift \( (\sigma w)(t) = w(t + 1) \).

\begin{proposition}
\( \Sigma_{MPUM} \) is well defined and autonomous. Its behavior

\[ B_{MPUM} = \text{span} \{ \sigma^i(\tilde{w}_i); \quad i = 1, \ldots, n, \quad t \in Z_+ \} \]

is the smallest linear shift invariant finite dimensional and closed subspace of \( l_2^f \) that contains \( \tilde{w}_i \) for all \( i = 1, \ldots, n \).
\end{proposition}
4 OPTIMAL APPROXIMATE MODELS

Proof. See e.g. [2, 8].

To represent $E_{MPUM}$ introduce the power series

$$\Theta(z) := \Theta_0 + \Theta_1 z + \cdots + \Theta_k z^k + \cdots$$

where $z \in \mathbb{C}$ and $\Theta_k$ are constant real matrices of dimension $g \times q$. Assume that $\Theta \in H_\infty$ (all entries of $\Theta(z)$ are analytic and bounded functions for $z \in \mathbb{C}$ with $|z| < 1$). Then $\Theta$ represents a linear time-invariant system through the difference equations

$$\Theta(\sigma)w = 0,$$

(3.2)

Formally, (3.2) defines the $l_2$ system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^t, B(\Theta)) \in S_2^2$ with

$$B(\Theta) := \{w \in l_2^+ | \Theta(\sigma)w = 0\}.$$  

(3.3)

Let $\tilde{B}$ denote the Laplace transform of $B$. Then $\Sigma$ is in the frequency domain equivalent described by

$$\tilde{B} = \{\tilde{w} \in \mathcal{H}_\infty^+ | (\Pi_+ \Theta w)(z) = 0, \ z \in \mathbb{C}\}.$$

Here, $\mathcal{H}_\infty^+$ denotes the image of $l_2^+$ under the Laplace transform and $\Pi_+$ denotes the canonical projection $\Pi_+ : L_2 \to \mathcal{H}_\infty^+$.

The most powerful unfalsified system $\Sigma_{MPUM}$ admits an $\mathcal{H}_\infty$ kernel representation which is obtained as follows

**Theorem 3.3** Let $W = \Theta_{MPUM}^{-1} B$ be a left coprime factorization over $H_\infty$ of $W$. Then $\Sigma_{MPUM} = (\mathbb{Z}_+, \mathbb{R}^t, B_{MPUM})$ where $B_{MPUM} = B(\Theta_{MPUM})$ or, equivalently, $B_{MPUM} = \ker \Pi_+ \Theta_{MPUM}$.

Once $\Sigma_{MPUM}$ is known, a system $\Sigma$ is unfalsified by the data (2.1) if and only if $\Sigma_{MPUM} \subseteq \Sigma$.

**Theorem 3.4** Let $\Sigma_{MPUM}$ be represented by $\Theta_{MPUM}$ and let $\Sigma = (\mathbb{Z}_+, \mathbb{R}^t, B(\Theta))$. Then the following statements are equivalent.

1. $\Sigma_{MPUM} \subseteq \Sigma$
2. $\Theta = \Lambda \Theta_{MPUM}$ for some rational $\Lambda \in H_\infty$.

All unfalsified models $\Sigma \in S_2^2$ are therefore generated by $\Sigma_{MPUM}$ by premultiplying $\Theta_{MPUM}$ with rational elements in $H_\infty$. We will use this result in section 4 below.

4 Optimal approximate models

4.1 Model complexity

In this section $\Sigma = (\mathbb{Z}_+, \mathbb{R}^t, B)$ belongs to $S_2^2$ and it is supposed that $B = B(\Theta)$ with $\Theta$ a rational element in $H_\infty$.

**Definition 4.1** The complexity $c(\Sigma)$ of $\Sigma$ is a pair $c(\Sigma) = (m(\Sigma), n(\Sigma))$ where $m(\Sigma) = q - \text{rank}(\Theta)$ and $n(\Sigma)$ is the McMillan degree of $\Theta$.

In this definition $m(\Sigma) + n(\Sigma)$ is to be interpreted as the total degree of freedom to uniquely determine a trajectory in $\Sigma$. This consists of the dimension $m(\Sigma)$ of the input space in any (and hence any) input-output representation of $\Sigma$ and the dimension $n(\Sigma)$ of the space of initial conditions (or the state space dimension in any minimal state space representation of $\Sigma$).

Introduce a lexicographic ordering on system complexities as follows. Define $c(\Sigma_1) \leq c(\Sigma_2)$ if

$$\begin{cases} 
  m(\Sigma_1) = m(\Sigma_2), \ n(\Sigma_1) = n(\Sigma_2) \quad \text{or} \\
  m(\Sigma_1) < m(\Sigma_2), \ n(\Sigma_1) = n(\Sigma_2) \quad \text{or} \\
  m(\Sigma_1) = m(\Sigma_2), \ n(\Sigma_1) < n(\Sigma_2) 
\end{cases}$$

Since autonomous systems $\Sigma \in S_2^2$ have finite dimensional behavior, it follows that their corresponding kernel representations have full rank or, equivalently, $m(\Sigma) = 0$. Therefore, the least complex systems are the autonomous ones.

4.2 Misfits

The discrepancy between model and data is formalized by the definition of a misfit function between the data (2.1) and models in $S_2^2$. We assume that the data is represented by the matrix $W$ defined in (3.1).

The misfit is defined as follows

**Definition 4.2** The misfit between $B$ and $W$ is defined as

$$d(B, W) := \sup \left\{ <Wz, v> \right\} | v \in \tilde{B}^L, z \in \mathcal{H}_2^+ \}$$

(\text{Here,})

$$\tilde{B}^L = \{v \in \mathcal{H}_2^+ | <v, w> = 0 \quad \text{for all} \ w \in B\}$$

is the orthogonal complement of $\tilde{B}$ in $\mathcal{H}_2^+$ and $W$ is viewed as a multiplicative operator $W : \mathcal{H}_2^+ \to L_2, \ W : x \mapsto Wx$. 


Clearly \( d(B, W) \geq 0 \) and \( d(B, W) = 0 \) if, and only if \( \Sigma \) is unfalsified by the data. Note that the misfit is independent of representations of \( \Sigma \in S^2 \). In other words, \( d(B, W) \) is a non-parametric criterion.

The following theorem relates the misfit to the Hankel norm of a specific operator. See also [7] for other characterizations of the misfit \( d(\Theta, W) \).

**Theorem 4.3** Let \( \Sigma \in S^2 \) and let its behavior \( B = B(\Theta) \) where \( \Theta \) is co-inner, i.e. \( \Theta \Theta^* = I \). Then

\[
d(B, W) = \| \Pi + \Theta W \|_H
\]

where \( \| \cdot \|_H \) denotes the induced operator (or Hankel) norm of the composite function \( \Pi + \Theta W \) viewed as a mapping from \( \mathcal{H}_2 \) to \( \mathcal{H}_2^+ \).

Co-inner kernel representations of systems in \( S^2 \) indeed exist.

**Proposition 4.4** The behavior \( B \) of every system \( \Sigma \in S^2 \) admits a representation \( B = B(\Theta) \) with \( \Theta \) co-inner.

### 4.3 Optimal identification

The approximate modeling problem consists of finding low complexity models which minimize the misfit between model and data. Precisely, given the data (2.1) together with prescribed complexity \( (m, n) \), find systems \( \Sigma \in S^2 \) with \( c(\Sigma) \leq (m, n) \) such that the misfit \( d(B, W) \) is minimal. A complete solution for the case where \( m = 0 \) is given in the following result.

**Theorem 4.5** Let \( W \) be given by (3.1) and let \( \Sigma_{\text{MPUM}} \) be the most powerful unfalsified model for \( W \). Suppose that \( c(\Sigma_{\text{MPUM}}) = (0, N) \) and let

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N > 0
\]

denote the singular values of \( W \). Denote by \( W_n \in H^\infty \) an optimal Hankel norm approximant of \( W \) of McMillan degree \( \leq n \). Let \( \Sigma_n = \left( \mathbb{Z}_+, \mathbb{R}^q, B_n \right) \) be the most powerful unfalsified model associated with \( W_n \).

Then

1. \( \Sigma_n \) is autonomous.
2. \( d(B_n, W) = \sigma_{n+1} \).
3. \( d(B'_n, W) \geq d(B_n, W) \) for all \( \Sigma'_n \in S^2 \) with \( c(\Sigma'_n) \leq c(\Sigma_n) \).

4. if \( \Sigma_n \subseteq \Sigma'_n \) then \( d(B'_n, W) \leq \sigma_{n+1} \).

**Proof.** See [7].

Note that \( c(\Sigma_n) = (0, n^*) \) with \( n^* \leq n \) the McMillan degree of \( W_n \). Conclude from Theorem 4.5 that

\[
\Sigma_n = \arg \min \{ d(B, W) \mid \Sigma \in S^2, c(\Sigma) \leq (0, n) \}
\]

For given data (2.1), Theorem 4.5 leads to the following constructive method for the computation of a kernel representation of \( \Sigma_n \):

1. Compute an optimal Hankel norm approximant \( W_n \in H^\infty \) of \( W \) with McMillan degree \( n \) [4].
2. Let \( W_n = \Theta_{\text{MPUM}, n}^{-1} W_n \) be a left coprime factorization over \( H^\infty \) of \( W_n \).
3. Put \( B_n = B(\Theta) := \ker \Theta_{\text{MPUM}, n}(\sigma) \).

The optimal approximate model is then given by \( \Sigma_n = \left( \mathbb{Z}_+, \mathbb{R}^q, B_n \right) \) whereas all models which have \( \Sigma_n \) as subsystem have misfit \( \leq \sigma_{n+1} \).

**5 Stabilization of systems**

In this section kernel representations of autonomous systems are used as descriptions of closed-loop behaviors. A “closed-loop” behavior consists of the interconnection of two \( X \) systems \( \Sigma_p \) and \( \Sigma_c \) which will be referred to as the \textit{plant} and the \textit{controller}, respectively.

**Definition 5.1** Let \( \Sigma_p \) and \( \Sigma_c \) be elements of \( S^2 \). Their interconnection is defined as the system \( \Sigma_{\text{int}} := \Sigma_p \wedge \Sigma_c = \left( \mathbb{Z}_+, \mathbb{R}^q, B_{\text{int}} \right) \) where \( B_{\text{int}} = B_p \cap B_c \).

Hence, interconnection and intersection are synonymous. It is easily seen that \( \Sigma_{\text{int}} \) again belongs to \( S^2 \).

**Definition 5.2** The interconnection \( \Sigma_{\text{int}} = \Sigma_p \wedge \Sigma_c \) is called a feedback interconnection if \( \Sigma_{\text{int}} \) is autonomous.

\(^{1}\)In this input-output independent context the traditional looping configuration of input and output signals is not implied.
Definition 5.3 A feedback interconnection $\Sigma_{\text{int}} = (Z_+, \mathbb{R}^q, B)$ is said to be internally stable if

$$B = \{ w \in l_2^\infty(\mathbb{Z}_+, \mathbb{R}^q) \mid \Theta(\sigma)w = 0 \}$$

for all $\Theta$ for which $B = B(\Theta)$. Here, $l_2^\infty(\mathbb{Z}_+, \mathbb{R}^q)$ denotes the set of locally square summable time series $w : \mathbb{Z} \to \mathbb{R}^q$. In that case, $\Sigma_c$ is called a stabilizing system for $\Sigma_p$ and $\Sigma_p$ is said to be stabilized by $\Sigma_c$.

Stated otherwise, in an internally stable feedback interconnection the set of locally square summable solutions of (3.2) coincides with the set of square summable solutions of (3.2). In particular this implies that the locally square summable solutions $w(t)$ of a feedback interconnected system converge to zero as $t \to \infty$.

Let $\Theta$ be a non-singular $q \times q$ rational matrix with entries in $H_\infty$. Suppose that $\Theta$ represents a feedback interconnection $\Sigma_{\text{int}} = (Z_+, \mathbb{R}^q, B_{\text{int}})$, i.e. $B_{\text{int}} = B(\Theta)$. Since the behavior of both plant and controller contain $B_{\text{int}}$ as a subset we derive the following property as an immediate consequence of Theorem 3.4.

Proposition 5.4 Let $\Sigma_{\text{int}}$ be be a feedback interconnection of $\Sigma_p$ and $\Sigma_c$ and let its behavior $B_{\text{int}} = B(\Theta)$ with $\Theta \in H_\infty$. Then there exist $\Lambda_p, \Lambda_c \in H_\infty$ such that

$$B_p = B(\Lambda_p \Theta) \quad (5.1)$$

$$B_c = B(\Lambda_c \Theta). \quad (5.2)$$

Conversely, if $\Lambda = (\Lambda_p \Lambda_c)^T$ is a unit in $H_\infty$ then the interconnection of (5.1) and (5.2) yields the feedback interconnection $\Sigma_{\text{int}}$ with behavior $B_{\text{int}} = B(\Theta)$.

In order to investigate internal stability of interconnected systems it is common to introduce fictitious signals in a plant-controller configuration. Let $\Theta \in H_\infty$ be as in Proposition 5.4 and consider the equation

$$\dot{\tilde{w}} = \Theta(\sigma)w \quad (5.3)$$

Introduce the system $\Sigma_{\text{full}} = (Z_+, \mathbb{R}^{2q}, B_{\text{full}})$ with full behavior

$$B_{\text{full}} = \{(w, \tilde{w}) \in l_2^+ \mid \tilde{w} \in B_p\}$$

That is,

$$B_p = \pi_w(B_{\text{full}} \cap B_{\text{full},p})$$

We remark that the number $m(\Sigma_p) = q - p > 0$ corresponds to the dimension of the input space in any input-output representation of $\Sigma_p$.

Theorem 5.5 Let $\Sigma_{\text{int}}$ be a feedback interconnection and let $\Sigma_{\text{full}}$ be its associated full behavior. Then the following statements are equivalent.

1. $\Sigma_{\text{int}}$ is internally stable.
2. $\pi_0 B_{\text{full}} = l_2^+$.
3. there exists a non-singular $\Theta \in H_\infty$ such that $\Theta^{-1} \in H_\infty$ and $B_{\text{int}} = B(\Theta)$.
4. for all non-singular $\Theta \in H_\infty$ for which $B_{\text{int}} = B(\Theta)$ there holds $\Theta^{-1} \in H_\infty$.

Theorem 5.5 has the interpretation that internal stability is equivalent to the property that the fictitious signal $\tilde{w}$ in $B_{\text{full}}$ can be considered as a free variable in $l_2^+$. Since for all $\tilde{w} \in l_2^+$, $\pi_wB_{\text{full}}$ is autonomous, the variables $(w, \tilde{w})$ of $\Sigma_{\text{full}}$ can be partitioned into a $q$-dimensional input variable $\tilde{w}$ and a $q$-dimensional output variable $w = \Theta^{-1}\tilde{w}$ which belongs to $l_2^+$ whenever $\Sigma_{\text{int}}$ is internally stable. The poles of the feedback interconnection are the invariant zeros of $\Theta$ or, equivalently, the poles of $\Theta^{-1}$.

Suppose that $\Lambda_p \in H_\infty$ has rank $p < q$ and consider the systems $\Sigma_p$ and $\Sigma_c$ with behavior $B_p := B(\Lambda_p \Theta)$ and $B_c := B(\Lambda_c \Theta)$, respectively. Let $\Sigma_c$ and $\Sigma_p$ be defined analogously. We address the question to characterize the class of controllers $\Sigma_c \in S_\Sigma^p$ which stabilize $\Sigma_p$. First observe that $\Sigma_p$ can be viewed as the interconnection $\Sigma_{\text{full}} \land \Sigma_{\text{full},p}$ where

$$B_{\text{full},p} := \{(w, \tilde{w}) \in l_2^+ \mid \tilde{w} \in B_p\}$$

Suppose that $\Lambda_p \in H_\infty$ has rank $p < q$ and consider the systems $\Sigma_p$ and $\Sigma_c$ with behavior $B_p := B(\Lambda_p \Theta)$ and $B_c := B(\Lambda_c \Theta)$, respectively. Let $\Sigma_c$ and $\Sigma_p$ be defined analogously. We address the question to characterize the class of controllers $\Sigma_c \in S_\Sigma^p$ which stabilize $\Sigma_p$. First observe that $\Sigma_p$ can be viewed as the interconnection $\Sigma_{\text{full}} \land \Sigma_{\text{full},p}$ where

$$B_{\text{full},p} := \{(w, \tilde{w}) \in l_2^+ \mid \tilde{w} \in B_p\}$$

That is,

$$B_p = \pi_w(B_{\text{full}} \cap B_{\text{full},p})$$

We remark that the number $m(\Sigma_p) = q - p > 0$ corresponds to the dimension of the input space in any input-output representation of $\Sigma_p$.

Theorem 5.6 Suppose that $\Sigma_{\text{int}}$ is an internally stable feedback interconnection with behavior $B_{\text{int}} = B(\Theta)$. Then the following statements are equivalent.

1. $\Sigma_p \land \Sigma_c$ is an internally stable feedback interconnection.
2. $\Sigma_p \land \Sigma_c$ is an internally stable feedback interconnection.
3. $\Lambda := [\Lambda_p \Lambda_c]^T$ is non-singular and $\Lambda^{-1} \in H_\infty$.
The Laplace transform \( \hat{B}_p \) of \( B_p \) is given by
\[
\hat{B}_p = \{ \hat{w} \in \mathcal{H}_2^+ | \Pi_+ \Lambda_p \Theta \hat{w} = 0 \}
\]
\[
= \{ \hat{w} \in \mathcal{H}_2^+ | < \hat{w}, \Theta^* \Lambda_p^* \hat{v} >= 0 \forall \hat{v} \in \mathcal{H}_2^+ \}
\]
\[
= \{ \hat{w} \in \mathcal{H}_2^+ | \hat{w} \in [\Theta^* \Lambda_p^* \mathcal{H}_2^+]^{-1} \}
\]
\[
= [\text{im } \Theta^* \Lambda_p^*]^{-1}
\]

where \( \Theta^* \in \mathcal{H}_\infty \) denotes the dual of \( \Theta \in \mathcal{H}_\infty \). Therefore, the behavior of \( \Sigma_p \) can equivalently be represented as the orthogonal complement (in \( \mathcal{H}_2^+ \)) of the \( \mathcal{H}_2^+ \) image of \( \Theta^* \Lambda_p^* \).

The following result provides a characterization of stabilizing systems for \( \Sigma_p \).

**Theorem 5.7** The following are equivalent

1. \( \Sigma_c \) stabilizes \( \Sigma_p \).
2. \( \hat{B}_p^+ + B_c^+ = \mathcal{H}_2^+ \)
3. \( B_c = B(\Lambda_c \Theta) \) where \( \Lambda := \begin{pmatrix} \Lambda_c \\ \Lambda_p \end{pmatrix} \) is a non-singular \( q \times q \) matrix and \( (\Lambda \Theta)^{-1} \in \mathcal{H}_\infty \).

The above Theorem 5.7 provides a characterization of all stabilizing controllers of the linear time-invariant system \( \Sigma_p \). Note that input/output representations of \( \Sigma_p \) and/or \( \Sigma_c \) are not necessary to provide such a characterization.

**References**


