On the nonexistence of periodic tilings with cubistic cross-polytopes
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1. Introduction

The existence of periodic tilings of n-space \( n \geq 2 \) with congruent cubistic analogues of the cross-polytopes of radius \( e \geq 1 \) has been investigated by several authors:

GOLOM B and WEL CH ([6]) gave examples for the case \( n = 2 \) (\( e \) arbitrary) and for the case \( e = 1 \) (\( n \) arbitrary). They also proved the nonexistence for the parameter values \( (n,e) = (3,2) \) and \( n \geq 3 \), \( e \geq \rho_n \) for some function \( \rho_n \) of \( n \), the numerical values of which were not known. They conjectured that \( \rho_n = 2 \) for all \( n \geq 3 \).

Explicit bounds were found by POST ([9], [10]) who showed among other things, that \( \rho_n = 2 \) (\( 3 \leq n \leq 6 \)), \( \rho_7 \leq 3 \), \( \rho_n \leq \frac{3\sqrt{2} - 4\sqrt{3} - 2}{2} \) (\( n \geq 8 \)). Cf. also Müller ([8]).

Further research was done by ASTOLA ([1], [2]), BASSALYGO ([3]) and LENSTRA ([7]), but the results obtained by these authors all concern tilings with periods having a specific kind of prime decomposition.

In this paper we shall show among other things that \( \rho_n \leq 3 \) (\( 8 \leq n \leq 10 \)) and that \( \frac{\rho_n}{n} \leq 0,3735 \) as \( n \to \infty \).

2. Basic concepts. Generating functions

The cubistic cross-polytope of radius \( e \geq 0 \) in \( \mathbb{R}_n \), with center
\[ (x_1, \ldots, x_n) \in \mathbb{Z}_n \] is defined to be the union of all unit cubes centered in those points \((y_1, \ldots, y_n) \in \mathbb{Z}_n\) that satisfy the inequality
\[ \sum_{i=1}^{n} |x_i - y_i| \leq e. \]

A sketch of cubistic cross-polytopes for \( n = 2 \) and for \( n = 3 \) is given in Figure 1.

The volumes \( S_{n,e} \) of these polytopes have the generating function (cf. [4], [9])
\[ S_n(z) := \sum_{e=0}^{\infty} S_{n,e} z^e = \frac{(1+z)^n}{(1-z)^{n+1}} \]
and the numbers of boundary cubes $B_{n,e} := S_{n,e} - S_{n,e-1}$ satisfy the relation

$$B_{n}(z) := \sum_{e=0}^{\infty} B_{n,e} z^e = \frac{(1+z)^n}{(1-z)^n}.$$

Finally, we observe that the function $S(z,w)$ defined by

$$S(z,w) := \sum_{n=0}^{\infty} \sum_{e=0}^{\infty} S_{n,e} z^n w^e = \frac{1}{1-z-w-zw},$$

is a symmetric function in $z$ and $w$, so that

$$\sum_{n=0}^{\infty} S_{n,e} w^n = \frac{(1+w)^e}{(1-w)^{e+1}}.$$

This last identity enables us to find a recursive relation for $S_{n,e}$ ($e$ fixed), viz.

$$S_{0,e} = 1$$

$$S_{1,e} = 2e + 1$$

$$(n+1)S_{n+1,e} = (2e+1)S_{n,e} + nS_{n-1,e} \quad (n \geq 1).$$

For a discussion of these properties see [9].

Now let us consider some fundamental tiling properties that have to be explored:

i) Cubistic cross-polytopes are unions of unit cubes. For a tiling, of course, in every vertex point $2^n$ of these cubes must meet.

ii) Given a vertex point, the cubes belonging to a fixed cubistic cross-polytope, that share this vertex point, exhibit a metric structure that can be seen as a Hamming sphere in $\{0,1\}^n$, when we identify in the natural way the centers of the $2^n$ cubes meeting in that vertex point with the binary sequences of length $n$. For example, the type of a vertex point in a cubistic cross-polytope, i.e. the number of cubes meeting in that vertex point is one of the numbers

$$\sum_{j=0}^{t} \binom{n}{j} \quad (t = 0, \ldots, n-1) \quad (\text{see Fig. 2}).$$

Combination of i) and ii) indicates that it is important to know how many Hamming spheres of various sizes can be combined to form a decomposition of $\{0,1\}^n$. 
We conclude this section by giving the generating function for the number of vertex points of various types in one orthant of a given cubistic cross-polytope (see [9]).

Let $q_{n,e,t}$ denote the number of vertex points of type $\sum_{j=0}^{t} \binom{n}{j}$ in one orthant of a cubistic cross-polytope of radius $e$ in $n$-space. Then

$$\sum_{e=0}^{\infty} q_{n,e,t} z^e = \frac{z^t}{(1-z)^n}.$$ 

3. Optimal packing of Hamming spheres in $\{0,1\}^8$

Hamming spheres in binary 8-space have the following volumes:

<table>
<thead>
<tr>
<th>radius</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>volume</td>
<td>1</td>
<td>9</td>
<td>37</td>
<td>93</td>
<td>163</td>
<td>219</td>
<td>247</td>
<td>255</td>
</tr>
</tbody>
</table>

A packing $\Pi$ of $\{0,1\}^8$ is defined to be a partition of $\{0,1\}^8$ into disjoint Hamming spheres.

A packing $\Pi$, consisting of $m_i$ spheres of radius $i$ ($i = 0, \ldots, 7$) say, is called optimal if no packing exists with $n_i$ spheres of radius $i$, satisfying $n_0 < m_0$, $n_1 > m_1$ and $n_i = m_i$ ($i \neq 0,1$). We shall investigate systematically, which combinations $(m_i)_{i=0}^{7}$ admit optimal packings.

I. 255.1 (i.e. $m_7 = 1$, $m_0 = 1$, $m_i = 0$ ($i \neq 0,7$)) trivial, take antipodal centers.

II. 247.9 ($m_6 = 1$, $m_1 = 1$, $m_i = 0$ ($i \neq 1,6$)) trivial, antipodal centers.

IIIa. 219.37 trivial, antipodal centers.

IIIß. 219.9.28 ($m_5 = 1$, $m_1 = 1$, $m_0 = 28$, $m_i = 0$ ($i \neq 0,1,5$)). Assume that $(1 1 1 1 1 1 1 1)$ is the radius-5-center. All radius-1-centers must have weight $\leq 1$, hence cannot have mutual distance $\geq 3$. So $m_1 \leq 1$. Obviously, a packing with the given $m_i$-values exists.

IVa. 163.93. Trivial, antipodal centers.

IVß. 163.37.56. Let $(1 1 1 1 1 1 1 1)$ be the radius-4-center. Radius-2-centers must have weight $\leq 1$ (so there is at most one radius-2-center), and w.l.o.g. we have to distinguish between the following two cases:

i) $(r = 4)$ (1 1 1 1 1 1 1 1)
   $(r = 2)$ (1 0 0 0 0 0 0 0)

ii) $(r = 4)$ (1 1 1 1 1 1 1 1)
    $(r = 2)$ (0 0 0 0 0 0 0 0).
In case i) let an additional radius-1-center have weight $a$ in the first coordinate, weight $b$ in the remaining 7 coordinates. Then we get the inequalities
\[
\begin{align*}
(1-a) + (7-b) &\geq 6, \\
(1-a) + b &\geq 4,
\end{align*}
\]
so $9 - 2a \geq 10$, a contradiction.

In case ii) let a radius-1-center have weight $a$. The resulting inequalities
\[
\begin{align*}
8 - a &\geq 6, \\
a &\geq 4
\end{align*}
\]
yield $8 \geq 10$, which is impossible.

So there cannot be an additional radius-1-sphere.

IVy. 163.9.4.57. Assuming that $(1 1 1 1 1 1 1 1)$ is the radius-4-center we must locate the radius-1-centers in points of weight $\leq 2$. In order to keep their distances $\geq 3$ these points must have disjoint supports, and at most one of them can have weight 1. So there can be 4 radius-1-spheres and no more.

Va. 93.2.70. Assume that $(1 1 1 1 1 1 1 1)$ is radius-3-center. The other radius-3-center must have weight 1 or 0, and we apply the same method as we did in IV.

Case i) yields for an additional radius-1-center
\[
\begin{align*}
(1-a) + (7-b) &\geq 5, \\
(1-a) + b &\geq 5
\end{align*}
\]
so $9 - 2a \geq 10$, a contradiction.

Case ii) implies
\[
\begin{align*}
8 - a &\geq 5, \\
a &\geq 5
\end{align*}
\]
hence $8 \geq 10$, a contradiction.

V8. 93.37.9.63. Assume that $(1 1 1 1 1 1 1 1)$ is the radius-3-center. Radius-2-centers then must have weight $\leq 2$. so only one of them can be located, and without loss of generality there are three cases to be considered. We still use the same kind of argumentation as we did in IV.

Case i) $(r = 3) (1 1 1 1 1 1 1)$
$(r = 2) (1 1 0 0 0 0 0 0)$
$(r = 1) a b$ (weights)

Now
\[
\begin{align*}
(2-a) + (6-b) &\geq 5, \\
(2-a) + b &\geq 4
\end{align*}
\]
so $10 - 2a \geq 9$,

and the solutions are $(a,b) = (0,2), (0,3)$. Because $A(6,4,2) + A(6,4,3) = 3 + 4 = 7$ (cf. [5]) we see that at most 7 radius-1-spheres are possible in this case.
Case ii) \[(r = 2) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \]
\[(r = 2) \ (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \]
\[(r = 1) \ a \quad b \quad \text{(weights)} \]

We find
\[
\begin{align*}
(1-a) + (7-b) &\geq 5 \\
(1-a) + b &\geq 4
\end{align*}
\]

with one single solution \((a, b) = (0, 3)\).

According to [5], \(A(7,4,3) = 7\), and the incidence matrix of the Fano-plane locates the 7 radius-1-centers.

Case iii) \[(r = 3) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \]
\[(r = 2) \ (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \]
\[(r = 1) \ a \quad \text{(weight)} \]

Hence
\[
\begin{align*}
8-a &\geq 5 \\
a &\geq 4
\end{align*}
\]

\(V_y. \ 93.9^112.155\). Assume that \((1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)\) is the radius-3-center. So all radius-1-centers must have weight \(\leq 3\). According to [11] (cf. [5]) a maximal set of weight-3 radius-1-centers with distances \(\geq 3\) contains 8 elements and admits a unique maximal (i.e. size 4) set of weight-2 radius-1-centers under the distance \(\geq 3\) condition.

Example.

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

Obviously, only one radius-1-center can have weight 0 or weight 1. However, such a center would not allow a maximal weight-2-set. So the packing is apparently optimal.

\(V_y. \ 37^4.1108\). Three radius-2-centers in \(\{0,1\}^8\) always exhibit the distance pattern 5,5,6, but allow a fourth radius-2-center uniquely. W.l.o.g. we may locate the centers

\[
\begin{align*}
(r = 2) \ & (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \\
(r = 2) \ & (1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \\
(r = 2) \ & (0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0) \\
(r = 2) \ & (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1)
\end{align*}
\]
Let \((r = 1)\) a b c (weights).

This yields the inequalities

\[
\begin{align*}
(2 - a) + (3 - b) + (3 - c) &\geq 4 \\
(2 - a) + b + c &\geq 4 \\
a + (3 - b) + c &\geq 4 \\
a + b + (3 - c) &\geq 4 .
\end{align*}
\]

From the last three inequalities it follows that \(c \geq 1, a \geq 1, b \geq 2,\) a contradiction with the first inequality.

VI\(\beta. \) 37' .9' .64. Without loss of generality we may assume (see VI\(\alpha\)) the radius-2-centers to be located as follows:

\[
\begin{align*}
(r = 2) &\quad (111 1111111) \\
(r = 2) &\quad (1100000000) \\
(r = 2) &\quad (001110000)
\end{align*}
\]

let \((r = 1)\) a b c (weights) .

Hence

\[
\begin{align*}
(2 - a) + (3 - b) + (3 - c) &\geq 4 \\
(2 - a) + b + c &\geq 4 \\
a + (3 - b) + c &\geq 4 ,
\end{align*}
\]

and we get the solutions

\[
\begin{array}{c|cccccc}
a & 1 & 0 & 0 & 0 & 1 & 0 \\
b & 1 & 1 & 0 & 0 & 0 & 1 \\
c & 2 & 2 & 2 & 3 & 3 & 3 \\
\end{array}
\]

\(\uparrow\) \(\uparrow\) \(\uparrow\)

Class (1) (2) (3)

Since all points in class (3) have relative distances \(\leq 2\) only one radius-1-center can be chosen in class (3).

Class (2) can contain at most 3 radius-1-centers that w.l.o.g. can be located in the points

\[
\begin{align*}
(0 0 1 0 0 0 1 1) \\
(0 0 0 1 0 1 0 1) \\
(0 0 0 1 1 1 0 0).
\end{align*}
\]

In the same way, class (1) can contain at most 2 * 3 centers, that however can be chosen compatible with the choice made above in an essentially unique way.
Since this configuration is unique and does not leave place for a center in class (3) the optimality of the given packing is proved.

VIy. 37, 9, 14, 56. Now we have to consider four cases: the radius-2-centers can have distance 5, 6, 7 or 8.

Case i) \((r = 2) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)\) 
\((r = 2) \ (1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0)\)

let \((r = 1) \ a \ b \) (weights).

The resulting inequalities

\[(3 - a) + (5 - b) \geq 4\]
\[(3 - a) + b \geq 4\]

have the solutions

\[
\begin{array}{c|cccccc}
  a & 0 & 0 & 0 & 0 & 1 & 1 \\
  b & 1 & 2 & 3 & 4 & 2 & 3 \\
  C_2 & 4 & 1 & 3 & 0 & 1 & 0 \\
  C_3 & 0 & 3 & 1 & 4 & 0 & 1 \\
\end{array}
\]

In this diagram the entries \(C_2\) and \(C_3\) denote the number of points in \(\{0, 1\}^8\) that have weight 0 in the first 3 coordinates, weight 2 resp. 3 in the last 5 coordinates, that are covered by radius-1-spheres centered in the points given by \((a,b)\).

Let \(P_0\) be the number of radius-1-centers chosen with \(a = 0\), and \(P_1\) the corresponding number with \(a = 1\). Addition of the \(C_2\) and the \(C_3\) values yields

\[4P_0 + P_1 \leq \binom{5}{2} + \binom{5}{3} = 20.\]

On the other hand, since a maximal set of weight-2 and weight-3 vectors in \(\{0, 1\}^5\) that have minimum distance \(\geq 3\), contains 4 elements, we see in addition that

\[P_1 \leq 3 \times 4 = 12.\]

Hence \(P_0 + P_1 \leq 14.\)
Case ii) \( (r = 2) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \)
\( (r = 2) \ (1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \)

Let \( (r = 1) \ a \ b \) (weights)

Now the inequalities are
\[(2 - a) + (6 - b) \geq 4\]
\[(2 - a) + b \geq 4\]

and have the solutions
\[
\begin{array}{c|cccc}
  a & 1 & 0 & 0 & 0 \\
  b & 3 & 3 & 2 & 4 \\
  C_3 & 1 & 1 & 4 & 4 \\
\end{array}
\]

The corresponding inequalities are
\[(1 - a) + (7 - b) \geq 4\]
\[(1 - a) + b \geq 4\]

and have the solutions \((a, b) = (0, 3), (0, 4)\). Let a set of radius-1-centers be chosen. Replace the first coordinate of every center by a parity bit.

Then we get a set of weight-4-vectors, distance \( \geq 4 \). Since \( A(8, 4, 4) = 14 \) (cf. [5]) the result follows.

Case iii) \( (r = 2) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \)
\( (r = 2) \ (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \)

Let \( (r = 1) \ a \ b \) (weights)

Here \( C_3 \) denotes the number of points having weight 0 in the first 2 coordinates, weight 3 in the remaining coordinates, that are covered by a radius-1-sphere centered in a point of given \((a, b)\) weight.

Let \( P_i \) be the number of centers chosen in class i) \((i = 1, 2)\). Counting weight-3 vectors on the last 6 coordinates yields

\[ P_1 + 4P_2 \leq \binom{6}{3} = 20. \]

On the other hand, since \( A(6, 4, 3) = 4 \) (cf. [5]) we obtain

\[ P_1 \leq 2 \times 4 + 1 \times 4 = 12, \]

so that \( P_1 + P_2 \leq 14. \)

Case iv) \( (r = 2) \ (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \)
\( (r = 2) \ (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \)

The corresponding inequalities are
\[(1 - a) + (7 - b) \geq 4\]
\[(1 - a) + b \geq 4\]

and have the solutions \((a, b) = (0, 3), (0, 4)\). Let a set of radius-1-centers be chosen. Replace the first coordinate of every center by a parity bit. Then we get a set of weight-4-vectors, distance \( \geq 4 \). Since \( A(8, 4, 4) = 14 \) (cf. [5]) the result follows.
Now all radius-1-centers must have weight 4. Because $A(8,4,4) = 14$ (cf. [5]) we are able to locate 14 centers and no more.

This finally proves the optimality of $37^2.9.14.56$.

VI. Let the radius-2-center be chosen in $(111111111)$. Then the radius-1-centers all must have weight $\leq 4$. Choose one of these centers in $(000000000)$: The others must have weight 3 or 4. We add a parity bit and observe that $A(9,4,4) = 19$ (cf. [5]). So apparently 19 radius-1-centers can be located altogether. Since $A(8,3) = 20$ (cf. [5]) we cannot locate 20 centers besides the radius-2-center.

VII. A consequence of the fact that $A(8,3) = 20$ (cf. [5]).

Those packings of Hamming spheres in $\{0,1\}^8$ that are not optimal, are called suboptimal.

Altogether there are 15 optimal and 87 suboptimal combinations $(m_i)_{0}^{7}$.

4. Counting types by combinations in a period box in $\mathbb{R}_8$. An inductive principle

Referring to the classification of combinations introduced in section 3 we form the matrix $M$ as follows

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>255</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>247</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>219</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>163</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>93</td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>37</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>28</td>
<td>56</td>
<td>57</td>
<td>70</td>
<td>73</td>
<td>55</td>
<td>108</td>
</tr>
</tbody>
</table>

In every column the combination of an optimal packing is indicated by its contribution to the number of various types. Zeros are not written, and the suboptimal combinations (column $S$) are registered by an asterisk only, and not given in detail, since they do not play a significant role in the sequel.

For any given set $V$ of vertices of a tiling in $\mathbb{R}_8$ with cubistic cross-polytopes the inventory $\underline{t}$ of types can be expressed in terms of the matrix $M$ and the inventory $\underline{c}$ of combinations,

$$M\underline{c} = \underline{t},$$

($\underline{c}$ is a column vector of dimension 102, $\underline{t}$ a column vector of dimension 8).
Left-multiplication of this matrix equality by the row vector
\([-3, 5, 49, 105, -105, -49, -5, 3]\) transforms its left hand side into an obviously nonnegative expression (the optimal combinations give a nonnegative, the suboptimal combinations even a positive contribution each), so that apparently on the right hand side for the inventory of types we must have

\[3(t_1 - t_{255}) - 5(t_9 - t_{247}) - 49(t_{37} - t_{219}) - 105(t_{93} - t_{163}) \geq 0.\]

Let us now choose for \(V\) the set of vertices within a period box of a periodic tiling. Because of the equicontribution of all orthants of the cubistic cross-polytopes to the inventory of types in a period box the above inequality reduces to a necessary condition for the existence of a periodic tiling in \(\mathbb{R}_8^n\), i.e.

\[3(q_1 - q_{255}) - 5(q_9 - q_{247}) - 49(q_{37} - q_{217}) - 105(q_{93} - q_{163}) \geq 0 ,\]

where \(q_i :=\) the number of vertices of type \(i\) in one orthant of the cubistic cross-polytopes.

**Remark.** The choice of the special row vector for left-multiplication was made according to a linear programming argument in order to get a strong nonexistence result.

It should be kept in mind that the arguments in this section can be used only in \(\mathbb{R}_8^n\), but refer to periodic tilings with cubistic cross-polytopes, irrespective of the fact, whether these polytopes are congruent or not. This is very important, since a periodic tiling with congruent cubistic cross-polytopes in \(\mathbb{R}_n^n\) (\(n \geq 8\)) induces periodic tilings with the same period, but with not necessarily congruent cubistic cross-polytopes in an ensemble of 8-spaces. Those 8-spaces namely, that are obtained by keeping \((n - 8)\) integer-valued coordinates in \(\mathbb{R}_n^n\) fixed. We only have to check how many 8-dimensional cubistic cross-polytopes of different sizes are induced by a given radius-\(e\) cubistic cross-polytope in \(\mathbb{R}_n^n\) (\(n \geq 8\)) in order to get a correct value for \(q_i\) in the last inequality.

To be more specific: Every radius-\(e\) cubistic cross-polytope in \(\mathbb{R}_n^n\) induces \(B_{n-8,s}\) cubistic cross-polytopes of radius \(s\) in 8-space \((0 \leq s \leq e)\). This modifies the generating function of the number of vertex points of type \(\sum_{j=0}^{t} \binom{8}{j}\) pro orthant in the induced 8-dimensional sections of a radius-3 polytope in \(n\)-space to the form
\[ \sum_{e=0}^{\infty} q_{n,e} z^e = \frac{z^t (1+z)^{n-8}}{(1-z)^n}. \]

Now it is interesting to analyze the generating function of the expression in the quantities \( q_1, \ldots, q_{255} \), the nonnegativity of which is a necessary condition for the existence of a tiling.

This function has the form

\[
\frac{3(1-z^7) - 5(z^6 - 6) - 49(z^5 - 5) - 105(z^4 - 4)}{(1-z)^n} (1+z)^{n-8} = \frac{-4S_{n-2}}{(1-z)^6} + 10S_{n-4} \frac{(1+z)^4}{(1-z)^4} - 4S_{n-6} \frac{(1+z)^2}{(1-z)^2} + 1 \frac{(1+z)^{n-8}}{(1-z)^n-7} =
\]

\[
= -4S_{n-2} (1+z) + 10S_{n-4} (1+z) - 4S_{n-6} (1+z) + S_{n-8} (1+z).
\]

Hence, a necessary condition for the existence of a periodic tiling of \( \mathbb{R}_n \) (\( n \geq 8 \)) with radius-\( e \) cubistic cross-polytopes is

\[
-4S_{n-2,e} + 10S_{n-4,e} - 4S_{n-6,e} + S_{n-8,e} \geq 0.
\]

5. Computer results. Asymptotic estimates

In this section the last inequality of section 4 is applied.

For \( 8 \leq n \leq 50 \) it was found by computer that for a periodic tiling with radius-\( e \) polytopes \( e \leq \frac{10n - 19}{27} \) (so \( \rho_n \leq \frac{10n - 19}{27} \)). For large \( n \) and \( e \) asymptotic considerations justify that the successive volumes in this inequality are successive terms in a geometric progression. The equality

\[-4a^6 + 10a^4 - 4a^2 + 1 \geq 0 \]

is violated by \( a^2 \geq 2.07638 \). Now the recursive relation for \( S_{n,e} \) (cf. section 2) implies nonexistence when \( \frac{e}{n} \geq 0.3735 (n \to \infty) \). Hence \( \rho_n / n \leq 0.3735 (n \to \infty) \).

References


Figure 1. Cubistic Cross-polytopes for dimension 2 and 3.
Figure 2. Various type of vertex points in one orthant of a cubistic cross-polytope \((n = 3, e = 5)\)

\(\circ\): type 1, \(\bullet\): type 4, \(\oplus\): type 7.