A theory of generalized functions based on holomorphic semi-groups

de Graaf, J.

Published: 01/01/1983

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal ?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 12. Dec. 2018
A THEORY OF GENERALIZED FUNCTIONS BASED ON
HOLomorphic SEMI-GROUPS

Part C: Linear mappings, tensor products and kernel theorems.
CHAPTER 4. Characterization of continuous linear mappings between the spaces \( S_{X,A} \), \( T_{X,A} \), \( S_{Y,B} \) and \( T_{Y,B} \)

Let \( B \) be a nonnegative self-adjoint operator in a separable Hilbert space \( Y \). As before \( A \) is a nonnegative self-adjoint operator in a separable Hilbert space \( X \). In this chapter we shall derive conditions implying the continuity of linear mappings \( S_{X,A} \to S_{Y,B} \), \( S_{X,A} \to T_{Y,B} \), \( T_{X,A} \to T_{Y,B} \), \( T_{X,A} \to S_{Y,B} \).

Further we investigate which linear operators, defined on a subset of \( X \), can be continuously extended to operators on \( T_{X,A} \). The next theorem is an immediate consequence of the fact that \( S_{X,A} \) is bornological. Cf. Theorem 1.11. For completeness we give an ad hoc proof.

**Theorem 4.1.** A linear map \( Q : S_{X,A} \to V \), where \( V \) is an arbitrary locally convex topological vector space, is continuous

I. iff for each \( t > 0 \) the map \( Q e^{-tA} : X \to V \) is continuous;

II. iff for each null sequence \( \{u_n\} \subset S_{X,A} \), \( u_n \to 0 \) in \( S_{X,A} \), the sequence \( \{Q u_n\} \) is a null sequence in \( V \).

**Proof**

I. \( e^{-tA} \) is an isomorphism from \( X \) to \( X_t \). By the definition of the inductive limit, \( X_t \) is continuously injected in \( S_{X,A} \). So if \( Q \) is continuous it follows that \( Q e^{-tA} \) is continuous.

\( \Rightarrow \) Let \( Q_t \) denote the restriction of \( Q \) to \( X_t \). From the continuity of \( Q e^{-tA} \) on \( X \) follows the continuity of \( Q_t \) on \( X_t \). Now let \( \emptyset \subset 0 \) be open in \( V \). For each \( t > 0 \), \( Q_t^{-1}(\emptyset) \cap X_t = Q_t^{-1}(\emptyset) \) is an open \( 0 \)-neighbourhood in \( X_t \). Thus \( Q_t^{-1}(\emptyset) \) is open in \( S_{X,A} \).

II. Follows from I because null sequences in \( S \) are always null sequences in some \( X_t \), \( t > 0 \), and vice versa.

In the next theorem we characterize continuous linear mappings from \( S_{X,A} \) to \( S_{Y,B} \).
Theorem 4.2. Suppose $P : S_{X,A} \rightarrow S_{Y,B}$ is a linear mapping. The following seven conditions are equivalent.

I. $P$ is continuous with respect to the strong topologies of $S_{X,A}$ and $S_{Y,B}$.

II. $u_n \rightarrow 0$, strongly in $S_{X,A}$ implies $P u_n \rightarrow 0$, strongly in $S_{Y,B}$.

III. For each $\alpha > 0$ the operator $P e^{-\alpha A}$ is a bounded linear operator from $X$ into $Y$.

IV. For each $\alpha > 0$ and each $\psi \in B$ the operator $\psi(B) P e^{-\alpha A}$ is a bounded linear operator from $X$ into $Y$.

V. For each $t > 0$ there exists $\beta > 0$ such that $P e^{-tA}(X) \subset e^{-\beta B}(Y)$ and $e^{\beta B} P e^{-tA}$ is a bounded linear operator from $X$ into $Y$.

VI. There exists a dense linear subspace $\Xi \subset Y$ such that for each fixed $y \in \Xi$ the linear functional $L_{P,y}(f) = (P f, y)$ is continuous on $S_{X,A}$.

VII. For each $t > 0$ $(P e^{-tA})^*$ is a bounded linear operator from $Y$ to $X$.

(Remark: One has $(P e^{-tA})^*(Y) \subset S_{X,A}$).

Proof.

I $\Rightarrow$ II. See Theorem 4.1.

II $\Rightarrow$ III. If $\{x_n\}$ is a null sequence in $X$ then for any $\alpha > 0$ $\{e^{-\alpha A} x_n\}$ is a null sequence in $S_{X,A}$. By II $\{P e^{-tA} x_n\}$ is a null sequence in $S_{Y,B}$ and hence in $Y$.

III $\Rightarrow$ IV. The operator $\psi(B) P e^{-\alpha A}$ is closed and defined on the whole of $X$. Therefore it is bounded.

IV $\Rightarrow$ V. Let $U$ denote the unit ball in $X$. Because of IV the set $P e^{-\alpha A}(U)$ is bounded in $S_{Y,B}$. Now apply Theorem 1.6.

III $\Rightarrow$ VI. We can take $\Xi = Y$. Take $y \in Y$ fixed. For each $x \in X$ and each $t > 0$ we have

$$|L_{P,y}(e^{-tA} x)| = |(P e^{-tA} x, y)| \leq C_{t,y} \|x\|_X.$$  

Together with Theorem 4.1 the result follows.
According to Riess' theorem for each \( y \in E \) and each \( t > 0 \) there exists \( f_t \in X \) such that for each \( h \in X \)

\[
(*) \quad L_{p,Y}(e^{-tA}h) = (Pe^{-tA}h, y) \quad g = (h, f_t)_X
\]

Replacing \( h \) by \( e^{-tA}x \), \( x \in X \), we observe that \( f_{t+T} = e^{-tA}f_t \) so that \( f_t \in SX,A' \).

From (*) we obtain \( f_t = (Pe^{-tA})^* \). So \( D((Pe^{-tA})^*) \supseteq \Xi \) which is dense in \( Y \).

Since \( Pe^{-tA} \) is defined on the whole \( X \) the operator \( (Pe^{-tA})^* \) is defined on the whole \( Y \) and bounded. Repeating the argument with arbitrary \( y \in X \) shows \( (Pe^{-tA})^*y \in SX,A' \).

\[ VII \Rightarrow III. \quad Pe^{-tA} \text{ is bounded because } (Pe^{-tA})^* \text{ is bounded.} \]

\[ V \Rightarrow II. \quad \text{Trivial.} \]

The next corollary is important for applications.

**Corollary 4.3.** Suppose \( Q \) is a densely defined closable operator: \( X \to Y \). If \( D(Q) \to SX,A \) and \( Q(SX,A) \subseteq SY,S' \) then \( Q \) maps \( SX,A \) continuously into \( SY,S' \).

**Proof.** \( Q \) is closable iff \( D(Q^*) \) is dense in \( Y \). Since \( Ke^{-tA} : X \to Y \) is defined on the whole \( X \), its adjoint \( (Ke^{-tA})^* \) is bounded. The adjoint, however, is densely defined since on \( D(Q^*) \) one has \( (Ke^{-tA})^* = e^{-tA}Q^* \). Hence \( (Ke^{-tA})^* \) is defined on the whole \( Y \) and bounded. From this the boundedness of \( Ke^{-tA} \) follows. Application of Theorem 4.2 III yields the desired result.

**Theorem 4.4.** Let \( K : SX,A \to SY,S \) be a linear mapping. The following three conditions are equivalent.

I. \( K \) is continuous with respect to the strong topologies of \( SX,A \) and \( SY,S' \).

II. For each \( t > 0, a > 0, e^{-tB}Ke^{-aA} \) is a bounded linear operator from \( X \) into \( Y \).

III. For each \( t > 0, e^{-tB}K \) is a continuous map from \( SX,A \) into \( SY,S' \).

**Proof.**

I \( \Rightarrow II. \) Let \( \{x_n\} \) be a null-sequence in \( X \). Then \( \{e^{-aA}x_n\} \) is a null sequence in \( SX,A \). Since \( K \) is continuous, \( \{Ke^{-aA}x_n\} \) is a null sequence in \( SY,S' \), which
means that for every $t > 0$ \{(e^{-tB} Ke^{-A}x_n)\} is a null sequence in $Y$. Hence $e^{-YB} Ke^{-A}$ is bounded.

II $\Rightarrow$ I. Let $\{u_n\}$ be a null-sequence in $S_{X,A}$. Then for some $\alpha > 0$ \{e^{\alpha A}u_n\} is defined and is a null-sequence in $X$. But then $\{Ku_n\} = \{Ke^{-\alpha A}e^{\alpha A}u_n\}$ is a null-sequence in $T_{Y,B}$ since for each $t > 0$

$$(Ke^{-\alpha A}e^{\alpha A}u_n)(t) = (e^{-tB} Ke^{-\alpha A})e^{\alpha A}u_n \to 0 \text{ in } Y.$$ 

II $\Rightarrow$ III. Apply Theorem 4.2 with $P = e^{-tB}K$. 

Theorem 4.5. Let $\Gamma : T_{X,A} \to S_{Y,B}$ be a linear mapping. Let $\Gamma' : X \to Y$ denote the restriction of $\Gamma$ to $X$. The following five conditions are equivalent.

I. $\Gamma$ is continuous with respect to the strong topologies of $T_{X,A}$ and $S_{Y,B}$.

II. $\Gamma^* : Y \to X$ is a bounded operator and $\Gamma^*(Y) \subset S_{X,A}$.

III. There exists $t > 0$ such that $\Gamma^*(X) \subset e^{-tA}(X)$ and $e^{tA}\Gamma^*$ is a bounded operator from $Y$ into $X$.

IV. There exists $t > 0$ such that $\Gamma e^{tA}$ with domain $X_c \subset X$ is bounded as an operator from $X$ into $Y$.

V. There exists $t > 0$ and a continuous linear map $Q : S_{X,A} \to S_{Y,B}$ such that $\Gamma = Qe^{-tA}$.

Proof.

I $\Rightarrow$ II. From the continuity of $\Gamma$ it immediately follows that $\Gamma$ is continuous. Its adjoint $\Gamma^*$, defined by $(x,\Gamma^*y)_X = (\Gamma x,y)_Y$ for all $x \in X$, $y \in Y$, is a bounded operator as well. Now consider the dual operator $\Gamma' : T_{Y,B} \to S_{X,A}$ defined by

$$<F,\Gamma'G>_X = <\Gamma F,G>_Y \text{ for all } F \in T_{X,A}.$$ 

For fixed $G \in T_{Y,B}$ the right-hand side defines a continuous linear functional on $T_{X,A}$. $\Gamma^*$ is a restriction of $\Gamma'$ and therefore maps $Y$ into the set $S_{X,A} \subset X$.

II $\Rightarrow$ III. For each $\psi \in B$ the operator $\psi(A)\Gamma^* : Y \to X$ is bounded and $\psi(A)\Gamma^*(Y) \subset S_{X,A}$. Similar to the method employed in Theorem 4.2 the result follows.
Theorem 4.6. Let $\phi : T_{x,A} \to T_{y,B}$ be a linear mapping. Let $\phi_x : X \to T_{y,B}$ denote the restriction of $\phi$ to $X$.

The following six conditions are equivalent.

I. $\phi$ is continuous with respect to the strong topologies of $T_{x,A}$ and $T_{y,B}$.

II. For each $g \in S_{y,B}$ the expression $\langle g, \phi F \rangle_Y, F \in T_{x,A'}$ is a continuous linear functional on $T_{x,A'}$.

III. For each $t > 0$ the operator $e^{-tB}\phi$ is a continuous map from $T_{x,A}$ into $S_{y,B}$.

IV. For each $t > 0$ $(e^{-tB}\phi_x)^*(Y) \subset S_{x,A'}$.

V. For each $t > 0$ there exists $\beta > 0$ such that $(e^{-tB}\phi_x)^*(Y) \subset e^{-\beta A}(X)$ and $e^{\beta A}(e^{-tB}\phi_x)^*$ is a bounded operator from $Y$ into $X$.

VI. For each $t > 0$ there exists $\beta > 0$ such that $e^{-tB}\phi_x e^{\beta A} = e^{-tB}\phi_x e^{\beta A}$ on the domain $x_\beta \subset X$ is bounded as an operator from $X$ into $Y$.

Proof.

I $\implies$ II. Trivial.

I $\implies$ III. Trivial, because $e^{-tB} : T_{y,B} \to S_{y,B}$ is continuous.

III $\implies$ IV. Theorem 4.5, condition II.

IV $\implies$ V. Apply Theorem 4.5 to $e^{-tB}\phi_x$.

V $\implies$ VI. The adjoint of the bounded operator $e^{\beta A}(e^{-tB}\phi_x)^*$ is an extension of $(e^{-tB}\phi_x)e^{\beta A}$. Therefore the latter is bounded.

VI $\implies$ I. Let $\{F_n\}$ be a null-sequence in $T_{x,A}$. Then for each $t > 0$ there exists $\beta > 0$ such that we can write $(e^{-tB}\phi_x)(t) = e^{-tB}\phi_x e^{\beta A} F_n(\beta)$. This converges to zero as $n \to \infty$ because of the boundedness of $e^{-tB}\phi_x e^{\beta A}$. 


\textit{\textbf{C.5}}
II \to V. \langle g, \Phi F \rangle_Y \text{ has the representation } \langle e, F \rangle_X', \text{ see Theorem 3.2.IV, here } \\
f = \phi' g \in S_{X,A}. \text{ Taking } F = u \in S_{X,A} \text{ we observe that } (u, \phi' g)_X = (u, \phi' g)_Y = \langle \Phi u, g \rangle_Y \text{ is, as a function of } g, \text{ a continuous linear functional on } S_{Y,B}. \\
\text{Then with Theorem 4.2.VI it follows that } \phi' \text{ maps } S_{Y,B} \text{ continuously into } \\
S_{X,A}. \text{ Then, by Theorem 4.2.V for each } t > 0 \text{ there exists } \beta > 0 \text{ such that } \\
e^{\beta t} \Phi e^{-tB} \text{ is a bounded operator. But } e^{\beta A} \Phi e^{-tB} = e^{\beta A}(e^{-tB} \Phi)' \text{ because for all } x \in X, y \in Y \\
(\Phi e^{-tB} y, x)_X = \langle e^{-tB} y, \Phi x \rangle_X = \langle y, e^{-tB} \Phi x \rangle_Y = \langle (e^{-tB} \Phi) y, x \rangle_X. \qedhere

Theorem 4.7. Let the linear mappings \\
P : S_{X,A} \to S_{Y,B} \quad \phi : T_{X,A} \to T_{Y,B} \\
\Gamma : T_{X,A} \to S_{Y,B} \quad K : S_{X,A} \to T_{Y,B}

be continuous with respect to the strong topologies on the mentioned spaces. \\
Then the dual linear mappings \\
P' : T_{Y,B} \to T_{X,A} \quad \phi' : S_{Y,B} \to S_{X,A} \\
\Gamma' : T_{Y,B} \to S_{X,A} \quad K' : S_{Y,B} \to T_{X,A}

are also continuous with respect to the strong topologies.

Proof.

Compare Theorem 4.2.V with Theorem 4.6.VI. \\
Compare Theorem 4.5.III with Theorem 4.5.IV. \\
Look at Theorem 4.4.II. \qedhere

The interesting question arises which densely defined (possibly unbounded) 
operators from X into Y can be extended to a continuous mapping from T_{X,A} 
into T_{Y,B}.

Theorem 4.8. Let E be a linear map X \to D(E) \to Y with \overline{D(E)} = X. E can be 
extended to a continuous linear map \overline{E} : T_{X,A} \to T_{Y,B} \text{ iff } E \text{ has a densely 
defined adjoint } \overline{E}^* : Y \to D(\overline{E}^*) \to S_{Y,B} \to X \text{ with } \overline{E}^*(S_{Y,B}) \subseteq S_{X,A}. 

Proof.

(*) If $\tilde{E}$ exists as a continuous map, its dual operator $\tilde{E}'$ maps $S_{Y,B}$ into $S_{X,A}$.
For each $x \in D(E)$ and $g \in S_{Y,B}$ one has $<g,Ex>_Y = (g,Ex)_Y = <E'g,x>_X = (E'g,x)_X$. It follows that $E^* = \tilde{E}'$ and $E^*(S_{Y,B}) \subset S_{X,A}$.

(**) From Corollary 4.3 it follows that $E^*$ maps $S_{Y,B}$ continuously into $S_{X,A}$. Then by Theorem 4.7 the dual $(E^*)'$ maps $T_{X,A}$ continuously into $T_{Y,B}$.
However, $(E^*)'$ is an extension of $E$. $\square$

Corollary 4.9. A continuous linear map $Q : S_{X,A} \rightarrow S_{Y,B}$ can be extended to a continuous linear map $\overline{Q} : T_{X,A} \rightarrow T_{Y,B}$ iff $Q$ has a Hilbert space adjoint $Q^*$ with $D(Q^*) = S_{Y,B}$ and $Q^*(S_{Y,B}) \subset S_{X,A}$.

CHAPTER 5. Topological tensor products of spaces of type $S_{X,A}, T_{X,A}$

For two separable Hilbert spaces $X$ and $Y$ we consider the complex vector space consisting of all Hilbert-Smidt operators $Z$ from $X$ into $Y$. We shall denote this vector space by $X \otimes Y$. For any $Z \in X \otimes Y$ and any orthonormal basis $\{e_i\} \subset X$ we have

$$||Z||^2 = \sum_{i=1}^{\infty} ||Ze_i||_Y^2 < \infty.$$ 

The double norm $||\cdot||$ does not depend on the choice of the orthonormal basis $\{e_i\}$. We introduce an inner product in $X \otimes Y$ by

$$(Z,K)_{X \otimes Y} = \sum_{i=1}^{\infty} (Ze_i,Ke_i)_Y.$$ 

Endowed with this inner product $X \otimes Y$ is a Hilbert space. See [RS] Ch. VIII.10. Examples of elements in $X \otimes Y$ are $\xi \otimes \eta, \xi \in X, \eta \in Y$, defined by $(\xi \otimes \eta)f = (f,\xi)_X\eta$, for all $f \in X$, and finite linear combinations of these: $\sum_{j=1}^{N} (\xi_j \otimes \eta_j)$ with $\xi_j \in X, \eta_j \in Y$. The linear subspace of $X \otimes Y$ which consists of all HS operators of the type just mentioned will be denoted by $X \otimes_a Y$, i.e. the (sesquilinear) algebraic tensor product of $X$ and $Y$. $X \otimes_0 Y$ may be regarded as the completion of $X \otimes_a Y$ with respect to the double norm.
Therefore $X \otimes Y$ is called the completed (sesquilinear) topological tensor product of $X$ and $Y$. For later reference we mention the following properties taken from [RS] Ch. VIII.

Properties 5.1.

(a) $\forall x \in X \forall y, \eta \in Y \quad (\xi \otimes \eta)_{X \otimes Y} = (x, \xi) \chi(\eta, y) \gamma$.

(b) $\forall \lambda \in \mathbb{C} \forall \xi \in X \forall \eta \in Y \quad \lambda (\xi \otimes \eta) = (\bar{\lambda} \xi) \otimes \eta = \xi \otimes (\lambda \eta)$.

Thus the canonical mapping $X \times Y \to X \otimes Y$, defined by $[x; y] \to x \otimes y$ is anti-linear in $x$ and linear in $y$.

(c) $\forall z \in X \otimes Y \forall x \in X \forall y \in Y \quad (z, x \otimes y)_{X \otimes Y} = (Zx, y) \gamma$.

Let $H$ respectively $J$ denote bounded linear operators on $X$, respectively $Y$ into themselves. $H \otimes J$ denotes the linear mapping of $X \otimes Y$ into itself defined by $(H \otimes J)(x \otimes y) = (Hx) \otimes (Jy)$ and linear extension, followed by continuous extension.

(d) The uniform operator norms of $H$, $J$ and $H \otimes J$ are related by $\| H \otimes J \| = \| H \| \| J \|$.

(e) $\forall z \in X \otimes Y \quad (H \otimes J)z = JzH^\dagger$.

(f) $H$ and $J$ injective $\Rightarrow H \otimes J$ is injective.

The theory of closable tensor products of unbounded closable operators and the description of their properties in terms of corresponding properties of their factors presents greater difficulties. Only rather recently significant results have been attained [T].

Definition 5.2. Let $A$ with domain $D(A)$ be a densely defined closed linear operator in $X$. Let $B$ with domain $D(B)$ be the same in $Y$. On $D(A) \otimes A (B) \subset X \otimes Y$ we introduce the operator $A \otimes A I + I \otimes A B$ by $(A \otimes A I + I \otimes A B)(x \otimes y) = (Ax) \otimes y + x \otimes (By)$ and linear extension. This extension is well defined and closable, [RS] p. 298.
Lemma 5.3. Let $A$ respectively $B$ be self-adjoint operators in $X$ respectively $Y$.

I. $A \otimes I + I \otimes B$ is essentially self-adjoint in $X \otimes Y$. We denote the unique self-adjoint extension by $A \otimes I + I \otimes B$ or, briefly, $A \boxdot B$.

II. $A \geq 0$ and $B \geq 0$ implies $A \boxdot B \geq 0$.

Proof. As in [W] section 8.5.

Theorem 5.4. On $X \otimes Y$ we have for $t \geq 0$

$$e^{-t(A \boxdot B)} = e^{-tA} \otimes e^{-tB}.$$  

Proof. As in [W] section 8.5.

Applying the results of the preceding chapters we can introduce the spaces $S_{X \oplus Y, A \boxdot B}$, $T_{X \otimes Y, A \boxdot B}$ and, by taking $A = 0$ or $B = 0$, the spaces $S_{X \oplus Y, A \boxdot I}$, $T_{X \otimes Y, I \boxdot B}$ etc.

Definition 5.5. The canonical sesquilinear map $\otimes : S_{X, A} \times S_{Y, B} \rightarrow S_{X \oplus Y, A \boxdot B}$ is defined by $[u; v] \rightarrow u \otimes v$. Here the symbol $\otimes$ is the same as in Properties 5.1. This definition is consistent because for $u \in S_{X, A}$, $v \in S_{Y, B}$ there exist $x \in X$, $y \in Y$ and $t > 0$ such that $u = e^{-tA}x$, $v = e^{-tB}y$. Further, $u \otimes v = (e^{-tA} \otimes e^{-tB})(x \otimes y) = (e^{-tA}x) \otimes (e^{-tB}y)$, so that $u \otimes v \in S_{X \oplus Y, A \boxdot B}$.

Theorem 5.6. $S_{X \oplus Y, A \boxdot B}$ is a complete topological tensor product of $S_{X, A}$ and $S_{Y, B}$. By this we mean:

I. $S_{X \oplus Y, A \boxdot B}$ is complete.

II. The canonical sesquilinear map $\otimes : S_{X, A} \times S_{Y, B} \rightarrow S_{X \oplus Y, A \boxdot B}$ is continuous.

III. The span of the image of $\otimes$, i.e. the algebraic tensor product $S_{X, A} \otimes S_{Y, B}$ is dense in $S_{X \oplus Y, A \boxdot B}$.

Proof.

I. The completeness follows from Theorem 1.11.

II. It is enough to check the continuity of $\otimes$ at $[0; 0]$. Let $W$ be a convex open neighbourhood of $0$ in $S_{X \oplus Y, A \boxdot B}$. Then for each $t > 0$, $W \cap (X \otimes Y)_t$
is an open 0-neighbourhood in $(X \otimes Y)_t$ and it contains an open ball centered at 0 and radius $r_t'$, $0 < r_t' < 1$. In $X_t$ respectively $Y_t$ we introduce open balls $A_t$ respectively $B_t$ centered at 0 and with radius both $r_t'$. Let

$$A = \bigcup_{t>0} A_t \subset S_{X,A} \quad \text{and} \quad B = \bigcup_{t>0} B_t \subset S_{Y,B}.$$  

Then $\otimes$ maps $A \times B$ in $W$ since

$$\|x \otimes y\|_{\min(t, \tau)} \leq \|x\|_t \cdot \|y\|_\tau \leq r_{\min(t, \tau)}$$

whenever $x \in A$, $y \in B$. Let $\hat{A}$ respectively $\hat{B}$ denote the convex hulls of $A$ respectively $B$. Then $\otimes$ maps $\hat{A} \times \hat{B}$ in $W$. The set $\hat{A}$ is convex and $\hat{A} \cap X_t$ contains an open neighbourhood in $X_t$. From Theorem 1.4.II it follows that $\hat{A}$ contains an open set $U_{\psi,\varepsilon}$. Similarly $\hat{B}$ contains an open set $V_{\chi,\delta}$. We conclude that $\otimes$ maps $U_{\psi,\varepsilon} \times V_{\chi,\delta}$ into $W$.

III. For each $t > 0$, $X_t \otimes Y_t$ is dense in $(X \otimes Y)_t$. From this the desired result follows.

Remark. Our strong topology on $S_{X \otimes Y, A \otimes B}$ is, generally speaking, not the projective tensor product topology. Cf. [SCH] p. 93. Therefore the universal factorization property for continuous sesquilinear maps on this space does not hold in general.

Definition 5.7. The canonical sesquilinear map $\otimes : T_{X,A} \times T_{Y,B} \rightarrow T_{X \otimes Y, A \otimes B'}$ $[F;G] \rightarrow F \otimes G$, is defined by $(F \otimes G)(t) = F(t) \otimes G(t)$.

Here $\otimes$ is the same as in Properties 5.1. The definition is consistent because

$$(F \otimes G)(t + \tau) = e^{-\tau A} F(t) \otimes e^{-\tau B} G(t) =$$

$$= (e^{-\tau A} \otimes e^{-\tau B}) F(t) \otimes G(t) = (e^{-\tau A} \otimes e^{-\tau B}) (F \otimes G)(t).$$

Theorem 5.8. $T_{X \otimes Y, A \otimes B'}$ is a complete topological tensor product of $T_{X,A}$ and $T_{Y,B}$. By this we mean:

I. $T_{X \otimes Y, A \otimes B'}$ is complete.
II. The canonical sesquilinear map $\otimes : T_{X,A} \times T_{Y,B} \to T_{X\otimes Y, A\otimes B}$ is continuous.

III. The span of the image of $\otimes$, i.e. the algebraic tensor product $T_{X,A} \otimes a T_{Y,B}$ is dense in $T_{X\otimes Y, A\otimes B}$.

Proof.

I. The completeness follows from Theorem 2.5.

II. For each $t > 0$ we have $\|F(t) \otimes G(t)\|_{X\otimes Y} = \|F(t)\|_X \|G(t)\|_Y$. From this the continuity at $[0;0]$ follows.

III. $X \otimes_a Y$ is dense in $X \otimes Y$ which is dense in $T_{X\otimes Y, A\otimes B}$.

Now mixed sesquilinear topological tensor products of type $S_{X,A} \otimes T_{Y,B}$, $T_{X,A} \otimes S_{Y,B}$ will be considered. The notation of [ETH], Ch. II, will be used.

Definition 5.9. We introduce the following linear subspace of $T_{X\otimes Y, I\otimes B}$:

$$T(S_{X\otimes Y, A\otimes I}, I \otimes B) = \{\phi \in T_{X\otimes Y, I\otimes B} \mid \forall t > 0 \phi(t) \in S_{X\otimes Y, A\otimes I}\}.$$ 

In this space we take the topology generated by the semi-norms

$$\rho_{t, \psi}(\phi) = \|(\phi(A) \otimes I)\phi(t)\|_{X\otimes Y}, \quad t > 0, \quad \psi \in B_+.$$

Definition 5.10. The canonical sesquilinear map $\otimes : S_{X,A} \times T_{Y,B} \to T_{X\otimes Y, I\otimes B}$ is defined by

$$f \otimes G : t \mapsto f \otimes G(t).$$

(It is clear that $\forall t > 0 f \otimes G(t) \in S_{X\otimes Y, A\otimes I}$.)

Theorem 5.11. $T(S_{X\otimes Y, A\otimes I}, I \otimes B)$ is a complete topological tensor product of $S_{X,A}$ and $T_{Y,B}$. By this I mean

I. $T(S_{X\otimes Y, A\otimes I}, I \otimes B)$ is complete.

II. The canonical sesquilinear mapping $\otimes : S_{X,A} \times T_{Y,B} \to T(S_{X\otimes Y, A\otimes I}, I \otimes B)$ is continuous.
III. $S_{X,A} \otimes_{a} T_{Y,B}$ is dense in $T(S_{X \otimes Y, A \otimes I} I \otimes B)$.

Proof

I. Let $\{ \phi_\alpha \}$ be a Cauchy net. The $\alpha$'s belong to a directed set $D$. First take $\psi = 1$. $\phi_\alpha$ tends to a limit point $\phi \in T_{X \otimes Y, I \otimes B}$ because the latter is complete. It remains to show that, for each $t > 0$, $\phi(t) \in S_{X \otimes Y, A \otimes I}$.

For each $\psi \in B^+$ and each $t > 0$, $(\psi(A) \otimes I) \phi_\alpha = (\psi(A) \otimes I) \phi$ converges in $X \otimes Y$. From the closedness of $\psi(A) \otimes I$ it follows that $\phi(t) \in D(\psi(A) \otimes I)$. This is true for each $\psi \in B^+$ and therefore by Theorem 1.10, for each $t > 0$, $\phi(t) \in S_{X \otimes Y, A \otimes I}$.

II. Let $\psi \in B^+$ and let $t > 0$. Then for $f \in S_{X,A}$ and $G \in T_{Y,B}$

$$\| (\psi(A) \otimes I)(f \otimes G(t)) \|_{X \otimes Y} \leq \| \psi(A) \|_{X} \| G(t) \|_{Y} .$$

From this inequality the continuity of $\otimes$ follows.

III. Since $S_{X \otimes Y, A \otimes I}$ is dense in $T(S_{X \otimes Y, A \otimes I} I \otimes B)$ and since $S_{X,A} \otimes_{a} S_{Y,B}$ is dense in $S_{X \otimes Y, A \otimes I}$, the assertion follows.

Definition 5.12. We introduce the following linear subspace of $T_{X \otimes Y, A \otimes I}$:

$$T(S_{X \otimes Y, I \otimes B, A \otimes I}) = \{ P \in T_{X \otimes Y, A \otimes I} \mid \forall t > 0 \; P(t) \in S_{X \otimes Y, I \otimes B} \} .$$

In this space we take the topology generated by the semi-norms

$$\sigma_{t, \psi}(P) = \| (I \otimes \psi(B)) P(t) \|_{X \otimes Y}, \quad t > 0, \; \psi \in B^+ .$$

Definition 5.13. The canonical sesquilinear map

$$\otimes : T_{X,A} \times S_{Y,B} \rightarrow T_{X \otimes Y, A \otimes I}$$

is defined by

$$F \otimes g : t \rightarrow F(t) \otimes g .$$

(It is clear that $\forall t > 0 \; F(t) \otimes g \in S_{X \otimes Y, I \otimes B}$.)

The proof of the following theorem runs the same as the proof of Theorem 5.11.
Theorem 5.14. \( T(S_{X \otimes Y, I \otimes B}, A \otimes I) \) is a complete topological tensor product of \( T_{X, A} \) and \( S_{Y, B} \). Hence

I. \( T(S_{X \otimes Y, I \otimes B}, A \otimes I) \) is complete.

II. The canonical sesquilinear mapping

\[ \Theta : T_{X, A} \times S_{Y, B} \to T(S_{X \otimes Y, I \otimes B}, A \otimes I) \]

is continuous.

III. \( T_{X, A} \odot S_{Y, B} \) is dense in \( T(S_{X \otimes Y, I \otimes B}, A \otimes I) \).

Next I introduce a second type of mixed topological tensor product.

Definition 5.15. We introduce the following linear subspace of \( T_{X \otimes Y, A \otimes I} \):

\[ S(T_{X \otimes Y, A \otimes I}, I \otimes B) = \bigcup_{t > 0} (I \otimes e^{-tB}) (T_{X \otimes Y, A \otimes I}) \cdot \]

Each of the spaces \( (I \otimes e^{-tB}) (T_{X \otimes Y, A \otimes I}) \) can be written as \( T_{X \otimes e^{-tB}(Y), A \otimes I} \).
According to Chapter II they are Fréchet spaces. Their semi-norms are given by

\[ \eta \to \| (I \otimes e^{tB}) \eta \|_{X \otimes Y}, \quad \eta \in (I \otimes e^{-tB}) (T_{X \otimes Y, A \otimes I}), \quad n \in \mathbb{N}. \]

The space \( S(T_{X \otimes Y, A \otimes I}, I \otimes B) \) is an inductive limit of the spaces \( (I \otimes e^{-tB}) (T_{X \otimes Y, A \otimes I}) \). For its topology we take the inductive limit topology.

On \( T(S_{X \otimes Y, A \otimes I}, I \otimes B) \times S(T_{X \otimes Y, A \otimes I}, I \otimes B) \) we introduce the pairing

\[ \langle \phi, \rho \rangle_B = \langle \phi (\varepsilon), (I \otimes e^{tB}) \phi \rangle_{X \otimes Y} \]

for \( \varepsilon > 0 \) sufficiently small. Finally, we define the embedding of \( S(T_{X \otimes Y, A \otimes I}, I \otimes B) \) into \( T(S_{X \otimes Y, I \otimes B}, A \otimes I) \) by

\[ (\text{emb} F)(t) = (e^{-tA} \otimes I) F. \]
Theorem 5.16.

I. \( T_{X,Y,A} \oplus S_{Y,B} \) is dense in \( S(T_{X,Y},A \oplus I \oplus B) \).

II. For each fixed \( \phi \in T(S_{X,Y},A \oplus I \oplus B) \) the linear functional \( F + \phi, P_B \) is continuous on \( S(T_{X,Y},A \oplus I \oplus B) \).

III. The embedding described in Definitions 5.15 is continuous.

Proof.

I. \( S_{X,Y,A} \oplus S_{Y,B} \) is dense in \( S_{X,Y,A \oplus B} \) which is dense in \( S(T_{X,Y},A \oplus I \oplus B) \).

II. It is sufficient to prove the continuity for restrictions to each space \( (I \oplus e^{-tB})T_{X,Y, A \oplus I} \). We have

\[
\langle \phi, P_B \rangle = \langle (I \oplus e^{-tB}) \phi, (I \oplus e^{tB})F \rangle_{X,Y}.
\]

From this the continuity follows.

III. Let \( t > 0 \). Let \( \psi \in B_+ \). Let \( \alpha > 0 \) be fixed. Then for \( W \in T_{X,Y, A \oplus I} \)

\[
\| (e^{-tA} \circ \psi(B)) \|_{X,Y} \leq \| I \oplus \psi(B) e^{-tB} W(t) \|_{X,Y}.
\]

So \( \text{emb} \circ (I \oplus e^{-tB}) \) is continuous from \( T_{X,Y, A \oplus I} \) into \( T(S_{X,Y, A \oplus I}) \).

From this the continuity of \( \text{emb} \) follows.

Definition 5.17. Similar to Definition 5.15 we introduce the space

\[
S(T_{X,Y, I \oplus B, A \oplus I}) = \bigcup_{t>0} (e^{-tA} \circ I)(T_{X,Y, I \oplus B})
\]

with the appropriate inductive limit topology.

On \( T(S_{X,Y, I \oplus B, A \oplus I}) \times S(T_{X,Y, I \oplus B, A \oplus I}) \) we introduce the pairing

\[
\langle P, T \rangle_A = \langle P(e), (e^{tA} \circ I)T \rangle_{X,Y}.
\]

The embedding of \( S(T_{X,Y, I \oplus B, A \oplus I}) \) into \( T(S_{X,Y, A \oplus I}, I \oplus B) \) is defined by

\[
(\text{emb } T)(t) = (I \oplus e^{-tB})T.
\]
Analogous to Theorem 5.16 we have

Theorem 5.18.

I. \( S_{X,A} \otimes T_{Y,B} \) is dense in \( S(T_{X \otimes Y, A \otimes I} + I \otimes B) \).

II. For each fixed \( P \in T(S_{X \otimes Y, A \otimes I} + I \otimes B) \) the linear functional \( T \mapsto \langle P, T \rangle_A \) is continuous on \( S(T_{X \otimes Y, A \otimes I} + I \otimes B) \).

III. The embedding described in Definition 5.17 is continuous.

Remark. For more details on the topological properties of the spaces \( T(S_{X \otimes Y, A \otimes I} + I \otimes B) \), etc., see [ETH].

CHAPTER 6. KERNEL THEOREMS

In this final chapter we show that the elements of the completed sesquilinear topological tensor products of the preceding chapter can, in a very natural way, be interpreted as linear maps of the types we discussed in Chapter 4. We give necessary and sufficient conditions on the semi-groups \( e^{-tA} \) and \( e^{-tB} \) which ensure that the topological tensor product comprise all continuous linear maps. In this case we say that a kernel theorem holds.

CASE a: Continuous linear maps \( T_{X,A} \otimes S_{Y,B} \). We consider an element \( \theta \in S_{X \otimes Y, A \otimes B} \) as a linear operator \( T_{X,A} \otimes S_{Y,B} \) in the following way. Let \( F \in T_{X,A} \). We define \( \theta F \) by

\[
\theta F = e^{-\varepsilon B} (e^{-\varepsilon B} \theta e^{\varepsilon A}) F(\varepsilon).
\]

For \( \varepsilon > 0 \) and sufficiently small this definition makes sense and does not depend on \( \varepsilon \).

Theorem 6.1.

I. For each \( \theta \in S_{X \otimes Y, A \otimes B} \) the linear operator \( \theta : T_{X,A} \otimes S_{Y,B} \) as defined by (a), is continuous.
II. For each $\theta \in S_{X^0Y,AB}$, $F \in T_{X,A}$, $G \in T_{Y,B}$

$$\langle \theta F, G \rangle_Y = \langle \theta F \circ G \rangle_{X^0Y}.$$ 

III. If for each $t > 0$ at least one of the operators $e^{-tA}$, $e^{-tB}$ is HS, then $S_{X^0Y,AB}$ comprises all continuous linear operators from $T_{X,A}$ into $S_{Y,B}$.

IV. $S_{X^0X,A^0A}$ comprises all continuous linear operators from $T_{X,A}$ into $S_{X,A}$ iff for each $t > 0$ the operator $e^{-tA}$ is HS.

Proof.

I. We shall prove that $\theta$ satisfies condition IV of Theorem 4.5. Since $\theta \in X \otimes Y$ we have $\theta = \theta$. Since $\theta \in S_{X^0Y,AB}$ we have for $t$ sufficiently small $e^{tB} \theta e^{tA} \in X \otimes Y$. Therefore, $\theta e^{tA} = e^{-tB}(e^{tB} \theta e^{tA})$ is bounded.

II. For $t$ sufficiently small $e^{tB} \theta e^{tA} \in X \otimes Y$, therefore by Properties 5.1.c

$$\langle \theta F, G \rangle_Y = \langle e^{-tB}(e^{tB} \theta e^{tA})F(\epsilon), G(\epsilon) \rangle_Y = ((e^{tB} \theta e^{tA})F(\epsilon), G(\epsilon))_Y =$$

$$= (e^{tB} \theta e^{tA}F(\epsilon) \otimes G(\epsilon))_{X^0Y} = \langle \theta F \circ G \rangle_{X^0Y}.$$ 

III. Let $\Gamma : T_{X,A} \rightarrow S_{Y,B}$ be continuous. By Theorem 4.5.V there exists $\tau > 0$ and continuous $Q : S_{X,A} \rightarrow S_{Y,B}$ such that $\Gamma = Qe^{-\tau A}$. By Theorem 4.2.V there exists $\beta > 0$ such that $e^{\beta B}Qe^{-\tau A}$ is a bounded operator. Put $\alpha = \frac{1}{2} \min(\beta, \frac{1}{2}\tau)$, then

$$= e^{-\alpha B}(e^{(\beta-\alpha)B}(e^{\beta B}Qe^{-\tau A})e^{-(\beta-\alpha)A})e^{-\alpha A}.$$ 

The operator between ( ) is bounded. Further the operator between { } is HS since $e^{-(\beta-\alpha)B}$ or $e^{-(\frac{1}{2}\tau-\alpha)A}$ is HS.

It follows that $\Gamma \in S_{X^0Y,AB}$.

IV. The if-part is a special case of III. For the only-if-part consider the special map $\Gamma = e^{-\alpha A} : T_{X,A} \rightarrow S_{X,A}$ for some $\alpha > 0$. In order that $\Gamma \in S_{X^0X,A^0A}$ it has to be HS.
CASE b: Continuous linear maps $S_{X,A} \rightarrow T_{Y,B}$

Let $K \in T_{X \oplus Y,A \oplus B}$. For $f \in S_{X,A}$ we define $Kf \in T_{Y,B}$ by

$$(Kf)(t) = e^{-(t-\varepsilon)B}K(\varepsilon)e^{\varepsilon A}f.$$  

For any $f \in S_{X,A}$ and $t > 0$ this definition makes sense for $\varepsilon > 0$ sufficiently small. Moreover, $(Kf)(t)$ does not depend on $\varepsilon$.

**Theorem 6.2.**

I. For each $K \in T_{X \oplus Y,A \oplus B}$ the linear operator $K : S_{X,A} \rightarrow T_{Y,B}$ defined by (b) is continuous.

II. For each $K \in T_{X \oplus Y,A \oplus B}$, $f \in S_{X,A}$, $g \in S_{Y,B}$

$$\langle g, Kf \rangle = \langle f \circ g, K \rangle_{X \oplus Y}.$$  

III. If for each $t > 0$ at least one of the operators $e^{-tA}$, $e^{-tB}$ is HS, then $T_{X \oplus Y,A \oplus B}$ comprises all continuous linear operators from $S_{X,A}$ into $T_{Y,B}$.

IV. $T_{X \oplus X,A \oplus A}$ comprises all continuous linear operators from $S_{X,A}$ into $T_{X,A}$ iff for each $t > 0$ the operator $e^{-tA}$ is HS.

**Proof.**

I. We use condition II of Theorem 4.4.

For each $t > 0$, $\alpha > 0$, $e^{-tB}Ke^{-\alpha A}$ is a bounded operator from $X$ into $Y$ because for $\varepsilon$ sufficiently small

$$e^{-tB}Ke^{-\alpha A} = e^{-(t-\varepsilon)B}K(\varepsilon)e^{-(\alpha-\varepsilon)A}.$$  

All operators in the last expression are bounded.

II. For each $t > 0$, $K(\tau)$ is a HS-map. For $\varepsilon$ sufficiently small, with Properties 5.1.c

$$\langle g, Kf \rangle_{Y} = (e^{B}g, (e^{-\varepsilon B}Ke^{-\varepsilon A})(e^{\varepsilon A}f))_{Y}$$

$$\langle e^{A}f \circ e^{-\varepsilon B}g, K(\varepsilon) \rangle_{X \oplus Y} = ((e^{A} \circ e^{B})(f \circ g), K(\varepsilon))_{X \oplus Y} = \langle f \circ g, K \rangle_{X \oplus Y}.$$  

III. Let $L : S_{X,A} \to T_{Y,B}$ be continuous. According to Theorem 4.4.11 the operators $e^{-tB}$ and $e^{-tA}$ are bounded for each $t > 0$. However, if $e^{-tB}$ or $e^{-tA}$ is HS for each $t > 0$, then $e^{-tB}L_T^{-tA}$ is HS for each $t > 0$ and it defines an element in $T_{XoY,AMB}$. A simple verification shows that this element reproduces $L$ by recipe (b).

IV. The if-part is a special case of III. For the only-if-part consider the special map $L = \text{emb} = I$. Here $I$ is the identity map. $e^{-tA}L_Ie^{-tA} = e^{-2tA}$ can be considered as an element of $S_{XoY,AMB}$ iff $e^{-tA}$ is HS for all $t > 0$.

CASE c: Continuous linear maps: $S_{X,A} \to S_{Y,B}$.

Let $P \in T(S_{XoY,1oB,A \& I})$. For $f \in S_{X,A}$ we define $Pf \in S_{Y,B}$ by

\[(c) \quad Pf = P(\varepsilon)e^{\varepsilon f}.
\]

$Pf \in S_{Y,B}$ since $P(\varepsilon) \in S_{XoY,1oB}$. The definition makes sense for $\varepsilon$ sufficiently small and does not depend on the choice of $\varepsilon$.

Theorem 6.3.

I. For each $P \in T(S_{XoY,1oB,A \& I})$ the linear operator $P : S_{X,A} \to S_{Y,B}$ defined by (c) is continuous.

II. For each $P \in T(S_{XoY,1oB,A \& I})$, each $f \in S_{X,A}$, each $G \in T_{Y,B}$

\[<Pf,G>_X = <P,f \circ G>_A.
\]

III. If for each $t > 0$ at least one of the operators $e^{-tA}$, $e^{-tB}$ is HS, then $T(S_{XoY,1oB,A \& I})$ comprises all continuous linear operators from $S_{X,A}$ into $S_{Y,B}$.

IV. Consider the special case $X = Y$ and $B = A$. The space $T(S_{XoX,1oA,A \& I})$ comprises all continuous linear operators from $S_{X,A}$ into itself iff for each $t > 0$ the operator $e^{-tA}$ is HS.
Proof.

I. We use condition II of Theorem 4.2.
Let \( f_n \to 0 \) strongly in \( S_{X,A} \). For some \( \varepsilon > 0 \), \( e^{\varepsilon A} f_n \to 0 \) in \( X \).
\( P(\varepsilon) \in S_{X \otimes Y, I \otimes B} \), therefore there exists \( \delta > 0 \) such that \( e^{\delta B} P(\varepsilon) \) is a bounded operator. But then \( e^{\delta B} P_f e^{\delta B} P(\varepsilon) e^{\varepsilon A} f_n \to 0 \) in \( Y \), which shows that \( P_f e^{\varepsilon A} f_n \to 0 \) strongly in \( S_{Y,B} \).

II. For \( \beta \) and \( \delta \) sufficiently small and positive we have
\[
\langle P_f, e^{\varepsilon A} f \otimes G \rangle_A = \langle P(\varepsilon), e^{\varepsilon A} f \otimes I \rangle (f \otimes G) >_{X \otimes Y}
\]
\[
= \langle P(\varepsilon), e^{\varepsilon A} f \otimes G \rangle_{X \otimes Y} = \langle e^{\delta B} P(\varepsilon) e^{\varepsilon A} f \otimes (\beta - \delta) A, G(\delta) \rangle_{X \otimes Y}
\]
\[
= \langle e^{\delta B} P(\varepsilon) e^{\varepsilon A} f, (\beta - \delta) A, G(\delta) \rangle_Y
\]
\[
= \langle P(\varepsilon), e^{\varepsilon A} f, G \rangle_Y = \langle P_f, G \rangle_Y.
\]

III. Let \( Q : S_{X,A} + S_{Y,B} \) be continuous. According to Theorem 4.2.5 for each \( t > 0 \) there exist \( \beta(t) > 0 \) such that \( e^{\beta(t)B} Q e^{-tA} \) is a bounded map from \( X \) into \( Y \). Now because of the assumption on \( e^{-tA}, e^{-\beta B} \), we find that \( Q e^{-tA} = e^{-\beta(t)B} (e^{\beta(t)B} Q e^{-tA}) \) is an element of \( T(S_{X \otimes Y, I \otimes B}, A \otimes I) \). It reproduces the operator \( Q \) if the recipe (c) is applied.

IV. The if-part is a special case of III. For the only-if-part consider the identity map \( I : S_{X,A} \rightarrow S_{X,A} \). In order that \( I e^{-tA} \) as a function of \( t \) is an element of \( T(S_{X \otimes X, I \otimes A}, A \otimes I) \) the operator \( e^{-tA} \) has to be HS for all \( t > 0 \).

**CASE d:** Continuous linear maps: \( T_{X,A} \to T_{Y,B} \).
Let \( \phi \in T(S_{X \otimes Y, A \otimes I}, I \otimes B) \). For \( F \in T_{X,A} \) we define \( \phi F \in T_{Y,B} \) by
\[
\phi F(t) = \phi(t) e^{\varepsilon(t)A} F(\varepsilon(t)).
\]
This definition makes sense for \( t > 0 \) and \( \varepsilon(t) > 0 \) sufficiently small.
\( \phi F(t) \in S_{Y,B} \) since \( \phi(t) \in S_{X \otimes Y, A \otimes I} \). Moreover
\[
e^{-tB} (\phi F)(t) = e^{-tB} \phi(t) e^{\varepsilon(t)A} F(\varepsilon(t)) = \phi(t + \tau) e^{\varepsilon(t)A} F(\varepsilon(t)) = (\phi F)(t + \tau) .
\]
Theorem 6.4.

I. For each \( \phi \in T(S_{X_0Y}, A_{0I}, I \otimes B) \) the linear operator \( \phi : T_{X,A} + T_{Y,B} \) defined by (d) is continuous.

II. For each \( \phi \in T(S_{X_0Y}, A_{0I}, I \otimes B), F \in T_{X,A'}, g \in S_{Y,B'} \)

\[ <g, \phi F>_{Y} = <g, F \otimes g>_{B}. \]

III. If for each \( t > 0 \) at least one of the operators \( e^{-tA} \), \( e^{-tB} \) is HS, then \( T(S_{X_0Y}, A_{0I}, I \otimes B) \) comprises all continuous linear operators from \( T_{X,A} \) into \( T_{Y,B} \).

IV. Consider the special case \( Y = X \) and \( B = A \). The space \( T(S_{X_0X}, A_{0I}, I \otimes A) \) comprises all continuous linear operators from \( T_{X,A} \) into itself iff for each \( t > 0 \) the operator \( e^{-tA} \) is HS.

Proof.

I. We use Theorem 4.6.III. For each \( t > 0, e^{-tB} \in S_{X_0Y}, A_{0I} \). Then according to case a, \( e^{-tB} \) is a continuous linear map from \( T_{X,A} \) into \( S_{Y,B} \).

II. For \( \alpha \) and \( \delta \) sufficiently small and positive

\[ <\phi, F \otimes g>_{B} = <\phi(\alpha), (I \otimes e^{\delta B})(F \otimes g)>_{X_0Y} = \]

\[ = <\phi(\alpha), F \otimes e^{\delta B}g>_{X_0Y} = (e^{\delta B} \phi(\alpha)e^{\delta A}, F(\delta) \otimes e^{(\alpha-\delta)B})_{X_0Y} = \]

\[ = (e^{\delta B} \phi(\alpha)e^{\delta A}F(\delta), e^{(\alpha-\delta)B}g)_{Y} = (\phi(\alpha)e^{\delta A}F(\delta), e^{\delta B}g)_{Y} = \]

\[ = <g, \phi F>_{Y}. \]

III. Let \( \Psi : T_{X,A} + T_{Y,B} \) be continuous. According to Theorem 4.6.VI for each \( t > 0 \) there exists \( \beta(t) > 0 \) such that \( e^{-tB} \Psi e^{\beta(t)A} \) is a densely defined and bounded operator from \( X \) into \( Y \). If one of the operators \( e^{-\alpha A}, e^{-\alpha B} \) is HS for arbitrary small positive \( \alpha \) it follows that \( e^{-tB} \Psi e^{\beta(t)A} \) is HS for \( t > 0 \), because

\[ e^{-tB} \Psi e^{\beta(t)A} = e^{-tB} \Psi e^{\beta(t)A} e^{-\beta(t)A}. \]
Here \( \Psi e^{\beta(t)A} \) denotes the extension of \( \Psi e^{\beta(t)A} \) to the whole of \( X \).

Since
\[
e^{-tB} \Psi e^{\beta(t)A} = e^{-tB} (e^{-tB} \Psi e^{\beta(t)A}) e^{-\beta(t)A}
\]

it follows that \( e^{-tB} \Psi \in S_{X^{\otimes} Y, A^{\otimes} I} \). Hence \( e^{-tB} \Psi \) as a function of \( t \), belongs to \( T(S_{X^{\otimes} Y, A^{\otimes} I}, I \otimes B) \). By recipe (d) the operator \( \Psi \) is reproduced.

IV. The if-part is a special case of III. For the only-if-part consider the identity map \( I \). In order that \( e^{-tA} \), as a function of \( t \), can be considered as an element in \( T(S_{X^{\otimes} X, A^{\otimes} I}, I \otimes A) \) the operator \( e^{-tA} \) should be HS for all \( t > 0 \).

Remark. In [ETH] a kernel theorem for extendable continuous linear mappings has been stated and proved.

REFERENCES
See part A.

ACKNOWLEDGEMENT
The author thanks Dr. S.J.L. van Eyndhoven for contributing to the revision of the manuscript.