Gelfand-Shilov spaces of infinitely differentiable functions on the positive real line
van Berkel, C.A.M.; van Eijndhoven, S.J.L.

Published: 01/01/1992

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
RANA 92–07
April 1992
GELFAND-SHILOV SPACES OF
INFINITELY DIFFERENTIABLE
FUNCTIONS ON THE POSITIVE
REAL LINE
by
C.A.M. van Berkel
S.J.L. van Eijndhoven
Gelfand-Shilov spaces of infinitely differentiable functions on the positive real line

by

C.A.M. van Berkel and S.J.L. van Eijndhoven

Abstract

In this paper a number of characterizations of the spaces $S_{a,\text{even}}$, $S_{b,\text{even}}$ and $S_{c,\text{even}}$ consisting of all even functions in the Gelfand-Shilov spaces $S_{a}$, $S_{b}$ and $S_{c}$, respectively, is presented. Introducing $G_{a}$, $G_{b}$ and $G_{c}$ as the spaces of all $C^{\infty}$ functions $f$ on $\mathbb{R}^{+}$ with the property that $x \mapsto f(x^{2})$ belongs to $S_{a,\text{even}}$, $S_{b,\text{even}}$ and $S_{c,\text{even}}$, each of the mentioned characterizations leads to a description of the functions $f$ in $G_{a}$, $G_{b}$ and $G_{c}$ in terms of growth conditions on $f$ and its derivatives $f^{(l)}$, $l \in \mathbb{N}$. As such the paper yields refinements and extensions of results in a recent paper by Duran [Du1].

AMS Classifications 46F12, 46F10, 26F05
Introduction

The Gelfand-Shilov spaces have been subject of investigation by a lot of mathematicians in quite different fields. We mention Gong-Zhing Zhang [Zh] who showed the connection between the spaces $S_{\alpha}^\beta$, $\alpha \geq \frac{1}{2}$, and Hermite expansions, Kashpirovskii [Ka], who proved the intersection result $S_{\alpha}^\beta = S_{\alpha} \cap S^\beta$, De Bruijn [Br], who employed the space $S_{1/2}^{1/2}$ as a test space for a theory of generalised functions and Goodman [Go], who showed that the spaces $S_{1/k}^{1/k}$ and $S_{1/k}^{1/2}$ are analyticity domains of unitary representations on $L_2(\mathbb{R})$ of certain nilpotent Lie groups. More recently, Van Eijndhoven [E1] characterized the Gelfand-Shilov spaces as intersections of analyticity domains of the operators $|Q|^{1/\alpha}$ and $|P|^{1/\beta}$ where $Q$ denotes the self-adjoint operator of multiplication by $x$ in $L_2(\mathbb{R})$ and $P$ the self-adjoint differentiation operator $id/dx$ in $L_2(\mathbb{R})$, thus extending Goodman’s results.

Furthermore, Ter Elst and Van Eijndhoven [EE] proved that the spaces $S_{\alpha}^{\beta k/k+1}$ are the Gevrey domains of certain symmetric differential operators in $L_2(\mathbb{R})$.

For $f \in S_{\alpha}^\beta$ its Fourier transform $\mathcal{F}f$ belongs to $S_{\beta}^\alpha$. So the spaces $S_{\alpha}^\beta$ are Fourier invariant and therefore suitable as test function spaces in distribution theories involving the Fourier transformation. The spaces $S_{\alpha,even}^\beta$, $S_{even}^\beta$ and $S_{\alpha,even}^\beta$ play a similar role when dealing with the Hankel transformations. In [EG] it was proved that $S_{\alpha,even}^\beta$ for $\frac{1}{2} \leq \alpha < 1$ remains invariant under some modified form of the usual Hankel transformations as defined in [Mos], p. 397. An extended version of this result can be found in [Du2] and in [E2].

In [EB] a characterization of the spaces $S_{\alpha,even}^\beta$, $S_{even}^\beta$ and $S_{\alpha,even}^\beta$ in terms of the growth behaviour of a $C^\infty$-function on $\mathbb{R}^+$ and of its Hankel transforms was proved. It turned out that the Hankel transformations are bijections from $S_{\alpha,even}^\beta$ onto $S_{even}^\alpha$ and from $S_{\alpha,even}^\beta$ onto $S_{\alpha,even}^\beta$. A characterization of these spaces by means of the operators $x$ and $\frac{1}{z} \frac{d}{dz}$ in $L_2(\mathbb{R}^+)$ was proved by [Du1].

Duran’s paper is partly based on results in [EB]. However these results do not seem fully exploited. Moreover Duran’s characterizing conditions can be somewhat weakened and put in a context closer to the results in [EB] by employing properties of the Hankel transformations as stated therein.

The paper is divided into three sections. In Section 1 we introduce some notation and present some auxiliary results. In Section 2 characterizations of the Schwartz space $S^+$ of $C^\infty$-functions on $\mathbb{R}^+$ and the space $S_{even}$ of all even functions in the Schwartz space $S$ of $C^\infty$ functions on $\mathbb{R}$. In Section 3 we consider the spaces $S_{\alpha,even}^\beta$, $S_{even}^\beta$ and $S_{\alpha,even}^\beta$ and we present a number of characterizations of these spaces. Besides we introduce and describe the spaces $G_{\alpha}$, $G_{\beta}$ and $G_{\alpha}^\beta$ (notation in accordance with Duran’s).
1. Notation and auxiliary results

The test space in the theory of tempered distributions is the space $\mathcal{S}$ of all $C^\infty_0$ functions on $\mathbb{R}$ of rapid decrease. So an infinitely differentiable function $f$ on $\mathbb{R}$ belongs to $\mathcal{S}$ if for all $k, l \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} |x^k f^{(l)}(x)| < \infty.$$ 

In order to get a feel for the strength of the theory of tempered distributions it is helpful to have several characterizations of $\mathcal{S}$. The characterizations (1.1), (1.2) and (1.3) we present here, are known, cf. [Go], [Jo], [Si], [El], or can be derived easily from the mentioned literature.

(1.1) Let $f$ be an infinitely differentiable function on $\mathbb{R}$.

- $f \in \mathcal{S}$ iff $x^k f \in L_2(\mathbb{R})$ and $f^{(l)} \in L_2(\mathbb{R})$ for all $k, l \in \mathbb{N}_0$;
- $f \in \mathcal{S}$ iff $x^k f \in L_2(\mathbb{R})$ and $f^{(l)} \in L_\infty(\mathbb{R})$ for all $k, l \in \mathbb{N}_0$.

In the above characterizations, growth conditions and differentiability conditions occur separately.

Let $\mathcal{F}$ denote the Fourier transformation on $L^2(\mathbb{R})$, 

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ixy} dy.$$ 

Then $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$ and $\mathcal{F}$ diagonalizes the differentiation operator, 

$$\mathcal{F}f' = ix \mathcal{F}f.$$ 

With the Fourier transformation, $\mathcal{S}$ admits the following characterization.

(1.2) Let $f$ be a square integrable function on $\mathbb{R}$.

- $f \in \mathcal{S}$ iff $x^k f \in L_2(\mathbb{R})$ and $x^l \mathcal{F}f \in L_2(\mathbb{R})$ for all $k, l \in \mathbb{N}_0$;
- $f \in \mathcal{S}$ iff $x^k f \in L_\infty(\mathbb{R})$ and $x^l \mathcal{F}f \in L_\infty(\mathbb{R})$ for all $k, l \in \mathbb{N}_0$.

Let $Q$ denote the self-adjoint operator of multiplication by the identity function in $L_2(\mathbb{R})$ with maximal domain $D(Q)$ and let $\mathcal{P} = \mathcal{F}Q\mathcal{F}^*$ denote the differentiation operator in $L_2(\mathbb{R})$ with $D(\mathcal{P}) = \mathcal{F}(D(Q))$. Then characterization (1.2) says
\[ S = D^\infty(Q) \cap D^\infty(P) \]

where

\[ D^\infty(Q) = \bigcap_{n=1}^{\infty} D(Q^n) \quad \text{and} \quad D^\infty(P) = \bigcap_{n=1}^{\infty} D(P^n) . \]

In the third characterization of \( S \) the Hermite basis in \( L_2(\mathbb{R}) \) is used. For \( n \in \mathbb{N}_0 \) the Hermite functions \( \psi_n \) are defined by

\[ \psi_n(x) = \frac{(-1)^n}{\sqrt{\pi^n n! 2^n}} e^{\frac{1}{2}x^2} \left( \frac{d}{dx} \right)^n (e^{-x^2}) . \]

They form an orthonormal basis in \( L_2(\mathbb{R}) \). The following characterization of \( S \) was proved:

\( (1.3) \) Let \( f \) be a square integrable function on \( \mathbb{R} \).

\[ - f \in S \iff \sup_{n \in \mathbb{N}_0} n^k |(f, \psi_n)_{L_2(\mathbb{R})}| < \infty \text{ for all } k \in \mathbb{N}_0. \]

The Hermite functions \( \psi_n \) satisfy the differential equation \( \left( - \frac{d^2}{dx^2} + x^2 \right) \psi_n = (2n + 1) \psi_n, \ n \in \mathbb{N}_0 \), so that characterization (1.3) can be reformulated as

\[ S = D^\infty(P^2 + Q^2) . \]

As we see from characterization (1.2), \( S \) is Fourier invariant. So its dual space \( S' \), i.e. the space of tempered distributions, is Fourier invariant, too.

From characterization (1.1) it follows that each tempered distribution can be written as the sum of a regular and an irregular distribution: For each \( \Phi \in S' \) there exist bounded functions \( g_1, g_2 \) on \( \mathbb{R} \) and \( k, l \in \mathbb{N} \) such that

\[ \Phi(f) = \int_{-\infty}^{\infty} x^k f(x) \, g_1(x) \, dx + \int_{-\infty}^{\infty} f^{(l)}(x) \, g_2(x) \, dx \ , \ f \in S . \]

From characterization (1.3) it follows that each tempered distribution can be represented by a Hermite expansion: For each \( \Phi \in S' \),

\[ \Phi = \sum_{n=0}^{\infty} \overline{\Phi(\psi_n)} \psi_n \]

to be understood as
\[ \Phi(f) = \sum_{n=0}^{\infty} (f, \psi_n)_{L^2(\mathbb{R})} \Phi(\psi_n), \quad f \in S. \]

In the paper [EG1] the space \( S_{\text{even}} \) consisting of all even functions \( f \) in \( S \), was studied, and the following description of \( S_{\text{even}} \) was derived.

(1.4) The even extension of an infinitely differentiable function \( f \) on \( \mathbb{R}^+ \) belongs to \( S \) iff for all \( k, l \in \mathbb{N}_0 \),

\[
\sup_{x > 0} |x^k \left( \frac{1}{x} \frac{d}{dx} \right)^l f(x)| < \infty.
\]

Another characterization was obtained via the Laguerre basis \((L_n^\nu)_{n \in \mathbb{N}_0} \) which is orthonormal in the Hilbert space \( X_\nu = L^2(\mathbb{R}^+, x^\nu + x) dx \) where \( \nu \geq -\frac{1}{2} \) arbitrarily. Therefore, define

\[
\mathcal{L}_n^\nu(x) = \left( \frac{2 \Gamma(n + 1)}{\Gamma(n + \nu + 1)} \right)^{1/2} x^{-\frac{3}{2}x^2} L_n^\nu(x^2)
\]

with \( L_n^\nu \) the \( n \)-th Laguerre polynomial of order \( \nu \),

\[
L_n^\nu(x) = \frac{1}{n!} x^{-\nu} e^x \left( \frac{d}{dx} \right)^n \left( x^{\nu+1} e^{-x} \right).
\]

For \( \nu = -1/2 \) we have

\[
\mathcal{L}_n^{-\frac{1}{2}} = (-1)^n \sqrt{2} \psi_{2n}, \quad n \in \mathbb{N}_0.
\]

So replacing \( \psi_n \) by \( \mathcal{L}_n^{-\frac{1}{2}} \) in (1.3) we obtain a characterization of \( S_{\text{even}} \) in terms of expansions with respect to the basis \((\mathcal{L}_n^{-\frac{1}{2}})\). Since for \( \alpha, \beta \geq -\frac{1}{2} \),

\[
L_n^\alpha = \sum_{m=0}^{n} \frac{(\alpha - \beta)m}{m!} L_n^{\beta-m},
\]

as a quite natural consequence we have, cf. [E2]:

(1.5) Let \( \nu \geq -\frac{1}{2} \). The even extension of \( f \in X_\nu \) belongs to \( S_{\text{even}} \) iff for all \( k \in \mathbb{N}_0 \),

\[
\sup_{n \in \mathbb{N}_0} n^k |(f, L_n^\nu)_{X_\nu}| < \infty.
\]

With \( J_\nu \), the \( \nu \)-th order Bessel function, introducing the Hankel transformation on \( X_\nu \) by
\[
(H_\nu f)(x) = \int_0^\infty (xy)^{-\nu} J_\nu(xy) f(y) y^{2\nu+1} dy, \quad f \in X_\nu,
\]

we have

\[
H_\nu H_n^\nu = (-1)^n H_n^\nu,
\]

cf. [MOS], p. 244. So \(H_\nu\) is a unitary operator on \(X_\nu\) and, by (1.5). \(S_{\text{even}}\) remains invariant under \(H_\nu\) for each \(\nu \geq -\frac{1}{2}\). (Observe that \(H_{-\frac{1}{2}}\) is the Fourier-cosine-transformation.) A characterization of \(S_{\text{even}}\) in terms of Hankel transforms was proved in [EB]:

(1.6) Let \(\nu \geq -\frac{1}{2}\). The even extension of a measurable function \(f\) on \(\mathbb{R}^+\) belongs to \(S\) iff for all \(k, l \in \mathbb{N}_0\),

\[
x^k f \in L_\infty(\mathbb{R}^+) \quad \text{and} \quad x^l H_\nu f \in L_\infty(\mathbb{R}^+) .
\]

For \(\nu = -\frac{1}{2}\) this characterization is a reformulation of (1.3). For future use we shall reprove (1.6) with \(L_\infty(\mathbb{R}^+)\) replaced by \(X_\nu\). First we present some auxiliaries.

(1.7) Lemma

Let \(\nu \geq -\frac{1}{2}\) and let \(f \in X_\nu\) such that also \(x^{\nu+3/2} f \in X_\nu\). Then \(f \in L_1(\mathbb{R}^+, x^{2\nu+1} \, dx)\) and \(H_\nu f\) is continuous.

Proof

Using Cauchy’s inequality,

\[
\int_0^\infty |f(y)| y^{2\nu+1} dy \leq \left( \int_0^\infty (1 + y) |y^{2\nu+1} f(y)|^2 dy \right)^{1/2} \left( \int_0^\infty \frac{1}{(1 + y)^2} dy \right)^{1/2}
\]

\[
\leq \|f\|_{X_\nu} + 2 \|x^{\nu+3/2} f\|_{X_\nu},
\]

so that \(f \in L_1(\mathbb{R}^+, x^{2\nu+1} \, dx)\).

Since \(z^{1/2} J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} \cos z\) and since for \(\nu \geq -\frac{1}{2}\),

\[
z^{-\nu} J_\nu(z) = \frac{2}{\pi^{1/2}} \frac{1}{\Gamma(\nu + 1/2)} (\frac{1}{2})^\nu \int_0^1 (1 - t^2)^{-\nu-\frac{1}{2}} \cos(zt) dt,
\]

cf. [MOS, p. 79], we have sup \(z^{-\nu} J_\nu(z)\) < \(\infty\), and so the integral

\[
\int_0^\infty (xy)^{-\nu} J_\nu(xy) f(y) y^{2\nu+1} dy
\]
converges absolutely and uniformly on $\mathbb{R}^+$. $\square$

From [EB, Proposition (1.3)] we take the following result:

(1.8) **Proposition**

Let $-\frac{1}{2} \leq \mu < \nu$ and let $f \in L_2(\mathbb{R}^+, (1+t^2)^\alpha (1+t)^{-1} \, dt)$ with $\alpha > \nu + 1$. Then $H_\nu f \in X_\mu$ and for almost all $x > 0$,

$$H_\mu (H_\nu f) (x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu - \mu)} \int_0^\infty f(\xi) \left( \xi^2 - x^2 \right)^{\nu-\mu-1} \xi \, d\xi .$$

$\square$

We come to the following adaption of Lemma (1.5) in [EB].

(1.9) **Lemma**

Let $-\frac{1}{2} \leq \mu < \nu$ and let $W(x)$ be a nonnegative function on $\mathbb{R}^+$ such that $(1-x)^{-\nu+1} W(x)$ is nondecreasing on $[b, \infty)$ for some $b > 0$. Let $f \in X_\nu$ such that

$$x^k f \in X_\nu \quad \text{and} \quad x^l H_\nu f \in X_\nu ,$$

for all $k, l \in \mathbb{N}_0$ and suppose in addition that $W(x) H_\nu f \in X_\nu$. Then

$$(1+x)^{-\nu+1} W(x) H_\mu f \in X_\mu .$$

**Proof**

Let $\mu \in \mathbb{R}$ with $-\frac{1}{2} \leq \mu < \nu$. Since $x^l H_\nu f \in X_\nu$ for all $l \in \mathbb{N}_0$, it follows from Lemma (1.7), that $f = H_\nu (H_\nu f)$ is continuous on $\mathbb{R}^+$. Continuity of $f$ together with the fact that $x^k f \in X_\nu$ for all $k \in \mathbb{N}_0$, yields $x^k f \in X_\mu$ for all $k \in \mathbb{N}_0$. Thus we obtain continuity of $H_\mu f$ on $\mathbb{R}^+$ by Lemma (1.7).

We conclude that

$$(1+x)^{-\nu+1} W(x) H_\mu f \in L_2([0, b], x^{2\nu+1} \, dx) .$$

Furthermore, since $x^l H_\nu f \in X_\nu$ for all $l \in \mathbb{N}_0$,

$$H_\nu f \in L_2(\mathbb{R}^+, (1+y^2)^{\nu+2} y^{2\nu+1} \, dy)$$

and so replacing $f$ by $H_\nu f$ in Proposition (1.8),

$$H_\mu (H_\nu f) (x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu - \mu)} \int_0^\infty \left( \xi^2 - x^2 \right)^{\nu-\mu-1} \xi \, d\xi .$$
The proof is completed by the following estimations:

\[
\left( \frac{\Gamma(\nu - \mu)}{2^{\nu-\mu+1}} \right)^2 \int_b^\infty |(1 + x)^{-(\nu-\mu+1)} W(x) (\mathcal{H}_\nu f) (x)|^2 x^{2\nu+1} \, dx
\]

\[
= \int_b^\infty (1 + x)^{-(\nu-\mu+1)} W(x) \int_0^\infty (\xi^2 - x^2)^{\nu-\mu-1} (\mathcal{H}_\nu f) (\xi) \xi \, d\xi |x|^{2\nu+1} \, dx
\]

\[
\leq \int_b^\infty \int_0^\infty (1 + (1 + \xi)^{-(\nu-\mu+1)} W(\xi) (\xi^2 - x^2)^{\nu-\mu-1} (\mathcal{H}_\nu f) (\xi) \xi \, d\xi |x|^{2\nu+1} \, dx
\]

\[
\leq \int_b^\infty \int_0^\infty (1 + (1 + \xi)^{-(\nu-\mu+1)} W(\xi) (\xi^2 - x^2)^{\nu-\mu-1} (\mathcal{H}_\nu f) (\xi)^2 (\xi^2 - x^2)^{\nu-\mu-1} \xi^{2\nu+3} \, d\xi)
\]

\[
\cdot \int_0^\infty (\xi^2 - x^2)^{\nu-\mu-1} \xi^{-2\nu-1} \, dx x^{2\nu+1} \, dx
\]

\[
= \frac{1}{2} B(\nu - \mu, \mu + 1) \int_b^\infty x^{-1}(\int_0^\infty (1 + (1 + \xi)^{-(\nu-\mu+1)} W(\xi) (\mathcal{H}_\nu f) (\xi)^2 \xi^{2\nu+3} \, d\xi)
\]

\[
\cdot (\xi^2 - x^2)^{\nu-\mu-1} \xi^{-1} \, dx
\]

\[
= \frac{1}{2} B(\nu - \mu, \mu + 1) \int_b^\infty |(1 + (1 + \xi)^{-(\nu-\mu+1)} W(\xi) (\mathcal{H}_\nu f) (\xi)|^2 \xi^{2\nu+3} \, d\xi
\]

\[
\cdot (\xi^2 - x^2)^{\nu-\mu-1} \xi^{-1} \, dx
\]

\[
\leq \frac{B(\nu - \mu, \mu + 1)}{4 b^2 (\nu - \mu)} \int_b^\infty |W(\xi) (\mathcal{H}_\nu f) (\xi)|^2 \xi^{2\nu+1} \, d\xi \quad \Box
\]

As a consequence of Lemma (1.9) there is the following extension of characterization (1.6).

(1.10) Theorem
Let \( \nu \geq -\frac{1}{2} \) and let \( f \in X_\nu \). Then the even extension of \( f \) belongs to \( S \) iff for all \( k, l \in \mathbb{N}_0 \),
\[ x^k f \in X_\nu \quad \text{and} \quad x^l H_\nu f \in X_\nu. \]

**Proof**

Let the even extension of \( f \) belong to \( S \). Then by (1.6),

\[ x^k f \in L_\infty(\mathbb{R}) \quad \text{and} \quad x^l H_\nu f \in L_\infty(\mathbb{R}), \]

for all \( k, l \in \mathbb{N}_0 \), from which the necessity of the condition follows. For sufficiency let for all \( k, l \in \mathbb{N}_0 \), \( x^k f \in X_\nu \) and \( x^l H_\nu f \in X_\nu \). If \( \nu = -\frac{1}{2} \), characterization (1.2) yields the wanted result (Recall that for an even function \( f \in L_2(\mathbb{R}) \), \( (Ff)|_{\mathbb{R}^+} = H_{-\frac{1}{2}}(f|_{\mathbb{R}^+}) \)).

If \( \nu > -\frac{1}{2} \) note first that from the fact that \( x^l H_\nu f \in X_\nu \), \( l \in \mathbb{N}_0 \), it follows that \( f \) is continuous and therefore that \( x^k f \in X_{-\frac{1}{2}} \) for all \( k \in \mathbb{N}_0 \), from the fact that \( x^k f \in X_\nu \), \( k \in \mathbb{N}_0 \). Applying Lemma (1.9) with \( W(x) = x^p, p > \nu + 3/2 \) and \( \mu = -\frac{1}{2} \) yields

\[(1 + x)^{-\left(\nu + \frac{3}{2}\right)} x^p H_{-\frac{1}{2}} f \in X_{-\frac{1}{2}}.\]

So with \( \rho = l + \nu + 3/2 \), \( l \in \mathbb{N}_0 \),

\[ x^l H_{-\frac{1}{2}} f \in X_{-\frac{1}{2}}. \]

For \( \mu, \nu \geq -\frac{1}{2} \) the linear mappings \( H_\mu H_\nu \) are bijective on \( S_{\text{even}} \). By Proposition (1.8) for \( f \in S_{\text{even}} \) and \( -\frac{1}{2} \leq \mu < \nu \)

\[(H_\mu H_\nu f)(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} \int_\mathbb{R} (\xi^2 - x^2)^{\nu-\mu-1} f(\xi) \xi \, d\xi.\]

We see that for \( \nu \geq -\frac{1}{2} \) and \( k \in \mathbb{N}_0 \),

\[ H_{\nu+k} H_\nu = \left(-\frac{1}{x} \frac{d}{dx}\right)^k.\]

The operator \( H_{\nu+\alpha} H_\nu, \nu \geq -\frac{1}{2}, \alpha \geq -\nu - \frac{1}{2} \) interprets the fractional differentiation \( \left(-\frac{1}{x} \frac{d}{dx}\right)^\alpha \).
2. The spaces $S^+$ and $S_{\text{even}}$

(2.1) DEFINITION
The space $S^+$ consists of all infinitely differentiable functions on $\mathbb{R}^+$ with the property that for all $k, l \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}^+} |x^k f^{(l)}(x)| < \infty .$$

There exist relations between the space $S^+$ and the spaces $S$ and $S_{\text{even}}$ as indicated in the next theorem.

(2.2) THEOREM
Let $f$ be an infinitely differentiable function on $\mathbb{R}^+$.

(i) $f \in S^+$ iff there exists $g \in S$ such that $f = g|_{\mathbb{R}^+}$ ,

(ii) $f \in S^+$ iff the function $h : x \mapsto f(x^2)$ belongs to $S_{\text{even}}$.

Proof

(i) Let $f \in S^+$. From Borel's theorem, see [Zu, Chapter 1, Exercise 1], the existence of a function $\varphi$ on $\mathbb{R}$ with compact support can be derived such that

$$\varphi^{(l)}(0) = \lim_{x \to 0} f^{(l)}(x) .$$

Now define $g$ on $\mathbb{R}$ by

$$g(x) = \begin{cases} f(x) , & x > 0 , \\ \varphi(x) , & x \leq 0 . \end{cases}$$

Sufficiency of the condition is trivial.

(ii) Let $h$ on $\mathbb{R}$ be defined by $h(x) = f(x^2)$, $x \in \mathbb{R}$. Then for all $l \in \mathbb{N}_0$,

$$f^{(l)}(x^2) = \left( \left( \frac{1}{2x} \frac{d}{dx} \right)^l h \right)(x) .$$

and so the statement follows from characterization (1.4) of $S_{\text{even}}$. \hfill \Box

The connections between $S^+$, $S$ and $S_{\text{even}}$, as just mentioned and the characterizations of $S$ and $S_{\text{even}}$ yield characterizations of $S^+$ accordingly. For this, introduce the Hilbert space $Y_\nu = L_2(\mathbb{R}^+, x^\nu dx)$, the orthonormal basis $(\mathcal{L}_n^\nu)_{n \in \mathbb{N}_0}$ in $Y_\nu$, 9
\[ \mathcal{L}_n^{\nu}(x) = \frac{1}{\sqrt{2}} \mathcal{H}_n^{\nu}(\sqrt{x}) = \left( \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \right)^{1/2} e^{-\frac{x}{2}} L_n^{\nu}(x), \]

and the Hankel-Clifford transformation \( \mathcal{H}_\nu \) on \( Y_\nu \),

\[ (\mathcal{H}_\nu f)(x) = \frac{1}{2} \int_0^\infty (xy)^{-\frac{1}{2} \nu} J_\nu((xy)^{1/2}) f(y) y^{\nu} dy. \]

Then for \( f \in Y_\nu \), \( h \) defined by \( h(x) = f(x^2), x > 0 \), belongs to \( X_\nu \) with

\[ (\mathcal{H}_\nu f)(x^2) = (\mathcal{H}_\nu h)(x) \text{ and } \sqrt{2} \|f\|_{Y_\nu} = \|h\|_{X_\nu}, \]

so that

\[ \mathcal{H}_\nu \mathcal{L}_n^{\nu} = (-1)^n \mathcal{L}_n^{\nu}. \]

(2.3) Theorem

I. Let \( f \) be infinitely differentiable on \( I\mathbb{R}^+ \).
- \( f \in S^+ \) iff \( x^k f \in L_\infty(I\mathbb{R}^+) \) and \( f^{(l)} \in L_\infty(I\mathbb{R}^+) \) for all \( k, l \in \mathbb{N}_0 \).

II. Let \( n \geq -\frac{1}{2} \) and let \( f \in Y_\nu \).
- \( f \in S^+ \) iff \( x^k f \in Y_\nu \) and \( x^l \mathcal{H}_\nu f \in Y_\nu \) for all \( k, l \in \mathbb{N}_0 \);
- \( f \in S^+ \) iff \( x^k f \in L_\infty(I\mathbb{R}^+) \) and \( x^l \mathcal{H}_\nu f \in L_\infty(I\mathbb{R}^+) \) for all \( k, l \in \mathbb{N}_0 \).

III. Let \( \nu \geq -\frac{1}{2} \) and let \( f \in Y_\nu \).
- \( f \in S^+ \) iff for all \( k \in \mathbb{N}_0 \),

\[ \sup_{n \in \mathbb{N}_0} n^k \|f, \mathcal{L}_n^{\nu}Y_\nu\| < \infty. \]

Proof

I. From the definition of \( S^+ \) necessity of the condition follows. For its sufficiency let \( f \) satisfy \( x^k f \in L_\infty(I\mathbb{R}^+) \) and \( f^{(l)} \in L_\infty(I\mathbb{R}^+) \) for all \( k, l \in \mathbb{N}_0 \). Then there is a function \( \varphi \) on \( I\mathbb{R} \) with compact support such that

\[ \varphi^{(l)}(0) = \lim_{x \to 0} f^{(l)}(x). \]
Define the function $g$ on $\mathbb{R}$ by
\[
g(x) = \begin{cases} 
    f(x), & x > 0, \\
    \varphi(x), & x \leq 0.
\end{cases}
\]

Since $\varphi$ has compact support, $x^k g \in L_\infty(\mathbb{R})$ and $g^{(l)} \in L_\infty(\mathbb{R})$. So by characterization (1.1), $g \in S$ and $f = g|_{\mathbb{R}^+} \in S^+$.

II and III. These assertions are consequences of the relation between $S_{\text{even}}$ and $S^+$ as stated in Theorem (2.2) (ii) and of the characterizations of $S_{\text{even}}$ given in (1.5), (1.6) and in Theorem (1.10).

Remarks

- For $f \in Y_{\nu}$ the function $h : x \mapsto f(x^2)$ belongs to $X_{\nu}$ and $(H_{\nu}h)(x) = (H_{\nu}f)(x^2)$. So by Proposition (1.8), for $f \in S^+$ and $\nu, \mu \geq -\frac{1}{2}$ with $\nu > \mu$,
\[
(H_{\nu}H_{\nu}f)(x) = \frac{2^{\nu-\nu}}{\Gamma(\nu-\mu)} \int_0^\infty (t-x)^{\nu-\mu-1} f(t) \, dt.
\]
Therefore, for each $\nu \geq -\frac{1}{2}$ and $k \in \mathbb{N}_0$,
\[
H_{\nu+k}H_{\nu}f = 2^k \left(-\frac{d}{dx}\right)^k f, \quad f \in S^+.
\]

- The space $S^+$ remains invariant under the Hankel-Clifford transformations $H_{\nu}$, $\nu \geq -\frac{1}{2}$.

- The space $S^+$ is interesting also because of its behaviour with respect to the Laplace transformation. It maps $S^+$ onto the space of all analytic functions on the open right half plane which are infinitely differentiable on the imaginary axis. See [Du2,3].

In the next characterization we connect the space $S^+$ with the operators $x^k$ and $x^{l/2} \left(\frac{d}{dx}\right)^l$.

(2.4) Theorem
Let $\nu \geq -\frac{1}{2}$. For an infinitely differentiable function $g$ on $\mathbb{R}^+$ the following conditions are equivalent

(i) $g \in S^+$;

(ii) $x^{(k+l)/2} g^{(l)} \in L_\infty(\mathbb{R}^+)$ for all $k, l \in \mathbb{N}_0$;
(iii) \( x^{(k+1)/2}g^{(l)} \in Y \) for all \( k, l \in \mathbb{N}_0 \);
(iv) \( x^{k/2}g \in Y \) and \( x^{l/2}g^{(l)} \in Y \) for all \( k, l \in \mathbb{N}_0 \);
(v) \( x^{k/2}g \in L_\infty(\mathbb{R}^+) \) and \( x^{l/2}g^{(l)} \in L_\infty(\mathbb{R}^+) \) for all \( k, l \in \mathbb{N}_0 \).

**Proof**

The implications (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) and (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (v) need no explanation. We prove that (iv) \( \Rightarrow \) (i) and (v) \( \Rightarrow \) (iv). Assume \( g \) satisfies condition (iv).

Define \( f \) on \((-1, \infty)\) by \( f(x) = g(x+1) \). Then \( x^k f \in Y \), \( k \in \mathbb{N}_0 \), since

\[
\int_{-1}^{\infty} |x^k f(x)|^2 \, dx \leq \int_{-1}^{\infty} |(x+1)^{k-1/2}g(x+1)|^2 (x+1)^\nu \, dx \\
\leq \int_{1}^{\infty} |x^{k+1}g(x)|^2 x^\nu \, dx \leq \|x^{k+1}g\|_{Y}^2,
\]

with \( k_\nu = k \) for \( \nu \geq 0 \) and \( k_\nu = k + 1 \) for \(-\frac{1}{2} < \nu < 0\). Further \( f^{(l)} \in Y \) for \( l \in \mathbb{N} \), since

\[
\int_{0}^{\infty} |f^{(l)}(x)|^2 \, dx \leq \int_{0}^{\infty} (x+1)^{l+\nu} |g^{(l)}(x+1)|^2 \, dx \\
\leq \|x^{l/2}g^{(l)}\|_{Y}^2.
\]

Since \( f \) is infinitely differentiable on \((-1, \infty)\) there is an infinitely differentiable function \( \phi \) on \( \mathbb{R} \) with compact support such that \( \phi^{(l)}(0) = f^{(l)}(0) \). Define \( h \) on \( \mathbb{R} \) by

\[
h(x) = \begin{cases} 
  f(x), & x \geq 0 \\
  \phi(x), & x < 0.
\end{cases}
\]

Then

\[
\int_{-\infty}^{\infty} |x^k h(x)|^2 \, dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |h^{(l)}(x)|^2 \, dx < \infty.
\]

So by (1.2), \( h \in S \) and hence \( f \in S^+ \).

For \( l \in \mathbb{N}_0 \) we have by Cauchy-Schwartz's inequality,

\[
\int_{0}^{1} |g^{(l)}(y)| y^{\nu+1} \, dy \leq \|y^{l/2}g^{(l)}\|_{L^\infty} \frac{1}{\sqrt{1+\nu+1}}.
\]
So, using \( f \in S^+ \), we obtain \( g^{(l)} \in L_1(\mathbb{R}^+, y^{l+\nu} \, dy) \). It follows that \( \mathcal{H}_{l+\nu} g^{(l)} \) is continuous with

\[
(\ast) \quad (\mathcal{H}_{l+\nu} g^{(l)}) (x) = \frac{1}{2} \int_0^\infty (xy)^{-\frac{1}{2}(l+\nu)} J_{l+\nu}((xy)^{1/2}) \, g^{(l)}(y) \, y^{l+\nu} \, dy.
\]

In the next part of the proof we show that we can apply integration by parts in the above integral expression. First observe that \( (x + 1)^{(l+\nu)/2} f^{(l-1)} \in L_\infty(\mathbb{R}^+) \) and therefore \( x^{(l+\nu)/2} g^{(l-1)} \in L_\infty((1, \infty)) \), \( l \in \mathbb{N} \). Further, for \( 0 < x < 1 \) and \( l \in \mathbb{N} \) we have

\[
|x^{(l+\nu)/2} g^{(l-1)}(x)|^2 \leq |g^{(l-1)}(1)|^2 + \\
+ 2 \int_1^x |y^{l+\nu} g^{(l-1)}(y)| \frac{l+\nu}{2} y^{l+\nu} |g^{(l)}(y)| \, dy + \\
\leq |g^{(l-1)}(1)|^2 + (l+\nu) \int_1^x y^{l+\nu} |g^{(l)}(y)| \, dy
\]

\[
+ 2 \int_1^x |y^{(l-1)/2} g^{(l-1)}(y)| \frac{l+\nu}{2} y^{l+\nu} |g^{(l)}(y)| \, dy + \\
\leq |g^{(l-1)}(1)|^2 + (l+\nu) \|y^{(l-1)/2} g^{(l-1)}\|_{Y_\nu}^2 + \\
+ 2 \|y^{(l-1)/2} g^{(l-1)}\|_{Y_\nu} \|y^{l+\nu} g^{(l)}\|_{Y_\nu}.
\]

Hence for each \( l \in \mathbb{N} \),

\[
x^{(l+\nu)/2} g^{(l-1)} \in L_\infty(\mathbb{R}^+).
\]

Since \( \xi^{-\mu} J_\mu(\xi) \) is bounded on \( \mathbb{R}^+ \) for all \( \mu \geq -1/2 \), it follows that for each \( x \in \mathbb{R}^+ \),

\[
\{(xy)^{-\frac{1}{2}(l+\nu)} J_{l+\nu}((xy)^{1/2})\} \{g^{(l-1)}(y) y^{(l+\nu)/2}\} \, y^{(l+\nu)/2}
\]
tends to 0 as \( y \downarrow 0 \) or \( y \uparrow \infty \).

With the recurrence relation

\[
\frac{d}{d\xi} (\xi^\mu J_\mu) = \xi^\mu J_{\mu-1} ,
\]

13
we obtain through integration by parts
\[ \mathcal{H}_{\nu+1}g^{(l)} = (-\frac{1}{2})^l \mathcal{H}_{\nu+1-l}g^{(l-1)}. \]

Repeating this procedure yields
\[ \mathcal{H}_{\nu+1}g^{(l)} = (-\frac{1}{2})^l \mathcal{H}_{\nu}g. \]

Since \( \mathcal{H}_{\nu+1} \) is unitary on \( Y_{\nu+1} \) we have proved that for all \( l \in \mathbb{N}, \)
\[
\int_{0}^{\infty} x^l |(\mathcal{H}_{\nu}g)(x)|^2 x^\nu \, dx = 2^{2l} \int_{0}^{\infty} x^l |(\mathcal{H}_{\nu+1}g^{(l)})(x)|^2 x^\nu \, dx \\
= 2^{2l} \int_{0}^{\infty} x^l |g^{(l)}(x)|^2 x^\nu \, dx < \infty.
\]

We see that for all \( k, l \in \mathbb{N}_0, \)
\[
x^{k/2}g \in Y_\nu \quad \text{and} \quad x^{l/2} \mathcal{H}_{\nu}g \in Y_\nu,
\]
whence \( g \in S(\mathbb{R}^+) \) by Theorem 2.3.11 and the proof that (iv) implies (i) is complete.

Let \( g \) satisfy condition (v). Then clearly \( f \) on \( \mathbb{R}^+ \) defined by \( f(x) = g(x+1), \, x \in \mathbb{R}^+, \)
belongs to \( S^+ \). Hence for all \( k, l \in \mathbb{N}_0, \)
\[
\int_{0}^{\infty} |x^k f^{(l)}(x)|^2 \, dx < \infty.
\]

It follows that for all \( k, l \in \mathbb{N}_0, \)
\[
\int_{0}^{\infty} |x^{(k+l)/2} g^{(l)}(x)|^2 \, dx \leq \sup_{0 \leq x \leq 1} |x^{l/2} g^{(l)}(x)|^2 + \\
+ \int_{0}^{\infty} (x + 1)^{k+l} |f^{(l)}(x)|^2 \, dx < \infty,
\]
whence \( g \) satisfies condition (iv) with \( \nu = 0. \)

\[ \square \]

**Remark**
Characterizations of \( S^+ \) as in the preceding theorem are not in Duran's paper [Du1]. In fact Duran states that such a characterization cannot be valid. But his example \( f(t) = \sqrt{t} e^{-t} \) does not work since it does not satisfy condition (ii) in Theorem (2.4): For this consider its second derivative,
Clearly $tf^{(2)}$ is not bounded on $\mathbb{R}^+$.

The final characterization of $S^+$ presented here, is a consequence of Theorem (2.4) and the following result.

(2.5) **Lemma**
Let $\nu \geq -\frac{1}{2}$ and let $f$ be an infinitely differentiable function on $\mathbb{R}^+$ with $x^{k/2}f \in Y_\nu$ and $f^{(l)} \in Y_\nu$ for all $k, l \in \mathbb{N}_0$. Then for all $k, l \in \mathbb{N}_0$,

$$x^{k/2}f^{(l)} \in Y_\nu.$$ 

**Proof**
The proof is by induction with respect to $l$. For $\alpha > \nu + 1$,

$$x^{\alpha} |f(x)|^2 = 2 \operatorname{Re} \int_0^\infty [\alpha \xi^{\alpha-1}f(\xi) + \xi^{\alpha} f'(\xi)] \frac{f'(\xi)}{f(\xi)} d\xi$$

$$\leq 2\alpha \|\xi^{(\alpha-\nu-1)/2}f\|_{Y_\nu}^2 + 2\|\xi^{\alpha-\nu}f\|_{Y_\nu} \|f'\|_{Y_\nu}$$

and so for all $\beta > 0$,

$$x^{\beta} f(x) \to 0 \quad \text{as} \quad x \to \infty.$$ 

Moreover, for $x > 1$,

$$|f'(x) - f'(1)| = \left| \int_1^x f''(t) dt \right| \leq \|f''\|_{Y_\nu} q_\nu(x)$$

where

$$q_\nu(x) = \begin{cases} \sqrt{x} & \text{if } \nu \geq 0, \\ \left( \frac{x^{1-\nu}}{1-\nu} \right)^{1/2} & \text{if } -\frac{1}{2} \leq \nu < 0. \end{cases}$$

Now let $k \in \mathbb{N}$. Then

$$\int_1^\infty x^{k+\nu} |f'(x)|^2 dx = x^{k+\nu} f(x) \frac{f'(x)}{1} \bigg|_1^\infty$$

15
\[- \int_{1}^{\infty} \{(k + \nu) x^{k+\nu-1} f'(x) + x^{k+\nu} f''(x)\} f(x) \, dx\]

\[\leq |f(1)| |f'(1)| + (k + \nu) \|x^{k-1} f\|_{Y_\nu} + \|x^k f\|_{Y_\nu} \|f''\|_{Y_\nu},\]

and

\[\int_{0}^{1} x^{k+\nu} |f'(x)|^2 \, dx \leq \int_{0}^{1} x^{\nu} |f'(x)|^2 \, dx \leq \|f'\|^2_{Y_\nu},\]

so that

\[\int_{0}^{\infty} x^{k+\nu} |f'(x)|^2 \, dx < \infty.\]

We find that \(x^{k/2} f' \in Y_\nu\) for all \(k \in \mathbb{N}_0\).

So with \(f_1 = f'\) we have for all \(k, l \in \mathbb{N}_0\)

\[x^{k/2} f_1 \in Y_\nu \quad \text{and} \quad f^{(l)}_1 \in Y_\nu\]

which yield, repeating the previous step, \(x^{k/2} f'' = x^{k/2} f'_1 \in Y_\nu\), and so on. \(\square\)

(2.6) **Theorem**

Let \(\nu \geq -\frac{1}{2}\) and let \(f\) be an infinitely differentiable function on \(\mathbb{R}^+\). Then \(f \in S^+\) iff for all \(k, l \in \mathbb{N}_0\),

\[x^{k/2} f \in Y_\nu \quad \text{and} \quad f^{(l)} \in Y_\nu.\]

**Proof**

The necessity of the condition is evident from Theorem (2.4).

So let \(f\) satisfy the stated condition for some \(\nu \geq -\frac{1}{2}\). Then Lemma (2.5) says that \(x^{k/2} f^{(l)} \in Y_\nu\) for all \(k, l \in \mathbb{N}_0\). It follows that \(x^{k/2} f \in Y_\nu\) and \(x^{l/2} f^{(l)} \in Y_\nu\) for all \(k, l \in \mathbb{N}_0\). So \(f \in S^+\) by Theorem (2.4). \(\square\)

The above characterizations of \(S^+\) and the relation between \(S^+\) and \(S_{\text{even}}\) yield necessary and sufficient conditions on an infinitely differentiable function \(f\) on \(\mathbb{R}^+\) to have an even extension to \(S\). They are additional to the ones given before.

(2.7) **Theorem**

Let \(\nu \geq -\frac{1}{2}\) and let \(f\) be an infinitely differentiable function on \(\mathbb{R}^+\). Then the following statements are equivalent.
(i) The even extension of $f$ belongs to $S$;

(ii) $x^{(k+l)\left(\frac{d}{dz}\right)} f \in L_\infty(\mathbb{R}^+) \text{ for all } k, l \in \mathbb{N}_0$;

(iii) $x^k f \in L_\infty(\mathbb{R}^+)$ and $x^l \left(\frac{d}{dz}\right)^\nu f \in L_\infty(\mathbb{R}^+)$ for all $k, l \in \mathbb{N}_0$;

(iv) $x^{(k+l)\left(\frac{d}{dz}\right)} f \in X_\nu$ for all $k, l \in \mathbb{N}_0$;

(v) $x^k f \in X_\nu$ and $x^l \left(\frac{d}{dz}\right)^\nu f \in X_\nu$ for all $k, l \in \mathbb{N}_0$;

(vi) $x^k \left(\frac{d}{dz}\right)^\nu f \in X_\nu$ for all $k, l \in \mathbb{N}_0$;

(vii) $x^l \left(\frac{d}{dz}\right)^\nu f \in X_\nu$ for all $k, l \in \mathbb{N}_0$.

Finally we present two very elegant relations which put corresponding results in [Du1] in a wider context.

(2.8) Lemma

(i) Let $f \in S^+$. Then for all $k, l \in \mathbb{N}_0$,

$$
\|x^{(k+l)/2} \left(\frac{d}{dz}\right)^\nu f\|_{Y_\nu} = 2^{k-l} \|x^{(k+l)/2} \left(\frac{d}{dz}\right)^k \mathcal{H}_\nu f\|_{Y_\nu}.
$$

(ii) Let $f \in S_{\text{even}}$. Then for all $k, l \in \mathbb{N}_0$,

$$
\|x^{(k+l)} \left(\frac{d}{dz}\right)^\nu f\|_{X_\nu} = 2^{k-l} \|x^{(k+l)} \left(\frac{d}{dz}\right)^k \mathcal{H}_\nu f\|_{X_\nu}.
$$

Proof

Since for $\mu \geq -\frac{1}{2}$, $\mathcal{H}_\mu$ is unitary on $Y_\mu$, we have for $\alpha > -(\nu + 1/2)$,

$$
\|t^{\alpha/2} f\|_{Y_\mu} = \|f\|_{Y_{\nu+a}} = \|\mathcal{H}_{\nu+a} f\|_{Y_{\nu+a}} = \|t^{\alpha/2} \mathcal{H}_{\nu+a} f\|_{Y_\nu}.
$$

Thus we derive

$$
\|x^{(k+l)/2} \left(\frac{d}{dz}\right)^\nu f\|_{Y_\nu} = \|x^{(k+l)/2} \mathcal{H}_{\nu+k+l} \left(-\frac{d}{dz}\right)^k \mathcal{H}_{\nu+l} f\|_{Y_\nu} = (\frac{1}{2})^k \|x^{(k+l)/2} \mathcal{H}_{\nu+k+l} \mathcal{H}_{\nu+l} f\|_{Y_\nu} = (\frac{1}{2})^{l-k} \|x^{(k+l)/2} \left(-\frac{d}{dz}\right)^k \mathcal{H}_{\nu} f\|_{Y_\nu}.
$$

The proof of (ii) runs the same. \qed
3. Gelfand-Shilov spaces $S_{\alpha,\text{even}}^\beta$ and $G_{\alpha}^\beta$

The spaces $S_{\alpha}$, $S^\beta$ and $S_{\alpha}^\beta$ were introduced by Gelfand and Shilov as subspaces of $S$. Let us mention their definition and some of their properties. For $\alpha \geq 0$ the space $S_{\alpha}$ consists of all $f \in S$ for which there are constants $A > 0$ and $B_l > 0$ depending on $l$ and $f$, only, such that for all $k, l \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} |x^k f(l)(x)| \leq B_l A^k (k!)^\alpha.$$ 

$S_{\alpha}$ is nontrivial for all values of $\alpha$; $S_0$ consists of all infinitely differentiable functions on $\mathbb{R}$ with compact support.

For $\beta \geq 0$ the space $S^\beta$ consists of all $f \in S$ for which there are constants $B > 0$ and $A_k > 0$ depending on $k$ and $f$, only, such that for all $k, l \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} |x^k f(l)(x)| \leq A_k B^l (l!)^\beta.$$ 

For all values of $\beta$, $S^\beta$ is nontrivial; for $\beta < 1$, $S^\beta$ consists of entire analytic functions; $S^1$ consists of functions which are analytic on a strip around the real axis where the width of the strip depends on the particular function.

For $\alpha \geq 0$ and $\beta \geq 0$ the space $S_{\alpha}^\beta$ consists of all $f \in S$ for which there are constants $A, B, C > 0$ depending on $f$, only, such that for all $k, l \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} |x^k f(l)(x)| \leq C A^k B^l (k!)^\alpha (l!)^\beta.$$ 

The space $S_{\alpha}^\beta$ is nontrivial if $\alpha + \beta \geq 1$ with $\alpha > 0$ and $\alpha = 0$ and $\beta > 1$. Kashpirovskii proved the intersection result

$$S_{\alpha}^\beta = S_{\alpha} \cap S^\beta.$$ 

Refinement of this result can be found in [E1]. We observe that for $\alpha + \beta \geq 1$, $\alpha > 0$ and $0 \leq \beta < 1$ the space $S_{\alpha}^\beta$ consists of entire functions $f$ for which there are constants $K, a, b > 0$ such that

$$|f(x + iy)| \leq K \exp(-a|x|^{1/\alpha} + b|y|^{1/1-\beta}).$$

(3.1) **Theorem**

Let $\alpha > 0$ and $\beta > 0$, and let $f$ be a square integrable function on $\mathbb{R}$.

The statements in each of the following triples are equivalent

I  (i)  $f \in S_{\alpha},$
II (i) \( f \in S^\beta \),
(ii) for all \( k \in \mathbb{N}_0 \), \( x^k f \in L_\infty(\mathbb{R}) \) and there is \( t > 0 \) such that \( \exp(t x^\beta) \mathcal{F} f \in L_\infty(\mathbb{R}) \),
(iii) for all \( k \in \mathbb{N}_0 \), \( \sup_{x \in \mathbb{R}} |x^k f(x)| < \infty \) and there is \( B > 0 \) such that for all \( l \in \mathbb{N}_0 \),
\[
\sup_{x \in \mathbb{R}} |x^l(\mathcal{F} f)(x)| \leq B^{l+1}(l!)^\beta .
\]

III (i) \( f \in S_\alpha^\beta \),
(ii) there is \( t > 0 \) such that
\[
\exp(t |x|^{1/\alpha}) f \in L_\infty(\mathbb{R}) \quad \text{and} \quad \exp(t |x|^{1/\beta}) \mathcal{F} f \in L_\infty(\mathbb{R}) ,
\]
(iii) there is \( A'B > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x \in \mathbb{R}} |x^k f(x)| \leq A^{k+1}(k!)^\alpha
\]
and
\[
\sup_{x \in \mathbb{R}} |x^l \mathcal{F} f(x)| \leq B^{l+1}(l!)^\beta .
\]

Proof
In all three cases I, II and III equivalence of (ii) and (iii) follows from simple estimations based on properties of the function \( x^k \exp(-tx^{1/\nu}) \) for \( \nu > 0 \), \( x > 0 \) and \( t \in \mathbb{R}^+ \). The remaining part of the proof can be found in [E1], p. 140.

(3.2) Theorem
In Theorem (3.1) in all statements \( L_\infty(\mathbb{R}) \) can be replaced by \( L_2(\mathbb{R}) \) and therefore, the \( L_\infty \)-norm by the \( L_2 \)-norm.

Proof
This is a consequence of the Sobolev inequality
\[
\sup_{x \in \mathbb{R}} |f(x)| \leq \frac{1}{\sqrt{2}} \left[ ||f||_{L_2(\mathbb{R})} + ||x \mathcal{F} f||_{L_2(\mathbb{R})} \right]
\]
and other straightforward estimations.

In [EB], we studied the spaces $S_{\alpha,\text{even}}$, $S_{\beta,\text{even}}^\alpha$ and $S_{\alpha,\text{even}}^\beta$. The role played by the Fourier transformation in the characterization of the spaces $S_{\alpha}$, $S_{\beta}^\alpha$ and $S_{\alpha,\text{even}}^\beta$ is taken by the Hankel transformations $H_\nu$, $\nu \geq -\frac{1}{2}$.

(3.3) THEOREM

Let $\nu \geq -\frac{1}{2}$ and let $\alpha > 0$, $\beta > 0$. For $f \in X_\nu$ the statements in each of the following triples are equivalent.

I

(i) $f$ extends to an even function in $S_{\alpha}$;
(ii) there exists $t > 0$ such that $\exp(tx^{1/\alpha})f \in L_\infty(\mathbb{R}^+)$ and for all $l \in \mathbb{N}_0$, $x^l H_\nu f \in L_\infty(\mathbb{R}^+)$;
(iii) there exists $A > 0$ such that for all $k \in \mathbb{N}_0$, 
$$\sup_{x > 0} |x^k f(x)| \leq A^{k+1}(k!)^\alpha$$

and for all $l \in \mathbb{N}_0$, $\sup_{x > 0} |x^l (H_\nu f)(x)| < \infty$.

II

(i) $f$ extends to an even function in $S_{\beta}$;
(ii) for all $k \in \mathbb{N}_0$, $x^k f \in L_\infty(\mathbb{R}^+)$ and there exists $t > 0$ such that
$$\exp(tx^{1/\beta}) H_\nu f \in L_\infty(\mathbb{R}^+);$$
(iii) for all $k \in \mathbb{N}_0$, $\sup_{x > 0} |x^k f(x)| < \infty$ and there exists $B > 0$ such that for all $l \in \mathbb{N}_0$, 
$$\sup_{x > 0} |x^l (H_\nu f)(x)| \leq B^{l+1}((l!)^\beta.$$  

III

(i) $f$ extends to an even function in $S_{\alpha,\text{even}}^\beta$;
(ii) there exists $t > 0$ such that
$$\exp(tx^{1/\alpha}) f \in L_\infty(\mathbb{R}^+) \text{ and } \exp(tx^{1/\beta}) H_\nu f \in L_\infty(\mathbb{R}^+);$$
(iii) there exists $A, B > 0$ such that
$$\sup_{x > 0} |x^k f(x)| \leq A^{k+1}(k!)^\alpha$$

and
$$\sup_{x > 0} |x^l (H_\nu f)(x)| \leq B^{l+1}((l!)^\beta.$$ 

Next we show the same characterization of $S_{\alpha,\text{even}}$, $S_{\beta,\text{even}}$ and $S_{\alpha,\text{even}}^\beta$ with $L_\infty(\mathbb{R}^+)$ replaced by $X_\nu$ and with the $L_\infty$-norm replaced by the $X_\nu$-norm. Via these new characterizations the properties of the Hankel transformations $H_\nu$ as stated in Lemma 2.8
can be applied fully.

(3.4) Theorem

Let $\alpha > 0$, $\beta > 0$ and let $\nu \geq -\frac{1}{2}$. For $f \in X_{\nu}$ the following conditions are equivalent.

I  
(i) $f$ extends to an even function in $S_{\alpha}$,
(ii) there exists $t > 0$ such that $\exp(tx^{1/\alpha}) f \in X_{\nu}$ and for all $l \in \mathbb{N}_0$, $x^l H_{\nu} f \in X_{\nu}$, 
(iii) there exists $A > 0$ such that for all $k \in \mathbb{N}_0$, 
\[ \|x^k f\|_{X_{\nu}} \leq A^{k+1}(k!)^\alpha \]
and for all $l \in \mathbb{N}_0$, $\|x^l H_{\nu} f\|_{X_{\nu}} < \infty$.

II  
(i) $f$ extends to an even function in $S^\beta$ ,
(ii) for all $k \in \mathbb{N}_0$, $x^k f \in X_{\nu}$, and there exists $t > 0$ such that 
\[ \exp(tx^{1/\beta}) H_{\nu} f \in X_{\nu} , \]
(iii) for all $k \in \mathbb{N}_0$, $\|x^k f\|_{X_{\nu}} < \infty$ and there exists $B > 0$ such that for all $l \in \mathbb{N}_0$, 
\[ \|x^l H_{\nu} f\|_{X_{\nu}} \leq B^{l+1}(l!)^\beta . \]

III  
(i) $f$ extends to an even function in $S^\beta_{\alpha}$ ,
(ii) there exists $t > 0$ such that 
\[ \exp(tx^{1/\alpha}) f \in X_{\nu}$ and $\exp(tx^{1/\beta}) H_{\nu} f \in X_{\nu}$ , 
(iii) there exist $A > 0$ and $B > 0$ such that for all $k, l \in \mathbb{N}_0$, 
\[ \|x^k f\|_{X_{\nu}} \leq A^{k+1}(k!)^\alpha \]
and 
\[ \|x^l H_{\nu} f\|_{X_{\nu}} \leq B^{l+1}(l!)^\beta . \]

Proof

In the triples I, II and III, the equivalence of (ii) and (iii) follows from the inequality
\[ \max_{x \geq 0} x^k \exp(-tx^{1/\gamma}) \leq (\gamma/t)^x (k!)^\gamma , \]
and a straightforward estimation of $\|x^n f\|_{X_{\nu}}$ in
I Let $f$ extend to an even function in $S_\alpha$. Then by Theorem (3.3) there exists $\tau > 0$ such that $\exp(\tau x^{1/\alpha}) f \in L_\infty(\mathbb{R}^+)$ and so for $0 < t < \tau$, $\varepsilon = \tau - t$,

$$
\| \exp(t x^{1/\alpha}) f \|_{X_\nu} \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \| x^{n/\gamma} f \|_{X_\nu} \cdot
$$

Further, the even extension of $f$ belongs to $S_{\text{even}}$ so that $x^k I H_\nu f \in X_\nu$, $k \in \mathbb{N}_0$, by Theorem (1.10). We conclude that (i) implies (ii). For the converse, let $f$ satisfy condition (ii) for some $t > 0$. Then as a consequence of Theorem (1.10) the even extension of $f$ belongs to $S_{\text{even}}$. Hence for all $l \in \mathbb{N}_0$, $x^l I H_{-\frac{1}{2}} f \in X_{-\frac{1}{2}}$ and so $f$ is continuous and bounded on $(0,\infty)$. In its turn this implies $\exp(t x^{1/\alpha}) f \in X_{-\frac{1}{2}}$. We conclude that the even extension $f_e$ of $f$ satisfies

$$
\exp(t |x|^{1/\alpha}) f_e \in L_2(\mathbb{R}) ,
$$

and for all $l \in \mathbb{N}_0$,

$$
x^l F f_e \in L_2(\mathbb{R}) .
$$

Hence $f_e \in S_\alpha$ by Theorem (3.2).

II Let $f$ extend to an even function in $S^\beta$. Then from Theorem 3.3 we see that $(IH_\nu f)_e \in S^\beta$ (Observe $f = IH_\nu(IH_\nu f)$.). So from I. (i) $\iff$ (ii), $(IH_\nu f)_e \in S^\beta$ if and only if there exists $t > 0$ such that

$$
\exp(t x^{1/\beta}) IH_\nu f \in X_\nu
$$

and for all $k \in \mathbb{N}_0$,

$$
x^k f = x^k IH_\nu(IH_\nu f) \in X_\nu ,
$$

III The even extension $f_e$ of $f$ belongs to $S^\beta_\alpha$ if and only if $f_e \in S_\alpha$ and $f_e \in S^\beta$. So the equivalence (i) $\iff$ (ii) is a straightforward consequence of I and II.  

We arrive at the point to present characterizations of $S^\beta_\alpha, S^\beta_{\text{even}}$ and $S^\beta_{\alpha,\text{even}}$ which are refinements and extensions of similar results derived by Duran [Du1].

(3.5) THEOREM
Let $\alpha > 0$, $\beta > 0$ and let $\nu \geq -\frac{1}{2}$. For an infinitely differentiable function on $\mathbb{R}^+$ the statements in each of the following triples are equivalent.
I (i) The even extension of \( f \) belongs to \( S_\alpha \),
(ii) there exists \( A > 0 \) such that for all \( k \in \mathbb{N}_0 \),
\[
\| x^k f \|_{X_\nu} \leq A^{k+1}(k!)^\alpha
\]
and for all \( l \in \mathbb{N}_0 \),
\[
\| x^l \left( \frac{d}{dz} \right)^l f \|_{X_\nu} < \infty,
\]
(iii) there are \( A > 0, B_l > 0 \) depending only on \( l \) and \( f \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\| x^{k+l} \left( \frac{1}{z} \frac{d}{dz} \right)^l f \|_{X_\nu} \leq B_l A^{k}(k!)^\alpha.
\]

II (i) The even extension of \( f \) belongs to \( S^\beta \),
(ii) for all \( k \in \mathbb{N}_0 \), \( \| x^k \|_{X_\nu} < \infty \) and there exists \( B > 0 \) such that for all \( l \in \mathbb{N}_0 \),
\[
\| x^l \left( \frac{1}{z} \frac{d}{dz} \right)^l \|_{X_\nu} \leq B^{l+1}(l!)^\beta,
\]
(iii) there exist \( B > 0, A_k > 0 \) depending only on \( k \) and \( f \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\| x^{k+l} \left( \frac{1}{z} \frac{d}{dz} \right)^l f \|_{X_\nu} \leq A_k B^l(l!)^\beta.
\]

III Here, in addition, we assume \( \alpha + \beta \geq 1 \).
(i) The even extension of \( f \) belongs to \( S^\beta_\alpha \),
(ii) there exist \( A > 0, B > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\| x^k f \|_{X_\nu} \leq A^{k+1}(k!)^\alpha \text{ and } \| x^l \left( \frac{1}{z} \frac{d}{dz} \right)^l f \|_{X_\nu} \leq B^{l+1}(l!)^\beta,
\]
(iii) there exist \( A > 0, B > 0, C > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\| x^{k+l} \left( \frac{1}{z} \frac{d}{dz} \right)^l f \|_{X_\nu} \leq C A^{k} B^l(l!)^\alpha (l!)^\beta.
\]

Proof
In Lemma (2.8), taking \( k = 0 \) we proved the following relation: For all \( \nu \geq -\frac{1}{2} \) and for all \( f \in S_{\text{even}} \),
\[
\| x^l \left( \frac{1}{z} \frac{d}{dz} \right)^l f \|_{X_\nu} = 2^{-l} \| x^l H_\nu f \|_{X_\nu}.
\]
So in each of the cases I, II and III equivalence of (i) and (ii) is a consequence of Theorem (3.4). Also in all three cases (iii) implies (ii). So there remains to show that (ii) implies (iii).
Let us first make some general observations. Let \( f \in S_{\text{even}} \) and let \( k \geq l \). Then an application of Leibniz's differentiation rule yields
\[
\|x^{k+l}(1 \frac{d}{d\alpha})^l f\|_{\mathcal{X}_\nu}^2 = \\
= (x^{2(k+l)}(1 \frac{d}{d\alpha})^l f, (1 \frac{d}{d\alpha})^l f)_{\mathcal{X}_\nu} \\
\leq \sum_{j=0}^{l} \binom{l}{j} 2^j \frac{\Gamma(k+l+\nu+1)}{\Gamma(k+l+\nu+1-j)} |(x^{2(k+l-j)}(1 \frac{d}{d\alpha})^{2l-j} f, f)_{\mathcal{X}_\nu}| \\
\sum_{j=0}^{l} \binom{l}{j} \binom{k+l+\nu}{j} j! 2^j \|x^{2k-j} f\|_{\mathcal{X}_\nu} \|x^{2l-j} (1 \frac{d}{d\alpha})^{2l-j} f\|_{\mathcal{X}_\nu}.
\]

For \( k > l \) we apply Lemma (2.8),
\[
\|x^{k+l}(1 \frac{d}{d\alpha})^l f\|_{\mathcal{X}_\nu} = 2^{k-l}\|x^{k+l}(1 \frac{d}{d\alpha})^k IH_\nu f\|_{\mathcal{X}_\nu},
\]
and then, replacing \( f \) by \( IH_\nu f \) and interchanging roles of \( k \) and \( l \) above, we derive
\[
\|x^{k+l}(1 \frac{d}{d\alpha})^l f\|_{\mathcal{X}_\nu}^2 \leq \\
\leq 2^{2k-2l} \sum_{j=0}^{k} \binom{k}{j} \binom{k+l+\nu}{j} j! 2^j \|x^{2l-j} IH_\nu f\|_{\mathcal{X}_\nu} \|x^{2k-j} (1 \frac{d}{d\alpha})^{2k-j} IH_\nu f\|_{\mathcal{X}_\nu} \\
= \sum_{j=0}^{k} \binom{k}{j} \binom{k+l+\nu}{j} j! 2^j \|x^{2l-j} (1 \frac{d}{d\alpha})^{2l-j} f\|_{\mathcal{X}_\nu} \|x^{2k-j} f\|_{\mathcal{X}_\nu}.
\]

From the considerations above we obtain the following estimation:
Let \( f \in S_{\text{even}} \) and let \( k, l \in \mathbb{N}_0 \). Put \( r = \min\{k, l\} \), then
\[
(*) \quad \|x^{k+l}(1 \frac{d}{d\alpha})^l f\|_{\mathcal{X}_\nu}^2 \leq \\
\leq 2^{2k+2l+\nu} \max_{0 \leq j \leq r} j! \|x^{2k-j} f\|_{\mathcal{X}_\nu} \|x^{2l-j} (1 \frac{d}{d\alpha})^{2l-j} f\|_{\mathcal{X}_\nu}.
\]

I For \( f \) satisfying I(ii) we get by (*)
\[
\|x^{k+l}(1 \frac{d}{d\alpha})^l f\|_{\mathcal{X}_\nu}^2 \leq \\
\leq 2^{2k+2l+\nu} \max_{0 \leq j \leq r} j! A^{2k-j+1}((2k-j)!)^{\alpha} \|x^{2l-j} (1 \frac{d}{d\alpha})^{2l-j} f\|_{\mathcal{X}_\nu}
\]

24
\[
\leq (2^{1+a}(1 + A)^{2k+1}(k!)^{2\alpha} \max_{0 \leq j \leq l} 2^{2l-j} f \| x^{2l-j} \left( \frac{d^l}{dx^l} \right)^{2l-j} f \|_{X_\nu} \\
=: \tilde{A}^{2k}(k!)^{2\alpha} \tilde{B}_l .
\]

II For \( f \) satisfying II(ii) we get similarly from (*)
\[
\| x^{k+l} \left( \frac{d}{dx} \right)^{l} f \|_{X_\nu}^2 \leq \leq (2^{1+\alpha}(1 + B))^{2l+1}(l!)^{2\beta} \max_{0 \leq j \leq k} 2^{2k+j} j! \| x^{2k-j} f \|_{X_\nu} \\
=: \tilde{B}^{2l}(l!)^{2\beta} \tilde{A}_k .
\]

III Finally for \( f \) satisfying III(ii) the estimation (*) yields
\[
\| x^{k+l} \left( \frac{d}{dx} \right)^{l} f \|_{X_\nu}^2 \leq 2^{2k+2l+j} \max_{0 \leq j \leq r} j! A^{2k-j+1} B^{2l-j+1} (2k-j)!^\alpha (2l-j)!^\beta \\
\leq 2^\alpha AB(2^{1+\alpha}(A + 1))^{2k} (2^{1+\beta}(B + 1))^{2l} (k!)^{2\alpha} (l!)^{2\beta} \\
\max_{0 \leq j \leq r} (j!)^{1-\alpha-\beta} \left( \frac{2k}{j} \right)^{-\alpha} \left( \frac{2l}{j} \right)^{-\beta} \\
=: \tilde{A}^{2k} \tilde{B}^{2l}(k!)^{2\alpha} (l!)^{2\beta} \tilde{C}^2 . \quad \Box
\]

As we have shown, the space \( S(\mathbb{R}^+) \) consists of all \( C^\infty \)-functions \( f \) on \( \mathbb{R}^+ \) with the property that the function \( x \mapsto f(x^2) \), \( x \in \mathbb{R} \), belongs to \( S_{\text{even}} \). Accordingly we introduce the spaces \( G_{2\alpha} \), \( G^{2\beta} \) and \( G_{2\alpha}^{2\beta} \). Here we follow Duran's notation in [Dul].

(3.6) Definition
Let \( f \) be an infinitely differentiable function on \( \mathbb{R}^+ \).

I \( f \in G_{2\alpha} :\Leftrightarrow x \mapsto f(x^2) \in S_{\alpha,\text{even}} \)

II \( f \in G^{2\beta} :\Leftrightarrow x \mapsto f(x^2) \in S_{\text{even}}^{\beta} \)

III \( f \in G_{2\alpha}^{2\beta} :\Leftrightarrow x \mapsto f(x^2) \in S_{\alpha,\text{even}}^{\beta} \).

25
By definition $G_{2\alpha}$, $G^{2\beta}$ and $G_{2\alpha}^{2\beta}$ are subspaces of $S^+$. The spaces $G_{2\alpha}$ and $G^{2\beta}$ are nontrivial for all values of $\alpha, \beta \geq 0$, $G_{2\alpha}^{2\beta}$ is nontrivial if $\alpha + \beta \geq 1$, $\alpha > 0$ and $\beta > 1$.

Theorems (3.3), (3.4) and (3.5) lead directly to characterizations of $G_{2\alpha}$, $G^{2\beta}$ and $G_{2\alpha}^{2\beta}$ be replacing $x^2$ by $x$, $\frac{1}{x}$ by $\frac{d}{dx}$, $H_\nu$ by $H_\nu$ and $X_\nu$ by $Y_\nu$.

(3.7) Theorem

Let $\alpha > 0$, $\beta > 0$ and let $\nu \geq -\frac{1}{2}$. Let $f$ be a function on $\mathbb{R}^+$, $f \in Y_\nu$.

I Each of the following conditions is necessary and sufficient for $f$ to belong to $G_{2\alpha}$.

(i) There exists $t > 0$ such that $\exp (t x^{1/2\alpha}) f \in L_\infty (\mathbb{R}^+)$ and for all $l \in \mathbb{N}_0$, $x^{l/2} H_\nu f \in L_\infty (\mathbb{R}^+)$,

(ii) there exists $A > 0$ such that for all $k \in \mathbb{N}_0$,

$$\sup_{x > 0} |x^{k/2} f(x)| \leq A^{k+1} (k!)^\alpha$$

and for all $l \in \mathbb{N}_0$, $\sup_{x > 0} |x^{l/2} (H_\nu f)(x)| < \infty$,

(iii) there exists $t > 0$ such that $\exp (t x^{1/2\alpha}) f \in Y_\nu$ and for all $l \in \mathbb{N}_0$, $x^{l/2} H_\nu f \in Y_\nu$,

(iv) there exists $A > 0$ such that for all $k \in \mathbb{N}_0$,

$$\|x^{k/2} f\|_{Y_\nu} \leq A^{k+1} (k!)^\alpha$$

and for all $l \in \mathbb{N}_0$, $\|x^{l/2} H_\nu f\|_{Y_\nu} < \infty$,

(v) there exists $A > 0$ such that for all $k \in \mathbb{N}_0$,

$$\|x^{k/2} f\|_{Y_\nu} \leq A^{k+1} (k!)^\alpha$$

and $f$ is infinitely differentiable on $\mathbb{R}^+$ with for all $l \in \mathbb{N}_0$,

$$\|x^{l/2} \left( \frac{d}{dx} \right)^{l} f\|_{Y_\nu} < \infty ,$$

(vi) $f$ is infinitely differentiable on $\mathbb{R}^+$ and there exist $A, B_l > 0$ depending only on $l$ and $f$ such that for all $k, l \in \mathbb{N}_0$,

$$\|x^{(k+l)/2} \left( \frac{d}{dx} \right)^{l} f\|_{Y_\nu} \leq A^k B_l (k!)^\alpha .$$

II Each of the following conditions is necessary and sufficient for $f$ to belong to $G^{2\beta}$.

(i) For all $k \in \mathbb{N}_0$, $x^{k/2} f \in L_\infty (\mathbb{R}^+)$ and there exists $t > 0$ such that $\exp (t x^{1/2\beta}) H_\nu f \in L_\infty (\mathbb{R}^+)$,

(ii) for all $k \in \mathbb{N}_0$, $\sup_{x > 0} |x^{k/2} f(x)| < \infty$ and there exists $B > 0$ such that for all $l \in \mathbb{N}_0$,
\[
\sup_{x > 0} |x^{1/2}(\mathcal{H}_{\nu} f)(x)| \leq B^{l+1}(l!)^\beta ,
\]

(iii) for all \( k \in \mathbb{N}_0 \), \( x^{k/2} f \in \mathcal{Y}_\nu \) and there exists \( t > 0 \) such that \( \exp(t x^{1/2}) \mathcal{H}_{\nu} f \in \mathcal{Y}_\nu \),

(iv) for all \( k \in \mathbb{N}_0 \), \( \|x^{k/2} f\|_{\mathcal{Y}_\nu} < \infty \) and there exists \( t > 0 \) such that for all \( l \in \mathbb{N}_0 \),
\[
\|x^{l/2} \mathcal{H}_{\nu} f\|_{\mathcal{Y}_\nu} \leq B^{l+1}(l!)^\beta ,
\]

(v) for all \( k \in \mathbb{N}_0 \), \( \|x^{k/2} f\|_{\mathcal{Y}_\nu} < \infty \) and \( f \) is infinitely differentiable on \( \mathbb{R}^+ \) with \( f \) for all \( l \in \mathbb{N}_0 \),
\[
\|x^{l/2} \left( \frac{d}{dx} \right)^l f\|_{\mathcal{Y}_\nu} \leq B^{l+1}(l!)^\beta ,
\]

(vi) \( f \) is infinitely differentiable on \( \mathbb{R}^+ \) and there exist \( A_k, B > 0 \) depending only on \( k \) and \( f \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\|x^{(k+1)/2} \left( \frac{d}{dx} \right)^l f\|_{\mathcal{Y}_\nu} \leq A_k B^{l}(l!)^\beta .
\]

III In addition we assume that \( \alpha + \beta \geq 1 \). Then each of the following conditions is necessary and sufficient for \( f \) to belong to \( G_{2a}^{b} \).

(i) There exists \( t > 0 \) such that 
\[
\exp(t x^{1/2a}) f \in L_\infty(\mathbb{R}^+) \text{ and } \exp(t x^{1/2b}) \mathcal{H}_{\nu} f \in L_\infty(\mathbb{R}^+) ,
\]

(ii) there exists \( A > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x > 0} |x^{k/2} f(x)| \leq A^{k+1}(l!)^\alpha \text{ and } \sup_{x > 0} |x^{l/2} (\mathcal{H}_{\nu} f)(x)| \leq A^{l+1}l!^\beta ,
\]

(iii) there exists \( t > 0 \) such that 
\[
\exp(t x^{1/2a}) f \in \mathcal{Y}_\nu \text{ and } \exp(t x^{1/2b}) \mathcal{H}_{\nu} f \in \mathcal{Y}_\nu ,
\]

(iv) there exists \( A > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\|x^{k/2} f\|_{\mathcal{Y}_\nu} \leq A^{k+1}(l!)^\alpha \text{ and } \|x^{l/2} \mathcal{H}_{\nu} f\|_{\mathcal{Y}_\nu} \leq A^{l+1}(l!)^\beta ,
\]

(v) there exists \( A > 0 \) such that for all \( k \in \mathbb{N}_0 \),
\[
\|x^{k/2} f\|_{\mathcal{Y}_\nu} \leq A^{k+1}(l!)^\alpha ,
\]
\( f \) is infinitely differentiable on \( \mathbb{R}^+ \) and there exists \( B > 0 \) such that for all \( l \in \mathbb{N}_0 \),
\[
\|x^{l/2} \left( \frac{d}{dx} \right)^l f\|_{\mathcal{Y}_\nu} \leq B^{l+1}(l!)^\beta ,
\]

(vi) \( f \) is infinitely differentiable on \( \mathbb{R}^+ \) and there exist \( A, B, C > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\|x^{(k+1)/2} \left( \frac{d}{dx} \right)^l f\|_{\mathcal{Y}_\nu} \leq C A^k B^l(l!)^\alpha (l!)^\beta ,
\]
Finally we shall show that in Theorem (3.7) condition (vi) can be replaced by the corresponding sup-norm condition.

For this, let $f \in S^+$. A straightforward estimation shows

$$
(*) \quad \|x^{(k+1)/2}(\frac{d}{dx})^{l}f\|_{Y_0} \leq \sup_{x \geq 0} |(1 + x)^{2}x^{(k+l)/2}f^{(l)}(x)| .
$$

Further,

$$
x^{1/2}f^{(l)}(x) = (-\frac{1}{2})^{l}x^{1/2}(\mathcal{H}_0 f)(x) = 
$$

$$
= \frac{1}{2}(-\frac{1}{2})^{l} x^{1/2} \int_{0}^{\infty} (xy)^{-1/2} J_l(\sqrt{xy}) (\mathcal{H}_0 f)(y) y^l \, dy .
$$

Since, cf. [MOS], p. 78,

$$
\sup_{\xi \in \mathbb{R}} |J_l(\xi)| \leq 1 ,
$$

we derive henceforth, applying Cauchy-Schwartz's inequality

$$
(**) \quad |x^{1/2}f^{(l)}(x)| \leq (\frac{1}{2})^{l} \int_{0}^{\infty} y^{1/2} |(\mathcal{H}_0 f)(y)| \, dy 
$$

$$
\leq (\frac{1}{2})^{l-1} (\|y^{1/2}\mathcal{H}_0 f\|_{Y_0} + \|y^{(l+2)/2}\mathcal{H}_0 f\|_{Y_0}) 
$$

$$
= (\frac{1}{2})^{l-1} (\|y^{1/2}f^{(l)}\|_{Y_0} + \|y^{(l+2)/2}f^{(l+2)}\|_{Y_0}) .
$$

Further, for $k \in \mathbb{N}$, $l \in \mathbb{N}_0$,

$$
(***) \quad |x^{(k+1)/2}f^{(l)}(x)|^2 = 2 \, \text{Re} \int_{0}^{\pi} \{x^{(k+1)/2}f^{(l)}(\xi) \} \cdot \{(k + l)/2 \, \xi^{(k+1)/2} f^{(l+1)}(\xi)\} \, d\xi
$$

$$
+ \xi^{(k+1)/2} f^{(l+1)}(\xi) \} \, d\xi 
$$

$$
\leq 2 \int_{0}^{\infty} \left( \frac{k + l}{2} \right) |\xi^{(k+1)/2} f^{(l)}(\xi)|^2 + |\xi^{(k+1)/2} f^{(l)}(\xi)| \, d\xi 
$$

$$
\leq (k + l) \|x^{(k+1)/2}f^{(l)}\|_{Y_0}^2 + \|x^{(k+1)/2}f^{(l)}\|_{Y_0} \|x^{(k+1)/2}f^{(l+1)}\|_{Y_0} .
$$

Starting from the inequalities $(*)$, $(**)$ and $(***)$ and applying some straightforward manipulations the following results turn out valid.

(3.8) **Theorem**

Let $\alpha > 0$, $\beta > 0$ and let $f$ be an infinitely differentiable function on $\mathbb{R}^+$. 

28
I \( f \in G_{2\alpha} \) iff there exist \( A, B_l > 0 \) depending only on \( f \) and \( l \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x>0} |x^{(k+l)/2} f^{(l)}(x)| \leq A^k B_l (k!)^\alpha.
\]

II \( f \in G^{2\beta} \) iff there exist \( A_k, B > 0 \) depending only on \( f \) and \( k \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x>0} |x^{(k+l)/2} f^{(l)}(x)| \leq A_k B^l (l!)^\beta.
\]

III \( f \in G^{2\beta} \) iff there exist \( A, B, C > 0 \) such that for all \( k, l \in \mathbb{N}_0 \),
\[
\sup_{x>0} |x^{(k+l)/2} f^{(l)}(x)| \leq C A^k B^l (l!)^\alpha (l!)^\beta.
\]

Here in addition we assume that \( \alpha + \beta \geq 1 \). \( \square \)
References


