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by

Willem Haemers and Mohan Shrikhande
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1. Introduction

Let $D(v,k,\lambda)$ be a symmetric design containing a symmetric design $D_1(v_1,k_1,\lambda_1)$ ($k_1 < k$). We call $D_1$ a subdesign of $D$. Assume

$$D = \begin{bmatrix} D_1 & R \\ S & T \end{bmatrix}$$

and let $x = \frac{(k - k_1)v_1}{v - v_1}$. We show that $k \geq (k_1 - x)^2 + \lambda$ (theorem 1). If equality holds, $D_1$ is called a tight subdesign of $D$. In the special case $\lambda_1 = \lambda$, our inequality reduces to that of R.C. Bose and S.S. Shrikhande [3] and tight subdesigns then correspond to their notion of Baer subdesigns. We give examples of tight subdesigns. We divide the possibilities for $(v,k,\lambda)$ designs, having Baer subdesigns into three cases, and give examples for each case.

2. Main results

Theorem 1. Let $D(v_1,k_1,\lambda_1)$ be a subdesign of $D(v,k,\lambda)$. Let $x = \frac{(k - k_1)v_1}{v - v_1}$. Then $k \geq (k_1 - x)^2 + \lambda$.

Proof. Let $D = \begin{bmatrix} D_1 & R \\ S & T \end{bmatrix}$. Then $x = \text{average row sum of } S$. Form

$$A = \begin{bmatrix} \mathbf{0} & D_1 & R \\ \mathbf{0} & S & T \\ D^t & S^t & 0 & 0 \\ R^t & T^t & 0 & 0 \end{bmatrix}$$

where $D^t$ denotes the transpose of $D$. Next, we construct the matrix $M$ consisting of the average row sums of $A$ corresponding to the given blocking. Then,
The eigenvalues of \( \begin{pmatrix} k & k-k_1 \\ k_1 & k-k_1 \\ k-x & k-x \\ x & k-x \end{pmatrix} \) are \( k \) and \( k_1-x \). Hence the eigenvalues of \( M \) are \( \pm k \) and \( \pm (k_1-x) \). The eigenvalues of \( A \) are \( \pm k \) and \( \pm \sqrt{k-\lambda} \). Using the result of [9] on the interlacing of the eigenvalues of \( M \) and \( A \), then gives \( \sqrt{k-\lambda} \geq (k_1-x) \). This yields \( k \geq (k_1-x)^2 + \lambda \).

Remark. It can be proved that if \( k = (k_1-x)^2 + \lambda \), then \( S \) has constant row sums.

Definition. \( D_1(v_1,k_1,\lambda_1) \) is a tight subdesign of \( D(v,k,\lambda) \) if \( k = (k_1-x)^2 + \lambda \).

Corollary ([3] or [10]). Let \( D_1(v_1,k_1,\lambda_1) \) be a subdesign of \( D(v,k,\lambda) \). Then \( k \geq (k_1-1)^2 + \lambda \).

This follows immediately upon noting that in this case \( x \geq 1 \).

If \( D_1(v_1,k_1,\lambda) \) is a subdesign of \( D(v,k,\lambda) \) and \( k = (k_1-1)^2 + \lambda \), then \( D_1 \) is called a Baer subdesign of \( D \) ([3]). For \( \lambda = 1 \), Baer subdesigns are just Baer subplanes of projective planes. In this case many things have been investigated [6].

Example 1. Let \( D \) be the design formed by the points and hyperplanes of \( PG(n,q), \) \( n > 3 \). Let \( X \) and \( Y \) be \( m \) and \( n-m-1 \) dimensional subspaces of \( PG(n,q) \), respectively, which do not have a point in common. The points of \( X \) and the hyperplanes containing \( Y \) form a subdesign of \( D \). This subdesign is not tight.

Example 2. Let \( H_1 \) be a regular Hadamard matrix of size \( 4n^2 \). Then \( H_1 \) is equivalent to a symmetric design \( D_1(4n^2,n(2n-1),n(n-1)) \). Put

\[
H = \begin{pmatrix}
H_1 & -H_1 & -H_1 & -H_1 \\
-H_1 & H_1 & -H_1 & -H_1 \\
-H_1 & -H_1 & H_1 & -H_1 \\
-H_1 & -H_1 & -H_1 & H_1 
\end{pmatrix}.
\]
Then $H$ is a regular Hadamard matrix of size $16n^2$ and is equivalent to a symmetric design $D(16n^2, 2n(4n+1), 2n(2n+1))$. It is easily checked that $D_1$ is a tight subdesign of $D$. For examples of regular Hadamard matrices cf. [8].

**Remarks.** Let $D_1(v_1, k_1, \lambda_1)$ be a tight subdesign of $D(v, k, \lambda)$. Then

(i) $k - \lambda$ is a square.

(ii) The complement of $D_1$ is a tight subdesign of the complement of $D$.

Using $\overline{D}$ to denote the complement of $D$, we then have

**Example 3.** Let $D_1(v_1, k_1, 1)$ be a Baer subplane of $D(v, k, 1)$. Then $\overline{D}_1(v_1, v_1 - k_1, v_1 - 2k_1 + 1)$ is a tight subdesign of $\overline{D}(v, v-k, v-2k+1)$.

**Theorem 2.** Let $D_1(v_1, k_1, \lambda)$ be a Baer subdesign of $D(v, k, \lambda)$. Then, one of the following holds:

a) $v = \lambda(\lambda^2 - 2\lambda + 2)$, $D$ has parameters $(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ and $D_1$ is the trivial design $(\lambda, \lambda, \lambda)$.

b) $v = \lambda^2(\lambda + 2)$, $D$ has parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ and $D_1$ is the trivial design $(\lambda + 2, \lambda + 1, \lambda)$.

c) $v > \lambda^2(\lambda + 2)$.

**Proof.** Let $D$ be a non-trivial design having a Baer subdesign $D_1$. Then $k < v - 1$ or equivalently $\lambda < k - 1$. Since $D_1$ is a Baer subdesign of $D$, we have

$$x = \frac{(k - k_1)v_1}{(v - v_1)} = 1.$$  

This gives

(1) \hspace{1cm} v = v_1(k - k_1 + 1) \\

and

(2) \hspace{1cm} k = (k_1 - 1)^2 + \lambda.

If $D_1$ is trivial then $v_1 = k_1 = \lambda_1$ or $v_1 = k_1 + 1 = \lambda_1 + 2$. Using (1) and (2) we see that these two trivial cases lead to (a) and (b), respectively. If $v_1 > k_1 + 1$ then (1) and (2) give

$$v > (k_1 + 1)((k_1 - 1)(k_1 - 2) + \lambda).$$

Using $k_1 > \lambda + 1$ we obtain $v > \lambda^2(\lambda + 2)$. \qed
We now give examples to show that in each of the above cases, there exist symmetric designs with Baer subdesigns.

Example 4. A symmetric design $D(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ has the parameters of the symmetric design of the points and planes of $\text{PG}(3, \lambda - 1)$ which exist for all prime powers $\lambda - 1$. Moreover the points on a given line and all planes containing it form a Baer subdesign $D_1(\lambda, \lambda, \lambda)$.

Example 5. From Ahrens and Szekeres [1], the existence of symmetric designs $D$ with parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ is known for all prime powers $\lambda$. From their construction it can be easily seen that $D$ has a Baer subdesign $D_1(\lambda + 2, \lambda + 1, \lambda)$, corresponding to a clique of size $\lambda + 2$ in the corresponding graph.

Before giving an example to show the existence of a design satisfying (c) of theorem 2, we make some observations:

If we consider designs $(v,k,\lambda)$ with $v > \lambda^2(\lambda + 2)$, then according to [5], p. 105 the only known examples are projective planes of prime power order and biplanes (= symmetric designs with $\lambda = 2$) on 37, 56 and 79 points; as far as we know meanwhile one other example is found, a $(71, 15, 3)$ design, see [2].

Note that if $D_1(v_1, k_1, \lambda)$ is a Baer subdesign of $D(v, k, \lambda)$, then $v$ cannot be prime. Thus if we are to find a Baer subdesign $D_1(v_1, k_1, \lambda)$ of $D(v, k, \lambda)$ which is not a Baer subplane, it is easily seen from above that $D(56, 11, 2)$ is the only possible candidate. Any Baer subdesign of $D$ has parameters $(7,4,2)$, whose of the complement of the Fano plane $(7,3,1)$. The next example shows that there is a $(56, 11, 2)$ design with a Baer subdesign.

Example 6. We follow [7] Denniston who gives constructions of $(56, 11, 2)$ designs some of which are based on Cameron's description [4] of biplanes. Namely, one block $b^*$ is fixed and all the other blocks are in 1-1 correspondence with the unordered pairs of points of $b^*$. Each point not on $b^*$ is represented by a disjoint union of polygons on the points of $b^*$. The block represented by $\{p, q\}$ is incident with $p$ and $q$ and with the points represented by graphs in which $p$ and $q$ are joined.
Let us represent the points of $b^*$ by $0,\ldots,10$ then according to [7] in at least two of the constructed biplanes (the "nice" one due to Gewirtz, Hall, Lane and Wales, and another design due to Assmus and others), there exist three points off $b^*$ whose polygons are

\[
\begin{align*}
(9 & 8 10 9) (0 2 4 6 0) (1 3 5 7 1) \\
(0 & 4 10 0) (9 2 8 6 9) (1 3 7 5 1) \\
(2 & 6 10 2) (9 4 8 0 9) (1 7 3 5 1).
\end{align*}
\]

It is easily seen that these 3 points together with the points 1,3,5 and 7 from $b^*$ form a $(7,4,2)$ design which is a Baer subdesign of the $(56,11,2)$ design.

Using the above example and remark ii) we have

**Example 7.** There exists a $D(56,45,36)$ which has the Fano plane ($=(7,3,1)$ design) as a tight subdesign.

References


