A study on mathematical models for turbulent flows

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A STUDY ON MATHEMATICAL MODELS FOR TURBULENT FLOWS

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A study on mathematical models for turbulent flows.

1 Introduction

Turbulence has kept engineers and scientists busy for more than a century. Despite the fact that it is quite a familiar notion, the definition of turbulence proves to be rather difficult. Hinze (1975) defines it as "an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned".

Basically a flow, whether laminar or turbulent, is described by the general conservation laws of mass, momentum and energy. These laws combined with relations describing medium properties (e.g. stress-deformation etc.) lead to a set of non-linear differential equations which in general may be solved by means of the numerical methods that are now available. However, when dealing with turbulence an extra problem is introduced, since in turbulent motion the length scales of the smallest eddies are of an order $10^{-3}$ times the largest length scale in the flow. For computation of a three dimensional area this would imply that a grid with approximately $10^9$ nodes is necessary to describe the problem. The core and speed of the computers available to date are still insufficient to handle the vast amount of calculations required for so many nodes. To overcome this problem models were developed to provide a mathematical description of the smallest eddies in a flow as a function of the mean flow variables. Some of these models, together with their applications and shortcomings are discussed in this paper.
1.1 Basic equations

The basic equations describing the transport of mass, momentum and energy in a flow are:

Mass: \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} = 0 \) \hspace{1cm} (1.1)

Momentum: \( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \mathbf{s} \) \hspace{1cm} (1.2)

with \( \mathbf{s} = -\rho \mathbf{I} + \mathbf{s}'' \) \hspace{1cm} (1.3)

Energy: \( \rho \frac{\partial e}{\partial t} = -\nabla \cdot \mathbf{q} + \phi + (\mathbf{s} \cdot \nabla) \cdot \mathbf{v} \) \hspace{1cm} (1.4)

\( \rho \) = density
\( \nabla \) = gradient operator
\( \mathbf{v} \) = velocity vector
\( p \) = static pressure
\( \mathbf{q} \) = gravitational force
\( \mathbf{s} \) = stress tensor
\( \mathbf{s}'' \) = viscous stress tensor
\( e \) = internal energy
\( \mathbf{q} \) = heat flux
\( \phi \) = heat source

These equations hold for a three dimensional, instationary, compressible laminar or turbulent flow. For the moment only the equations 1.1-1.3 are considered and the following assumptions are made:

- The flow is stationary
- The fluid is Newtonian
- The flow is incompressible, therefore \( \mathbf{s}'' = 2 \mu \mathbf{D} \). The rate of strain tensor \( \mathbf{D} \) is defined as: \( \mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \)
- Gravitational forces are negligible.
- The flow is isothermal.

Using these assumptions in the differential equations for mass and momentum results in:
\[ \nabla \cdot \mathbf{v} = 0 \quad (1.5) \]

\[ \nabla \cdot \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} \quad (1.6) \]

Writing eq. (1.6) dimensionless gives:

\[ \nabla \cdot \mathbf{v} = -\nabla p + 1/Re \, \nabla^2 \mathbf{v} \quad (1.7) \]

in which \( Re \) is defined as the Reynolds number. For very low Reynolds numbers the convective terms (I) of the Navier-Stokes equation may be neglected, in which case we have a Stokes flow. However if for relatively high Reynolds numbers all the viscous terms (II) are neglected, the "paradox of d'Alembert" which states that "a body placed in an ideal flow does not experience any resistance" is found. The first to overcome this problem was Prandtl by using his boundary layer theory. This will be discussed later.

### 1.2 Reynolds stresses

For sufficiently high Re-numbers the flow becomes turbulent. Due to the fluctuations of the mean velocity an extra shear stress is introduced in the mean momentum equation 1.6. This can be shown by assuming that the variables of the Navier-Stokes equation exist of a mean value and a fluctuation. Thus for the velocity in x-direction \( u = \bar{u} + u' \), where the prime denotes the fluctuation. (see appendix 1) The time averaged equations are:

\[ \nabla \bar{\mathbf{v}} = 0 \quad (1.8) \]

\[ \bar{\nabla} \cdot \bar{\mathbf{v}} = -1/\rho \, \bar{\nabla} \bar{p} + 1/\rho \, \nabla \bar{\mathbf{S}}^\tau + 1/\rho \, \bar{\nabla} \bar{\mathbf{T}} \quad (1.9) \]

where \( \bar{\mathbf{T}} \) and \( \bar{\mathbf{p}} \) are the mean values. \( \bar{\mathbf{T}} \) is defined as the turbulent- or Reynolds stress tensor.
The notation $T_{ij}$ indicates a stress acting on a surface $i$ in a direction $j$. The stress tensor may be interpreted as follows. Consider a flow element under the influence of turbulent stresses. The stress vector acting on a surface with normal vector $(1,0,0)$ is

$$
t = (T_{xx}, T_{xy}, T_{xz})$$

(See fig.(1.1)).

Suppose there is a mean velocity profile as shown in fig.(1.2). A flow element with a speed $\bar{u}(y_1)$ and a fluctuation $v' > 0$ moving up to a layer with speed $\bar{u}(y_2)$ causes a decrement in speed

$$\bar{u}(y_1) - \bar{u}(y_2) = -u'$$

Therefore $T_{xy} = -p'u'v'$ is positive according to the sign convention of fig.(1.1).

---

Fig. 1.1 Stresses active on a surface of a fluid element with normal vector $\hat{n}$

Fig. 1.2 Mean velocity profile (not to scale).
1.3 Closure problem

The mean mass and mean momentum equations (1.8) and (1.9) represent 4 equations for 13 unknowns:

- velocity vector $\mathbf{v}: u, v, w$
- pressure $p$
- 9 correlation components of the turbulent stress tensor $T_{ij}$.

However, since the stress tensor is symmetrical, the number of unknown components reduces to 6. A total of 10 unknowns remains which are described by only 4 equations. This now features the closure problem.

In order to proceed several models were developed to express the turbulent stress tensor as a function of the mean velocity field.

1.4 Turbulence models

The numerical procedures which are associated with turbulence models can be divided into integral and differential types. Differential models involve direct assumptions for the Reynolds stresses at a point and seek the solution of the governing equation in their partial differential equation form.

Integral methods involve the integral parameters of the shear layer momentum thickness, shape parameter, skin friction coefficient etc. One solves a system of ordinary differential equations (for two dimensional flows), whose dependent variables are the profile parameters and the independent variable is $x$. In three dimensional flows, the equations are the partial differential equations in the plane of the layer. The important distinction between calculation methods is the type of turbulence model rather than the type of numerical procedure.

Integral methods

A good example of an integral method is the approach of the momentum expression by Von Karman (Vossers 1978, Hinze 1975). The basis for this method is formed by a set of partial differential equations describing the flow in boundary layers (see appendix 2). These expressions were
derived by Prandtl who assumed that in a turbulent flow the viscous effects are negligible, except in a very thin layer near the wall. Outside this boundary layer the flow is almost ideal and the equation of Bernoulli holds. Prandtl's boundary layer equations are:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial y^2} \tag{1.11} \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.12} \]

\[ -\frac{1}{\rho} \frac{\partial p}{\partial x} = U_0 \frac{dU_0}{dx} \tag{1.13} \]

where \( U_0 \) is the main stream velocity outside the boundary layer.

Von Karman's approach starts by integrating (1.11) over the boundary layer thickness \( \delta \). Since the integrands outside the boundary layer vanish for \( y>\delta \), integration from zero to infinity is permissible. Thus (see Hinze 1975):

\[ \int_0^\infty \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - U_0 \frac{dU_0}{dx} \right) dy = \int_0^\infty v \frac{\partial^2 u}{\partial y^2} dy \tag{1.14} \]

This results in the ordinary differential equation for \( \delta \) and \( \delta^* \) as a function of \( x \) (Vossers 1978):

\[ \frac{d(\delta U_0^2)}{dx} + \delta^* U_0 \frac{dU_0}{dx} = \frac{\tau_0}{\rho} \tag{1.15} \]

Where the displacement thickness \( \delta^* \) is defined by:

\[ \delta^* U_0 = \int_0^\delta (U_0 - \bar{u}) dy \tag{1.16} \]

and the momentum loss thickness \( \theta \) in formula (1.15) is defined by:

\[ \theta U_0^2 = \int_0^\delta (U_0 - \bar{u}) U_0 dy \tag{1.17} \]

The first integral represents the loss of mass transport due to friction at the wall. The mass flow is reduced with an amount \( \delta^* U_0 \) compared to a flow without a boundary layer (see fig.1.3). It is as if the wall of an undisturbed flow undergoes a displacement \( \delta^* \), thus reducing the cross section of the channel or pipe. The second integral represents the loss of momentum in the vicinity of a wall. The variable \( \theta \) gives the displacement of a wall in an undisturbed flow causing a
loss of momentum which is equivalent to the loss in the "real" flow (see fig. 1.4).

The price one has to pay for facilitating equation (1.11) to get the ordinary differential equation is that because of the integration process no information about the velocity profile along the y-axis is available. On the contrary, one has to assume such a profile beforehand. Assuming $\bar{u} / U_0 = g(\varepsilon)$, with $\varepsilon = y / \delta$, an expression for $\delta^*$ and $\theta$ is found as a function of $\delta$, where $\delta$ itself is a function of $x$ (see Vossers 1978).

Fig. 1.3 Boundary layer thickness $\delta$ (not to scale).

Fig. 1.4 Displacement thickness $\delta^*$ and momentum loss thickness $\theta$. 
Differential methods

The advantage of differential methods is that the restrictions and inaccuracy that arise from the need to parameterize the velocity profiles are avoided. Differential methods introduce substantially more detailed information about the turbulent flow.

The turbulence models can be classified in several ways. The one most often used is that arranged in order of the number of differential equations solved in addition to the mean flow equations.

1. Zero equation models (i.e. no additional differential equations)
2. One equation models
3. Two equation models
4. Stress equation models

Most of the models (1) to (3) use the Boussinesq eddy viscosity model. In class (4) the Reynolds stress is obtained from a differential equation.

Before discussing the first three turbulence models as mentioned above, the Boussinesq hypothesis will be described.

1.5 The Boussinesq hypothesis

A way of tackling the closure problem is to treat the Reynolds stresses in a manner similar to the viscous stresses in Newtonian flows. Boussinesq (1877) introduced a "turbulent" or "eddy" viscosity \( \nu_t \) and defined the turbulent stress tensor \( \mathbf{T} \) as follows:

\[
T_{ij} = \rho \nu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{or} \quad T = 2 \rho \nu_t \mathbf{D}
\]  

(1.18)

This way he related the turbulent stress tensor to the mean velocity profile. Boussinesq assumed the eddy viscosity to be a constant at any point in the flow. Calculation of the trace of \( \mathbf{T} \) gives:

\[
\text{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33} = -\rho \left( u'^2 + v'^2 + w'^2 \right) = -2 \rho \nu_t k
\]

(1.19)

But according to (1.18)
Equation (1.20) suggests that the turbulent kinetic energy $k$, which is defined as $k = \frac{1}{2} \sum_i u_i^2$, in equation (1.19) would be zero, meaning that there is no turbulence at all. Therefore $\mathbf{I}$ is redefined as

$$\mathbf{I} = \mathbf{Q} \left( 2 v_t \mathbf{D} - \frac{2}{3} k \mathbf{I} \right) \quad (1.21)$$

where $\mathbf{I}$ is the unity tensor. Formula (1.21) together with the assumption that $v_t$ is a constant is known as the Bousinesq hypothesis.

In the next part of this section the necessity of abandoning the assumption that $v_t$ is a constant will be discussed. When considering a turbulent flow past a rigid body, two regions can be discerned. The first is the so called "wall region", which is directly affected by the condition of the wall. Beyond this wall region the flow is only indirectly affected by the wall and therefore this part is called the "outer region". The wall region itself is divided into three subregions (see fig. 1.5):

I Viscous subregion; the Re number is low. According to the Navier-Stokes equation convection is negligible and the turbulence fluctuations are almost zero. This region is very thin and has a thickness $\delta_1$.

II Buffer region; the Re-number is moderate. The viscous and turbulent shear stresses are of the same order.

III Turbulent region; the Re-number is high. In this case the shear stresses are almost completely due to turbulence. The viscous stresses are almost negligible. Let $\delta_t$ be the distance beyond which the flow is completely turbulent, then the bufferzone (II) is specified by $\delta_1 < y < \delta_t$.

From the Navier-Stokes equation in cylinder coordinates or by considering a cylindrical flow element in a fully developed flow, the expression for a pipe flow can be derived:

$$\frac{dp}{dx} = \frac{d}{dr} \left( r \frac{d}{dr} \left( r \frac{d}{dx} \right) \right) \quad (1.22)$$
Fig. 1.5 Mean velocity profile of a turbulent flow near a wall (not to scale).

Fig. 1.6a Total shear stress and turbulent shear stress in a pipe flow as function of distance to a wall (not to scale).

Fig. 1.6b Mean velocity profile and its derivative as function of distance to a wall (not to scale).
This holds for a fully developed flow whether turbulent or not. By integration it can be shown that $\tau$ is a linear function of the radius $r$ (see fig. 1.6). The symbol $\tau$ stands for the resulting shear stress due to both viscous and turbulent motion. Near the wall the flow may be regarded as two-dimensional. Experiments have learned that the correlation $-u'v'$ as a function of distance to a wall is roughly given by the graph of figure 1.6a. As seen before, the correlation is a measure for the turbulent shear stress (Vossers 1978) for which we have:

$$T_{yx} = \rho \nu_t \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\rho u'v'$$  \hspace{1cm} (1.23)

It will be shown in chapter 5 that in the wall region of a turbulent flow the velocity in the viscous sublayer is an approximately linear function of the distance to the wall. Outside this sublayer the velocity profile is approximated by a logarithmic velocity distribution. Qualitatively the velocity profile of the fully developed flow and its gradient will be as represented in fig. 1.6 B. By comparing fig. 1.6 A with 1.6 B in equation (1.23) it becomes plausible that $\nu_t$ will be a function of the space coordinates. An important improvement of Boussinesq's model would therefore be to assume that $\nu_t$ varies spatially. The unknowns in this new model are $\nu_t$ and $k$ from eq. (1.21). In order to proceed, an expression for $\nu_t$ and $k$ as a function of the mean flow variables must be found. The dimension of the eddy viscosity is $(m^2 / s)$. It is therefore reasonable to assume that $\nu_t$ can be written as

$$\nu_t = V_t \ell$$  \hspace{1cm} (1.24)

where $V_t$ is a characteristic turbulent velocity scale and $\ell$ a characteristic length scale. The models which are discussed here generally try to estimate $V_t$, $\ell$ and $k$. 
2.1 Zero equation models: Prandtl’s mixing length hypothesis (MLH)

2.1 General description

Consider a turbulent single shear flow with a mean velocity $\bar{u}$ in $x$ direction (see fig. 2.1). Suppose the flow is fully developed, so $\frac{\partial \bar{u}}{\partial x} = 0$. Prandtl’s idea is quite similar to that behind the kinetic gas theory where gas molecules are considered (Launder & Spalding 1972). The assumption of discrete particles leads to the velocity distribution of Stefan-Boltzman.

With a fluid however, it is difficult to regard the flow as the result of collisions between particles and it is therefore considered as a continuum. The behavior of so called "fluid lumps" or fluid elements is used to derive Prandtl’s mixing length theory.

Suppose fluid lumps move from several layers $y = \xi$ to other layers $y = \xi'$ where they mix with other fluid lumps, thus transferring momentum. This process is called diffusion. Taking the average of many fluid lumps, 'mixing' or transfer of momentum will take place at a certain average distance of the layer where the lump originated from. Suppose this distance to be $y = \xi'$. Taking the root mean square of the fluctuation $v'$ which carries the lumps from $y=\xi$ to $y=\xi'$ gives:

$$v' = \left\{ v'^2 \right\}^{1/2}$$

---

Fig. 2.1 Diffusion model: fluid lumps mix at an average distance $\xi'$. 

---
2.2

The difference in momentum in x-direction which a lump experiences by moving from \( y = 0 \) to \( y = 1' \) can be calculated by:

\[
\Delta \phi = \rho \, v' \, \left( u(1') - u(0) \right)
\]  

(2.1)

which is momentum per unit area per unit time.

The mean velocity \( u(1') \) is now estimated with a Taylor approximation around \( y = 0 \), neglecting second order terms. This neglect is only viable if

\[
l' \ll \left( \frac{du}{dy} \right) / \left( \frac{d^2u}{dy^2} \right) = x
\]

(2.2)

Assuming this is true then

\[
\Delta \phi = \rho \, l' \, v'' \, \frac{du}{dy}
\]

(2.3)

Since by Newton's law force is equal to change in momentum per unit time, \( \Delta \phi \) may be interpreted as a shear stress acting on the surface \( y=0 \), thus

\[
T_{yx} = \rho \, l' \, v'' \, \frac{du}{dy}
\]

(2.4)

But from the Reynolds stress tensor it follows that

\[
T_{yx} = -\rho \, u'v''
\]

(2.5)

The correlation coefficient \( c_{12} \) of \( u' \) and \( v' \) is defined as

\[
c_{12} = \frac{u'v''}{u'' \, v''}
\]

(2.6)

and has a constant value close to unity if the correlation is strong. Substituting (2.6) in (2.5) and subsequently comparing the result with (2.4) learns that:

\[
u'' \approx l' \, \text{abs} \left( \frac{du}{dy} \right)
\]

(2.7)

The absolute value sign is necessary because \( u'' \) cannot be negative according to the definition of root mean square. Assuming \( u'' \) and \( v'' \) to be of the same order it follows that \( v'' \) may be estimated by the same expression given in eq. (2.7), therefore:

\[
T_{yz} = \rho \, l' \, \text{abs} \left( \frac{du}{dy} \right) \, \frac{du}{dy}
\]

(2.8)

Comparing (2.8) to (1.18) and (1.24) gives:
\[ V_t = l_m \frac{\text{abs}(du)}{dy} \]
\[ \nu_t = l_m^2 \frac{\text{abs}(du)}{dy} \quad ; \quad l = l_m \]  
\[(2.9)\]

In this case contains the coefficient \( c_{12} \) together with the variable \( l' \) and is called the mixing length.

The variables still unknown in the closure problem are \( l_m \) and \( k \). Unfortunately there is no universal expression for \( l_m \) and its value has to be determined empirically. The value of \( k \) can either be estimated by using the definition of \( \nu_t \) according to the \( k \)-model (which will be discussed later) or used in an expression for a "new" pressure \( p' \). When calculating \( k \) as in the \( k \)-model the velocity scale \( V \) is taken equal to \( k^{1/2} \), thus:
\[ \nu_t = l_m k^{1/2} \quad \text{and} \quad k = \frac{\nu_t}{l_m^2} \]

where \( \nu_t \) is known from (2.9).

The second method starts by taking equation (1.91) and substituting
\[ \vec{\nu} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nabla \cdot (\vec{v} + \vec{v}_t) \]
\[(2.12)\]

where \( \vec{Q}' = \vec{Q} + 2/3k \)
\[(2.13)\]

Instead of calculating \( k \) directly, its value is taken into account as a boundary condition (Lavooy 1983).

Generalization of the MLH for two dimensional situations

It will be shown in chapter 3 that the transport equation for the turbulence kinetic energy \( k \) which is defined by
\[ k^{1/2} \overline{u_i'^2} \]  
(Einstein convention)

can be written as:
\[ \rho \sum_{j} \frac{\partial k}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{\mu_k}{k} \right) \]
\[ + \mu \sum_{j} \left( \frac{\partial \overline{u_i u_j}}{\partial x_j} \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_i}}{\partial x_j} \frac{\partial \overline{u_j}}{\partial x_i} \right) - c_\omega \rho l^{3/2} \]

Convection  Diffusion  Production  Dissipation
Now for a homogeneous turbulent flow the derivatives of the time averaged fluctuations are zero (Hinze 1975). Therefore we have:

\[
\mu \frac{\partial \bar{u}_i}{\partial x_j} \left( \frac{\partial \bar{u}_j}{\partial x_j} \right) = C_D V^3/2
\]

In the \( k \)-model the turbulence viscosity \( \nu_t \) is defined as:

\[
\nu_t = C \mu k^{1/2} \quad \text{therefore}
\]

\[
\nu_t = \frac{1}{m} \left( \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i} \right)^2 / 2 \quad \text{or}
\]

\[
\nu_t = 2^{1/2} \frac{1}{m} \left( \frac{1}{D^2} \right)^{1/2}
\]

(Lavooy 1983)

For a homogeneous single flow this reduces to the same expression as found in equation (2.9) where

\[
\frac{C}{D} \left( \frac{k}{l} \right)^{1/4} \quad \text{or}
\]

\[
\frac{1}{m} = \frac{C}{D} \left( \frac{k}{l} \right)^{1/4} \quad \text{or}
\]

From Hinze (1975) p.592, it can be learned that the production and dissipation of turbulence kinetic energy are in equilibrium if \( l << L_2 \) and \( L_2 << L_1 \). In this case the diffusive and convective terms may be neglected. It may be concluded that the MLH only works satisfactory in those cases where an equilibrium of turbulence exists.

### 2.2 Evaluation

#### 2.2.1 Assumptions of MLH

The calculation of the shear stress \( T_{12} \) is partly based on assumption (2.2). Suppose \( u \) to be of the order \( U_0 \), \( y \) will be of the order \( L_2 \) where \( U_0 \) and \( L_2 \) are a characteristic velocity and a characteristic length respectively (e.g. the diameter of a canal). Therefore \( X \) will be of the order \( L_2 \) and assumption (2.2) will only hold if the average mixing length \( l' << L_2 \). In many cases however, the large eddies in a flow have approximately the same dimensions as the length \( L_2 \) and in those cases (2.2) is not valid. Hence a mistake is introduced which is compensated by choosing \( l_m \) more or less arbitrarily. According to
Lavooy (1983), only for parallel or nearly parallel flows will this mistake not be a function of the main stream direction. Therefore the correction is only justified in these cases.

A second assumption made is that \( u'' \) and \( v'' \) are of the same order and that their correlation is good. This is only true for turbulent shear flows which are parallel or nearly parallel and fully developed.

### 2.2.2 Intermezzo: The turbulent transfer of scalar quantities

Only in relatively simple circumstances one can afford not to know the temperature field, for the temperature influences density and density influences velocity. In many flow problems a scalar quantity has to be taken into account because of its effect on the main stream.

Let \( \Phi \) be a scalar quantity, e.g. mass, temperature, molar fraction, then the general flux law may be written as:

\[
 J_\Phi = -\Gamma_{\Phi,t} \nabla \Phi \quad (\text{see Bird 1962, chap 16})
\]  

(2.14)

\( J_\Phi \) is a vector quantity with dimension (kg m\(^{-2}\)s\(^{-1}\))

\( \Gamma_{\Phi,t} \) is the so called turbulent exchange coefficient.

A specific example of the general flux law mentioned above is Fourier's law for the transfer of heat.

\[
 q = -\lambda \nabla T
\]

The mixing length theory assumes that \( \Gamma_{\Phi,t} \) may be written in a similar form as \( \nu_t \):

\[
 \Gamma_{\Phi,t} = \varphi \nu_t \nabla T
\]  

(2.15)

As in the boundary layer theory (Vossers 1978) where Prandtl's number is introduced as the ratio of momentum- and temperature layer thickness, a similar number may be defined here. In boundary layers we have:

\[
 \frac{\delta}{\delta T} = Pr^{1/2}
\]

In the MLH the turbulent Prandtl or Schmidt number is defined as:

\[
 \sigma_{\Phi,t} = \frac{\nu_t}{\Gamma_{\Phi,t}} = \frac{1}{\nu_t}
\]  

(2.16)

According to MLH, \( \nu_t \) can be taken as \( \nu \text{ abs}(\partial u / \partial y) \) thus

\[
 \Gamma_{\Phi,t} = \varphi \nu \text{ abs}(\partial u / \partial y)
\]  

(2.17)
2.6

We will need this relationship in chapter 3.

2.2.3 Shortcomings of MLH

1) The following relationship was derived:

\[ v_t = l m ^ 2 \text{abs}(\bar{\delta u} / \partial y) \]

Hence \( \mu_t \) vanishes whenever \( \bar{\delta u} / \partial y \) is zero. This would imply that along the centerline of a plane channel \( \mu_t \) is zero. Experiments (Lauder & Spalding 1972) however have shown \( \mu_t \) in this area to be about 20\% less than the maximum value in the flow. Fortunately this is not of much direct importance because the shear stress at the centerline is zero as was shown in section 1.5.

2) As with the viscosity \( v_t \), the turbulent exchange coefficient \( \mu_t \) vanishes whenever \( \bar{\delta u} / \partial y \) is zero. This leads to controversial results in some cases. For instance near the reattachment point in a recirculating flow behind a step \( \bar{\delta u} / \partial y \) is small indicating that \( \mu_t \) would be small. Experiments show that precisely at this point the heat exchange is large. The MLH fails here.

3) Influences of convection and diffusion on \( v_t \) are not accounted for. The model only regards local effects of turbulence. Generally the local level of velocity fluctuations is not only determined by local events but also by influences which may occur some distance upstream or downstream. For instance in a recirculating flow incidents downstream may affect the random fluctuation at any point.

4) The MLH only holds for flows in which an equilibrium exists between turbulence production and dissipation, so for parallel or nearly parallel flows where:

\[ u \gg y \]
\[
\frac{\partial u}{\partial y} \gg \frac{\partial v}{\partial x} \quad \text{or} \quad L_1 \gg L_2 \quad \text{where} \quad L_1 \quad \text{and} \quad L_2 \quad \text{are characteristic length scales in} \ x \ \text{and} \ y \ \text{direction. This is the mathematical formulation of a parallel flow. Only in this case, the mistake which is introduced by correction of} \ m \ \text{is not a function of the main stream direction.}
\]

Examples of flows which may or may not be approached with the MLH are given in the figures 2.2.

Fig. 2.2a Turbulent flow in a straight channel; the MLH may be used.
Fig. 2.2b Turbulent flow in a diverging channel; the MLH may be used also.
Fig. 2.2c Recirculating flow behind a step; the MLH fails here.
3 One equation models; k-model

3.1 General description

Before discussing the k-model in more detail, first the general thought will be described here. Prandtl (1945) and Kolmogorov (1942) both proposed that instead of relating the random velocity of turbulence to the mean velocity gradient as in equation (2.7), it would be more appropriate to choose a turbulent property. The turbulent viscosity is assumed to be related to the turbulence kinetic energy by:

\[ \nu_t \propto k^{1/2} \]  \hspace{1cm} (3.1)

where \( k = 1/2 \left( u_i' \right)^2 \)

One can expect this to provide a better formula for \( \nu_t \) for at the centerline of a pipe or channel flow, although the velocity gradient is zero, the turbulence kinetic energy is not.

In the k-model an additional transport differential equation for the turbulence kinetic energy is used to solve the closure problem of eq. (2.12). Writing in Einstein convention, the basic equations for the k-model are (see appendix 3):

Mass:

\[ \frac{\partial \tilde{u}_i}{\partial x_i} = 0 \]  \hspace{1cm} (3.2)

Momentum:

\[ - \frac{\partial \tilde{u}_i}{\partial x_j} = - \frac{\partial p''}{\partial x_i} + \frac{\partial}{\partial x_i} \left[ \nu + \nu_t \left( \frac{\partial u_i}{\partial x_j} \right) \right] \]  \hspace{1cm} (3.3)

with \( \rho'' = \rho + 2/3 \rho k \)

Viscosity:

\[ \nu_t = \frac{\mu_k}{\rho} \frac{\partial^2 k^{1/2}}{\partial t} \]  \hspace{1cm} (3.4)

(Another definition which is often used is \( \nu_t = k^{1/2} \bar{\nu}_m \) see Launder & Spalding)

Kinetic energy:
Here \( \sigma_{k,t} \) is the Schmid number for turbulence. Both \( \sigma \) and \( C_0 \) are considered to be constant. Again assumptions have to be made for \( l \) since this is the single unknown parameter in this set of equations. The formulae mentioned above provide 6 equations for 7 unknowns (including \( l \)). Since \( l \) is estimated algebraically, this set of equations forms an answer to the closure problem.

### 3.2 The transport equation for the turbulence kinetic energy \( k \)

Basically the first part of the derivation of the \( k \)-transport equation is rather straightforward. From the Navier-Stokes equation for incompressible isotropic flows an expression for \( k \) can be formulated (see appendix 3)

\[
\frac{\partial k}{\partial t} + \frac{\partial (k \bar{u}_i \bar{u}_i)}{\partial x_i} = \frac{\partial}{\partial x_k} \left( \frac{\partial k}{\partial x_k} \right) - \frac{\partial}{\partial x_j} \left( \frac{\partial k}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \left( \frac{\partial k}{\partial x_i} \right) - C_0 \frac{k^{3/2}}{l} \tag{3.5}
\]

(3.6)

Launder & Spalding (1972) state that for boundary layers and high Re-numbers the viscous term \( I \) may be neglected and the equation reduces to the following equation where \( k' = 1/2u_i^2 \) and \( k = 1/2 \bar{u}_i^2 \):

\[
\frac{\partial k}{\partial t} = -\frac{\partial}{\partial \bar{y}} (\bar{y} \bar{v}'k' + \bar{v}'p') - \frac{\partial}{\partial \bar{y}} (\bar{u}'v') \frac{\partial}{\partial \bar{y}} - \frac{\partial}{\partial \bar{x}_j} \left( \frac{\partial k}{\partial \bar{x}_j} \right) \tag{3.7}
\]

(1) \hspace{1cm} (2) \hspace{1cm} (3) \hspace{1cm} (4)
The terms (1) to (4) may now be interpreted as follows:

(1) \( \frac{\partial k}{\partial t} + \rho \frac{\partial}{\partial x} \frac{\partial k}{\partial x} = \) instationary and convective transport

(2) The second term may be interpreted as a diffusion term. To make this plausible the term is integrated across a boundary layer of a flow with a free stream on one edge and a wall on the other (see fig. 1.3). At the wall the turbulence kinetic energy is zero, at the free stream the kinetic energy is supposed to be zero also (see Hinze, 1975, chap. 7). For boundary layers, according to Prandtl \( \frac{\partial p}{\partial y} = 0 \). This means that the second term of the k-equation does not have any contribution to a change of \( k \), but merely redistributes the energy in space. Because of its diffusive nature it is hypothesised that term (2) can be written as part of a flux law, similar to (2.14). In the intermezzo it was shown that \( \Gamma_{k,t} \) can be expressed as in (2.15), thus:

\[ -\frac{\partial}{\partial y} (\rho v'k' + v'p') = c_0 k^{1/2} \frac{\partial k}{\partial y} = \mu k \frac{\partial k}{\partial y} \]  
\[ \sigma_{k,t} \frac{\partial k}{\partial y} \]  

(3) Production. If the k-equation is derived for the mean kinetic energy the same term is found but with the sign reversed. As seen in previous chapters \( -\rho u'v' \) gives the component \( T_{yx} \) of the Reynolds stress tensor. Therefore the third term is interpreted as the rate at which kinetic energy is drawn from the mean motion of the flow and converted into turbulence kinetic energy. This process is caused by the shear stress working against the mean velocity gradient. In conformation with section 1.5:

\[ -\rho u'v' = \mu \frac{\partial u}{\partial y} \]  

(4) Dissipation. The last term is interpreted as the rate at which energy is dissipated by action of viscosity. Because no expression is known (yet) the simplest combination of \( k \) and \( \mu \) giving dimensional consistency is used to get:

\[ \frac{\mu (\partial u)^2}{\partial x_j} = \rho C_o k^{3/2} \]  
\[ 1 \]
This leads to the following transport equation for the kinetic energy in boundary layers with high Re-number:

\[
\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x} \left( u \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial y} \left( v \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial z} \left( w \frac{\partial \theta}{\partial z} \right) = -\frac{\partial}{\partial y} \left( \chi \frac{\partial \theta}{\partial y} \right) - \frac{\partial}{\partial z} \left( \chi \frac{\partial \theta}{\partial z} \right) - C_D \frac{\theta^2}{\theta^1/2}
\]

(3.12)

A more general expression is given in appendix 2.

3.3 Estimation of \( C_D \) and \( l \)

In the vicinity of a wall - according to Launder & Spalding (1972) - the convective and diffusive terms in the \( k \)-equation are nearly always negligible. The production and dissipation of kinetic energy form an equilibrium. In this case the energy equation yields:

\[
T_{xy}^2 = (\mu_t \frac{\partial u}{\partial y})^2 = C_D \theta^1/2 \theta^2
\]

i.e. \( \frac{T_{xy}}{\theta k} = C_D^{1/2} \)

(3.13)

with \( \mu_t = k^{1/2} \).

By eliminating \( k \) from the dissipation term and using the definition of \( \mu_t \), this gives:

\[
T_{xy} = C_D^{-1/2} \theta^1/2 \left( \frac{\partial u}{\partial y} \right)^2
\]

(3.14)

The value of \( T_{xy}/(\theta k) \) in (3.13) has been determined experimentally by Launder & Spalding (1972) who used the results of these experiments to determine \( C_D \). They found the value of \( C_D \) to be about 0.08.

The resulting equation (3.14) may be recognised as equivalent to the expression found with the MLH. For this reason the MLH is sometimes referred to as the "local turbulence equilibrium model".

Again the length scale \( l \) has to be estimated algebraically. The following description may be used for boundary layer flows:

Near a wall:

\[
\frac{\partial u}{\partial y} = \left( \frac{T_{xy}}{\theta} \right)^{1/2} \frac{1}{\theta y} \quad \kappa = \text{constant}
\]

leading to

\[
l_m = C_D \theta^{1/4} \theta y
\]
3.4 Viscous subregion of a boundary layer flow (low Re number)

In the immediate vicinity of a wall the viscous terms which were neglected in section 3.2 must now be included in the k-equation. This affects the flow in two ways:

1) The diffusion of energy becomes appreciable.
2) The length scale of turbulence diminishes as the Reynolds number of turbulent motion "R" becomes low (Launder & Spalding 1972). R is defined as

\[ R = \frac{\rho}{\mu} \left( \frac{k}{\gamma} \right)^{1/2} \frac{y}{\mu} \]  

(3.15)

where \( y \) is the distance to a wall.

Wolfstein (1967) used different length scales for \( \mu_t \) and the dissipation term. He found that for boundary layers

\[ l_\mu \sim y(1 - e^{-AR}) \quad \text{and} \quad l_\nu \sim y(1 - e^{-BR}) \]

where A and B are constants.

These expressions for \( l \) show much similarity to the expressions used by Van Driest in the MLH. For a detailed description of the wall region see Hinze 1975.

3.6 Evaluation

1) A better measure of \( V_{\mu} \) is achieved by using \( k^{1/2} \) instead of \( l_{\mu} \text{abs}(\partial u/\partial y) \). At points where the velocity gradient is zero, the turbulence kinetic energy will not become zero and therefore a more realistic value for \( v_t \) is found.

2) For the same reason a better approach of \( \Gamma_{\mu,t} \) is realised.

3) Although \( k^{1/2} \) is a good velocity scale of turbulence and the use of a transport equation for \( k \) may prove advantageous, transport effects on the length scale turn out to be equally important (Launder and Spalding 1972).

4) The difficulty of finding a sufficiently precise algebraic prescription of \( l \) remains. In situations where convection and diffusion
are not important, the MLH gives equally good results rather more cheaply.
4.1 General description

Again before discussing the model in further detail the general idea of and the need for a two equation model will be discussed here. Two equation models calculate both $k$ and the length scale $l$ from transport equations. Subsequently $\mu_l$ is calculated from (3.4).

A transport equation for $l$ is needed because in many cases it is not possible or at least time consuming to find an algebraic length scale distribution. Once an expression is found, each new geometry would require a different function for $l$. For flows not described by the boundary layer equations (such as recirculating flows) the reason for using a transport equation for $l$ is even stronger. Under these circumstances the convective transport plays an important role and it is much more difficult than in boundary layer flows to determine the length scale by measurement. Another reason is that in complex cross sections it is often difficult to guess how diffusional influences will affect the length scale profile.

In general the method of constructing a length scale transport equation is as follows:
1) Define a variable $z = k^m l^n$ where $m$ and $n$ are constants.
2) Derive an exact equation for $z$ from the Navier-Stokes equation.
3) Express unknown correlations in terms of knowable quantities.
4) Deduce $l$ from $k$ and $z$

In the case of the $k,\varepsilon$-model $z$, $m$ and $n$ are chosen as follows:

\[
z = \varepsilon = k^{3/2} l^{-1} ; \text{thus } m = 3/2, n = -1 \quad (4.1)
\]

The extra transport equation for $\varepsilon$ which is used together with the set of equations (3.2)-(3.5) from the $k$-model is given by (see appendix 4):
The two equation model comprises therefore of a simultaneous solution of equations (3.2)-(3.5) and (4.2). "\( \varepsilon \)" is defined as the turbulence dissipation rate. Once \( k \) and \( \varepsilon \) are known, \( \lambda \) can be determined from (4.1).

Other definitions of \( \varepsilon \) are possible (Launder & Spalding 1972). The reason for presenting the \( k, \varepsilon \) -model here is that the method of arriving at this model is relatively simple, but the final result is almost equal to the result of the more complicated models.

The general form of the \( \varepsilon \)-equation for boundary layers at high Re-numbers is:

\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial \varepsilon}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left( \mu_k \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \right) = \frac{C_2 \varepsilon^2}{\mu k}
\]

with \( \varepsilon = \frac{k^{3/2}}{\lambda} \) (4.2)

4.2 Estimation of \( C_1 \), \( C_2 \) and \( \sigma_z \)

\( C_2 \) may be determined by considering the decay of turbulence behind a fine wire screen. In this case the variables are functions only of the stream wise coordinate \( x \), giving a set ordinary differential equations for \( k \) and \( \varepsilon \). Experiments show that for high Re-numbers the energy level varies with \( x^{-1} \). Therefore \( \frac{dk}{dx} \) varies with \( x^{-2} \). From the \( k \) and \( \varepsilon \) equations we get:

\[
\frac{\partial}{\partial x} \frac{\partial k}{\partial x} = - C_D \frac{\partial^2 k}{\partial \mu_t} \sim x^{-2}
\]

\[
\frac{\partial}{\partial x} \frac{\partial \varepsilon}{\partial x} = - C_D \frac{\partial^2 \varepsilon}{\partial \mu_t} \frac{k \varepsilon}{\mu_t}
\]

Using these results Launder and Spaldirig found that

\[
C_2 = C_D \left( m - \frac{n}{2} \right)
\]

(4.5)

\( C_1 \) is estimated from the \( k \) and \( \varepsilon \) equations for "near wall turbulence". According to section 3.4 we have:

\[
\frac{T_{xy}}{(\mu k)} = C_0^{1/2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \left( \frac{T_{xy}}{\mu} \right)^{1/2} \frac{1}{k y}
\]

Whence the same authors found
Again \( \sigma_z \) is assumed to be near unity. It is emphasized that in the \( z \) equation diffusion must not be neglected near the wall, for it may be one of the major terms in the equation.
5 Boundary conditions

5.1 Laminar and turbulent boundary layers; wall functions

5.1.1 Laminar boundary layer along a flat plate

For completeness the solutions of Prandtl's boundary layer equations for a laminar flow along a flat plate according to Blasius are mentioned here. Since for the main stream the velocity gradient $dU_0 / dx = 0$ also (See section 1.4 and appendix 2). The basic equations are:

$$\frac{u}{3x} + \frac{v}{dy} = \frac{d^2 u}{dy^2}$$ (5.1)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$ (5.2)

Defining the boundary layer thickness $\delta$ as the value of $y$ at which $u = 0.99U_0$ the following exact solutions are found (See Vossers 1978).

$$\frac{\delta}{X} = 5 \frac{Re_x^{-1/2}}{X} \quad \text{where} \quad \text{Re} = \frac{U_0 X}{v}$$ (5.3)

The shear stress at the wall:

$$\tau_0 = \mu \left( \frac{\delta u}{\delta y} \right)_{y=0} = 0.33 \frac{u^2}{U_0} \text{Re}_x^{-1/2}$$ (5.4)

The local friction coefficient $c_f$:

$$c_f = \frac{\tau_0}{1/2 \rho U_0^2} = 0.66 \text{Re}_x^{-1/2}$$ (5.5)

The average friction coefficient $C_f$ for one side of the plate:

$$C_f = \frac{1}{L} \int_0^L C_f \, dx = 1.33 \text{Re}_L^{-1/2} \quad \text{where} \quad \text{Re}_L = \frac{U_L}{v}$$ (5.6)

This result is according to the authors mentioned above, valid up to $Re_L$ numbers of about $5 \times 10^5$, above which value the boundary layer becomes turbulent.

5.1.2 Turbulent boundary layer along a flat plate at zero pressure gradient

The solution of the "turbulent boundary layer equations" requires a numerical approach. Because of the high velocity gradients in a relatively thin layer the direct solution by computer would involve the
Fig. 5.1a Relative turbulence intensities in a boundary layer along a smooth wall with zero pressure gradient (from Hinze, 1975).

\[ Re_b = \frac{U_b \delta}{\nu} = 7.5 \times 10^4 \]

\[ \frac{u^*}{U_0} \approx 0.037 \]

Fig. 5.1b Relative turbulence in the constant stress layer of a boundary layer along a smooth wall (Hinze, 1975).

Fig. 5.2a Distribution of turbulence kinetic energy and turbulence shear stress across a boundary layer. Here \( q^2 \) stands for \( \frac{1}{2} u_i^2 \).

(Hinze, 1975)

Fig. 5.2b Distribution of turbulence shear stress in the wall region of a boundary layer (Hinze 1975).
5.2

use of a very fine mesh and would therefore result in large computational times and storage (fig.5.1). A way around this problem is to use the so-called universal velocity distribution in order to describe the velocity profiles near a wall.

As seen before the wall region of a boundary layer may be divided into three subregions. The velocity \( \bar{u} \) in the boundary layer may be regarded as a function of the shear stress at the wall, the wall roughness \( y_r \), the kinematic viscosity \( v \) and the distance to the wall \( y \).

The wall friction velocity \( u^* \) is defined by

\[
\frac{\tau_0}{\rho} = \frac{u^*}{2} \quad \text{where} \quad \tau_0 = (S_{yx}^*) y = 0
\]

Hence from dimensional analysis it follows that

\[
\frac{-u}{v} = f\left( \frac{u^*}{v}, \frac{u^* y}{v} \right); \quad \text{for a smooth surface} \quad y_r = 0.
\]

In the viscous subregion the stresses are mainly determined by viscous effects and it is assumed that the velocity gradient is given by

\[
\mu \frac{\partial \bar{u}}{\partial y} = S_{yx} = \tau_0 = \text{constant} \quad \text{(see fig. 1.5, 1.6) whence}
\]

\[
\frac{-u}{v} = \frac{u^*}{v}
\]

So for \( y \to 0 \) we have a linear relationship for \( \bar{u}(y) \). In the buffer and turbulent region the total shear stresses are also assumed to be constant. Figure 5.2 gives evidence for this assumption to be reasonable since in this graph the turbulence shear stress near the wall is almost constant. The turbulent and viscous effects are approximately of the same order. The effective viscosity \( \nu_e = \mu + \nu_t \) however will not be a constant (as shown in section 1.5) since the eddies adjacent to the wall are smaller than those at a larger distance (See Hinze 1975). It is assumed that

\[
\nu_e = \frac{\mu_e}{\rho} = c u^* y \quad \text{so}
\]

\[
c u^* y \frac{\partial \bar{u}}{\partial y} = \frac{\tau_0}{\rho} = u^* \quad \text{or}
\]

\[
c y^* \frac{\partial \bar{u}}{\partial y} = 1 \quad \frac{-u}{u} = \frac{u^*}{\nu} \quad y^* = \frac{v u^*}{\nu}
\]
Fig. 5.3 Mean velocity distribution near smooth walls as measured by Ludwieg and Tillmann, Klebanoff and Diehl, Schultz-Grunow (Hinze 1975).

\[ \frac{U_1}{u^*} = 8.3 \left( \frac{u^* x_2}{v} \right)^{1/7} \]

\[ \frac{U_1}{u^*} = 2.44 \ln \frac{u^* x_2}{v} + 4.9 \]

Fig. 5.4 Development of boundary layer thickness along a flat plate.
Solving the differential equation gives:

\[ u^+ = \frac{1}{c} \ln y^+ + c_2 = A \ln y^+ + B \]  

(5.10)

In view of the assumptions made above the wall region is frequently called the constant stress layer. Outside the wall region in the "outer region" the condition of constant stress is no longer valid (See fig. 1.6). Experiments however (Hinze 1975), have shown that a logarithmic distribution also applies in this area. For this reason equation (5.10) is often referred to as the universal velocity distribution.

According to Hinze the logarithmic velocity distribution can be approximated by a power law in a large fraction of the boundary layer cross section:

\[ \frac{u^*}{u} = c \left( \frac{u^*}{v} \right)^{1/n} \]  

(5.11)

Some experimental data are represented in figure 5.3 and are compared with the logarithmic velocity distribution for \( A = 2.44 \) and \( B = 4.9 \). It appears that the theoretical curve forms a good average of the experimental results. However other investigators prefer different values for the constants or even different velocity distributions.

In the case of a flow along a rough wall corrections have to be made to account for the "roughness effects" at the wall. A "Reynolds number" is introduced such that

\[ \frac{u^* y}{v} < 5 \] means equivalent to a smooth surface.

\[ \frac{u^* y}{v} > 70 \] means equivalent to a completely rough surface.

To compare the turbulent boundary layer with a laminar one a velocity distribution according to the powerlaw is used:

\[ \frac{u^*}{v} = 0.7 \left( \frac{u^*}{v} \right)^{1/7} \]

Using the mean momentum equation of Von Kàrmàn it can be shown that for the boundary layer holds (Vossers 1978):

\[ \delta = 0.37 \ Re^{-1/5} \]

(5.12)
\[ \tau_0 = 0.029 \rho U_0^2 Re_x^{-1/5} \quad (5.13) \]
\[ c_f = 0.058 Re_x^{-1/5} \quad (5.14) \]
\[ C_f = 0.072 Re_L^{-1/5} \quad (5.15) \]

Equation (5.12) shows that the turbulent boundary layer thickness grows much faster than the laminar one (eq. 5.3). The development from laminar to turbulent layer occurs approximately at \( Re_x \approx 5 \times 10^5 \). Qualitatively the boundary layer thickness will be according to figure (5.4).

5.1.3 Turbulent flow through a straight circular pipe

For a fully developed flow through a pipe we have \( \bar{u}_r = \bar{u}_\theta = 0 \) and \( \partial u_x / \partial x = 0 \). As in a boundary layer flow (along a flat plate), in pipe flow also a wall region and an outer region may be defined. The flow in the wall region is not affected by the flow conditions far from the wall and may therefore be considered to be the same for boundary layer- and for pipe flow.

In the outer region of a pipe flow however, there is no interaction with a turbulence free stream as is the case with a boundary layer flow. Another dissimilarity is that in the fully developed pipe flow the conditions are independent of \( x \) (e.g. \( \partial k / \partial x = 0 \)). The total shear stress distribution in a boundary layer along a flat plate is approximately linear in the outer region, whereas as in the wall region it is almost constant (figure 5.2). It can be shown (Vossers 1978, Hinze 1975) that the distribution of the total shear stress in a pipe flow is exactly linear with the distance to the wall (figure 5.5). In the wall region of a pipeflow the shear stress is therefore essentially not constant, though the deviations from the wall shear stress \( \tau_0 \) may be only small because the wall region is relatively thin. For this reason, the wall region in pipe flows is usually also referred to as the constant stress layer, notwithstanding the incorrectness of this designation. The same logarithmic velocity distribution as found for a
Fig. 5.5 Distributions of turbulence kinetic energy and turbulence shear stress in pipe flow (Hinze 1975).

Fig. 5.6 Relative turbulence intensities in pipe flow (Hinze 1975).

Fig. 5.7 Relative turbulence intensities in pipe flow near the wall (Hinze 1975).
5.5

boundary layer flow applies for the turbulent region of a pipe flow (Hinze 1975).

To illustrate the above, some measurements of turbulence quantities in pipe flow as made by Laufer (1954) are presented here. Figures 5.6 and 5.7 show that the relative turbulence intensity distributions are of the same magnitude as the corresponding quantities in a boundary layer along a flat plate (the figures are from Hinze, 1975). One of the assumptions made when deriving the log law is that near the wall the total shear stress is approximately constant. Fig. 5.8 shows the actual turbulence shear stress. Figure 5.5 learns that at the center of the pipe flow the kinetic energy is smaller than near the wall, but not equal to zero. The production term of the kinetic energy transport equation is not negligible in the outer region of a pipe flow, but forms a balance with the diffusion and dissipation terms (Hinze 1975, p 716). In the free stream outside a boundary layer however, the turbulence kinetic energy is almost negligible as can be seen in figure 5.2. (figures from Hinze 1975).

5.2 Boundary conditions

In order to solve an elliptic second order differential equation 2 boundary conditions are necessary. For a one dimensional problem 2 boundary conditions in one border point or one boundary condition in 2 border points would suffice. However, in order to get an unambiguous solution of the problem the latter is necessary. If for instance the laminar Navier-Stokes equation together with the continuity equation has to be solved, 2 boundary conditions have to be supplied to describe velocity or stress along the border of the region considered. A more detailed description is given by Van Steenhoven (1978).

As an example, the problem of a flow over a backward facing step with recirculation is presented here. C. Taylor (1981) has studied this problem in case of both laminar and turbulent flow and under the latter circumstances using one- and two equation models (see fig. 5.9)
Fig. 5.8 Distribution of turbulence shear stress near the wall in pipe flow.

\[
\frac{u_{r_{max}}}{\nu} = 5 \times 10^3
\]

\[
\frac{u^*}{U_{r_{max}}} \approx 0.035
\]

Fig. 5.9 Flow past a backward facing step.

\[U_1 = 0\]
\[U_2 \rightarrow \text{eq. 5.16}\]

Fig. 5.10 Region where the universal velocity profiles are applied.

Fig. 5.11 The problem of "positive shear stress" near the reattachment point (not to scale).
5.2.1 Laminar flow

The boundary conditions used by Taylor are:

1) Upstream

\[ x = 0; \ 0 < y < R \quad u \ \text{specified} \]
\[ v = 0 \]

\( u \) is specified by utilising experimental results.

2) Downstream

\[ 0 < y < R \quad \frac{\partial u}{\partial x} = 0; \quad \frac{\partial v}{\partial x} = 0 \]

This simulates fully developed flow. The boundary should be chosen sufficiently far downstream for this condition to be acceptable.

3) On walls

\[ u = v = 0 \]

4) Pressure was specified as zero at one node point on the downstream boundary. All values are therefore relative to this point.

5.2.2 Turbulent flow

Near the wall universal laws of the type discussed in section 5.1.2 were used (see C. Taylor et.al. 1981). Let \( \lambda \) be the distance to the wall and \( \lambda' \) the distance from the wall where the finite element mesh begins (fig. 5.10), define:

\[
\lambda^* = \lambda \frac{u^*}{v} ; \quad u^* = \frac{r_0}{\sqrt{\nu}} \quad \text{where} \quad r = \left( \frac{\partial u}{\partial \lambda} \right) \quad (5.16)
\]

\( \tau_0 \) is assumed constant for \( 0 < \lambda < \lambda' \). The universal law (Davis 1972 and Taylor 1977) used by Taylor is:

\[ 0 < \lambda^* < 5; \quad u_{\perp}^* = \lambda^* \]
\[ 5 < \lambda^* < 30; \quad u_{\perp}^* = ( -3.05 + 5.0 \log \lambda^* ) \frac{\tau_0}{|\tau_0|} \quad (5.17) \]
5.7

\[ \Lambda^+ > 30; \quad u_i^+ = (-5.5 + 2.5 \log \Lambda^+) \frac{\tau_0}{|\tau_0|} \]

For regions as shown in figure 5.9 problems may arise when attempting to evaluate shear stresses near the reattachment point (see fig. 5.11). At node "i_1" the wall shear stress's sign is calculated correctly by using equation (5.16). At the node "i_2", the value of \( \frac{\partial u_i}{\partial x_2} \) is positive and thus gives rise to a positive shear stress \( \tau_0 \). In reality however this should be negative with a corresponding negative \( u_i^+ \) (at the wall \( \frac{\partial u_i}{\partial x_2} \) is negative so \( \tau_0 \) is negative). Without precautions such a movement would be exaggerated by each succesive iteration of the solution procedure leading, eventually, to a zero recirculating length. This can be overcome by calculating

\[ \tau = \mu \frac{\partial u_i}{\partial \lambda} \frac{u_i}{|u_i|} \text{ along the walls.} \quad (5.18) \]

5.2.2.1 One equation model, boundary conditions

Apart from the Navier-Stokes and the continuity equation the transport equation for the kinetic energy \( k \) has to be solved also. Therefore 3 boundary conditions along the border are necessary. The conditions used by Taylor are as follows:

1) Upstream:
   \[ x = 0; \quad 0 < y < R \]
   \( \bar{u} \) specified (experimental values)
   \( \bar{v} = 0 \)
   \( k \) specified by experimental values

2) Downstream:
   \[ 0 < y < R \]
   \( \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{v}}{\partial x} = 0 \)
   \( \frac{\partial k}{\partial x} = 0 \)

Again simulating fully developed conditions.
3) Walls: \( \bar{u} = \bar{v} = k = 0 \)

4) Pressure specified zero at one node point on the downstream boundary.

Solution procedure

The non linear terms such as \( u_j \frac{\partial u_i}{\partial x_j} \) are replaced by
\[
\frac{u_j}{u_i} \frac{\partial u_i}{\partial x_j},
\]
where "I" stands for the \( i^{th} \) iteration step, and the following iterative procedure is adopted:

1) Impose initial conditions on the velocity. In other words, choose an initial velocity field.

2) Evaluate the resulting distribution of wall shear stress using the relationship (5.18)

3) Use equation (5.17) and re-evaluate initial values of velocity along the boundary of the near wall region.

4) Determine the variation in \( k \) in the near wall region from equation (3.13) so \( k = \frac{\tau_0}{\varrho C_D^{1/2}} \)

5) Solve equations (3.2), (3.3) and (3.5) simultaneously to obtain the current iteration values for \( u_i \), \( p \) and \( k \).

6) Repeat the process from 2) until the required convergence criteria are satisfied.

5.2.2.2 Two equation model

In this model 4 differential equations have to be solved simultaneously, therefore 4 boundary conditions along the border of the region must be specified.

1) Upstream:
\[
x = 0; \ 0 < y < R \quad \bar{u} \text{ specified} \quad \bar{v} = 0
\]

\( \bar{u} \) and \( \bar{v} \) fully developed values of the one equation model.
These conditions were obtained by analysing flow in a channel of constant width equal to the width upstream of the step. Fully developed flow was simulated by imposing the downstream condition

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial k}{\partial x} = \frac{\partial \varepsilon}{\partial x} = 0; \ p = 0 \]

2) Downstream:

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial k}{\partial x} = \frac{\partial \varepsilon}{\partial x} = 0 \]

1 fully developed values of the one equation model

3) Walls: the universal laws were used. Apparently now specifications for \( k \) and \( \varepsilon \).

4) Again pressure specified zero at one node point on the downstream boundary.

Solution procedure (From Taylor et al.)

1) Set all initial values to zero and assume that the effective viscosity corresponds to the molecular viscosity.

2) Solve for \( u_1 \), \( p \) and \( k \) for a fixed distribution of \( \ell \).

3) Estimate near wall and boundary conditions for \( \varepsilon \) using

\[ \varepsilon = k^{3/2}/ \ell. \]

4) Solve for \( \varepsilon \) using fixed values of \( u_1 \), \( p \) and \( k \).

5) Test on convergence of \( \varepsilon \), if not converged repeat from 2).

6) Update \( \ell \) using \( \ell = k^{3/2}/ \varepsilon \)

7) Update \( \mu_e \) using the new \( k \) and \( \ell \) values.

8) Using equation (5.17) re-estimate the wall shear stress and therefore boundary conditions on \( u_1 \) and \( k \).

9) Repeat from 2) until convergence criteria are satisfied.

At first sight step 5 seems superfluous. An explanation for this step might be that by its omission the process may not converge. At the very first stroke \( u_1 \), \( p \) and \( k \) are chosen in step 2). Subsequently the first value of \( \varepsilon \) is estimated by 3). In step 4) the next value of \( \varepsilon \) is calculated by iteration. Test for convergence in step 5), if not converged then return to 2). A new value for \( u_1 \), \( p \) and \( k \) is determined by iteration etc. until convergence for \( \varepsilon \) is achieved in step 5). A new distribution for \( \ell \) and \( \mu_e \) is determined in 6) and 7). In step 8) the boundary conditions of the wall are taken into account. Again in a) the
value of $\varepsilon$ is tested for convergence. If not converged yet, the process is restarted from 2) using a new distribution of 1.

The coefficients assumed constant, used in the analyses of the two equation model by C. Taylor et al. (1981) are:

$C_\mu = 0.22, \sigma_k = 1.00, C_1 = 1.45, C_2 = 0.18, \sigma_\varepsilon = 1.3$ and $C_D = 0.092$ (Atkins 1974).
6 Conclusions

The turbulent viscosity $v_t$ is not a constant but varies spatially. In the models which have been discussed here it is assumed that $v_t = V_{t1}$.

The MLH can only be used with reasonable accuracy under circumstances where parallel or nearly parallel flows are considered. Therefore it is not possible to use this method for flows in which recirculation occurs. Neither is the method suitable for flows with a strong curvature. History effects are not accounted for. However the method is relatively simple and well known.

The k-model does have the advantage of a better measure for $V_{t1}$. However, it proves that the transport effects on the length scale are just as important as the transport effects on the turbulence kinetic energy. The extra effort which is needed to solve the equations incorporated in the model, only leads to better results than the MLH in some specific cases.

The k,ε- model provides a transport equation for both the kinetic energy and the length scale. This model is widely used in present day engineering. The constants which occur in the equation need to be changed to account for the effects such as curvature, low Re-number, near wall etc. (Nallasamy 1985)

Taylor et al. (1981) used both the one- and two equation models for solving a laminar and turbulent flow problem over a backward facing step with recirculation. They concluded that for this particular problem the results of the more complex k,ε - model were'nt any better than those of the k-model. The velocity profiles appear to be equally good, whilst the distribution of the turbulence energy seems slightly better with the one equation model.
Appendix 1: Reynolds stresses

Due to turbulence an extra stress tensor, the "turbulence" or Reynolds stress tensor appears in the Navier-Stokes equation. To see this the variables are split into a mean value and a fluctuation, so

\[ u = \bar{u} + u' \quad \text{and} \quad p = \bar{p} + p' \quad \text{etc.} \]

\[ \frac{T}{2} \]

where \( \bar{u} = \frac{1}{T} \int u(t) \, dt \), the time averaged value of \( u \).

Consider x-component of the two dimensional Navier-Stokes equation (1.2):

\[
\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial s_{xx}}{\partial x} + \frac{1}{\rho} \frac{\partial s_{yx}}{\partial y} \tag{A1.1}
\]

Now \[ \frac{\partial u}{\partial t} \]

\[ = \frac{\partial (\bar{u} + u')}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} = \frac{\partial u}{\partial t} \tag{A1.2} \]

and \[ u \frac{\partial u}{\partial x} = (\bar{u} + u') \frac{\partial (\bar{u} + u')}{\partial x} = \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} \]

Time averaging each term of (A1.1) and doing the same for the y- and z-component of equation (1.2) leads to:

\[
\frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} = - \frac{1}{\rho} \nabla \bar{p} + \frac{1}{\rho} \nabla \cdot \bar{\tau} + \frac{1}{\rho} \nabla \cdot \bar{\tau} \tag{A1.3}
\]

\[
\bar{u}' \bar{u}' \quad \bar{v}' \bar{v}' \quad \bar{w}' \bar{w}' \quad \bar{u}' \bar{v}' \quad \bar{v}' \bar{w}' \quad \bar{w}' \bar{w}' \tag{A1.4}
\]

\[ \bar{\tau} \]

is called the Reynolds stress tensor.
Appendix 2: Prandtl's boundary layer theory

For high Reynolds numbers it is assumed that the viscous effects are only significant in a very thin layer close to a wall. Suppose this layer to have a thickness $\delta$. Consider a body with a large radius and a characteristic length $L$. Assume $\delta \ll L$. Let $U_0$ be the velocity of the mainstream outside the boundary layer, then $u$ will be of the order $U_0$, and $\partial u/\partial x$ will be of the order $U_0/L$. From the continuity equation (1.5) we get:

$$v = \frac{\delta}{L} U_0$$

Substituting these results in the laminar N.S. equation gives (Vossers 1978):

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2} \quad (A2.1)$$

$$\frac{\partial p}{\partial y} = 0 \quad (A2.2)$$

(A2.2) states that the pressure in a boundary layer does not vary with $y$, so $p(x,y) = p(x,\delta)$. Outside the boundary layer the flow is supposed to be frictionless. Therefore Bernoulli’s equation holds:

$$-1 \frac{\partial p}{\partial x} = U_0 \frac{du_0}{dx} \quad (A2.3)$$

If a turbulent flow is considered an extra turbulent shear stress term is introduced in (A2.1) and (A2.2) (Hinze 1975, p591). The method of derivation for turbulent flow is similar to that for laminar flow. We get:

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x} \frac{\partial u'v'}{\partial y} + v \frac{\partial^2 u}{\partial y^2} \quad (A2.4)$$

$$\frac{\partial p}{\partial y} + \rho \frac{\partial v'}{\partial y} = 0 \quad (A2.5)$$
Appendix 3: Transport equation for k

Using the Einstein convention, the Navier-Stokes equation for incompressible flows may be written as follows:

\[
\frac{\partial}{\partial t} (\tilde{u}_i + u_i \tilde{u}_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i} [\mu (\tilde{u}_i + \tilde{u}_j) - \rho u_i u'_j] \tag{A3.1}
\]

and

\[
\frac{\partial}{\partial t} \tilde{u}_i + u_i \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i} [\mu (\tilde{u}_i + \tilde{u}_j) - \rho (u_i u'_j)] + \frac{\partial}{\partial x_k} \tilde{u}_i \frac{\partial u'_j}{\partial x_j} \tag{A3.2}
\]

giving two equations for \( i = 1, 2 \).

Time averaging \( \langle A3.1 \rangle \) results in:

\[
\frac{\partial}{\partial t} \tilde{u}_i + u_i \frac{\partial \tilde{u}_i}{\partial x_j} + \rho u_i u'_j \tilde{u}_i = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i} [\mu (\tilde{u}_i + \tilde{u}_j) - \rho (u_i u'_j)] + \frac{\partial}{\partial x_k} \tilde{u}_i \frac{\partial u'_j}{\partial x_j} \tag{A3.3}
\]

for clarity: \( u_i^* = u_i = \bar{u}_i + u'_i \)

The next step is to multiply \( \langle A3.3 \rangle \) by \( u'_j \) \(( j = 1, 2)\)

Write equation \( \langle A3.3 \rangle \) for \( u_j^* \), multiply by \( u_i' \). Adding this to the product of \( \langle A3.3 \rangle \) and \( u_j^* \) gives:

\[
\frac{\partial}{\partial t} (u_i u'_j) + \rho u_i \frac{\partial u_i}{\partial x_j} = -(u_i u'_j) - (u_i') \frac{\partial p}{\partial x_j} + \mu \left[ u_i \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_k} \right] \frac{\partial u_j}{\partial x_j} + \frac{\partial}{\partial x_k} (u_i u'_j) + \frac{\partial}{\partial x_k} \tilde{u}_i \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_k} \tilde{u}_i \frac{\partial u_i}{\partial x_j} \tag{3.5}
\]
This represents 4 equations. Taking the mean of (A3.5)

\[
\frac{\partial}{\partial t} \overline{u_i u_j} + \overline{\mu \partial^2 u_i} = - (\overline{u_i \partial \rho} + \overline{\mu \partial^2 \rho}) + \mu \overline{u_i \partial (\overline{u_j \partial u_i} + \overline{u_k \partial u_l})} + \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}
\]

(A3.6)

since: \( \overline{g u_i u_j} = 0 \)

and

\[
- \overline{g u_i u_j \partial u_i} - \overline{g u_j u_k \partial u_i} = - \overline{g \partial u_i u_j u_i}
\]

(A3.6) represents 4 equations for \( i = 1, 2 \) and \( j = 1, 2 \). Now considering only the two equations with \( i = j \), (A3.6) is multiplied by \( \delta_{ij} \), thus:

\[
\frac{\partial}{\partial t} \overline{u_i u_i} + \overline{\mu \partial^2 u_i} = - 2 (\overline{u_i \partial \rho}) + 2 \mu \overline{u_i \partial (\overline{u_i \partial u_i} + \overline{u_k \partial u_k})} + \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i}
\]

\[-2 \overline{g u_i u_i \partial u_i} - \overline{g \partial u_i u_i u_i}
\]

(A3.7)

The turbulence kinetic energy is defined by \( k = \frac{1}{2} (u_i^2) \).

Substituting the definition into (A3.7) and using

\[
\overline{\mu u_i \partial (\overline{u_i \partial u_i}) + \overline{u_i \partial^2 u_i}} = \mu \overline{\partial^2 u_i} - \mu \overline{(u_i^2)}
\]

gives

\[
\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}
\]

(A3.8)

This forms the exact transport equation for the turbulence kinetic energy \( k \) for an incompressible flow. The terms in (A3.8) are interpreted as follows:
A3.3

1) Using the Reynolds stress hypothesis and the equation for continuity we get:

\[-\rho u'u'_i \frac{\partial \bar{u}_i}{\partial x_i} = \mu \left( \frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) \frac{\partial x_k}{\partial x_i} - \frac{1}{3} u'^2 \delta_{ik} \frac{\partial \bar{u}_i}{\partial x_k} \]

\[= \mu \frac{\partial \bar{u}_i}{\partial x_k} \left( \frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k} \right) \]  

(A3.9)

2) The second term on the righthand-side of (A3.8) is a diffusion term. It is assumed that this term may be written as \( \bar{\gamma}_k = -l'_{k,t} \nabla k \) which is a general form of Fick's diffusion law. From (2.4) and (2.5):

\[ \bar{\gamma}_k = \rho \left( \frac{1}{m} k^{1/2} = \mu \right) \]

Hence:

\[ \frac{\partial k}{\partial x_i} \]

(A3.10)

3) The third term on the righthand side represents physically viscous dissipation of energy. The simplest combination of \( k \) and \( l \) giving dimensional consistency is:

\[ \mu \left( \frac{\partial \bar{u}_i}{\partial x_k} \right)^2 = \rho C_D k^{3/2} \text{ or } \frac{\partial (\bar{u}_i)^2}{\partial x_k} = C_D k^{3/2} \]

Substituting (A3.9) - (A3.11) in (A3.8) results in:

\[ \frac{\partial k}{\partial t} + \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k} \left[ (\mu + \mu) \frac{\partial k}{\partial x_k} \right] \]

(A3.12)
The k-equation for boundary layers

Consider a two dimensional body with a length L in a stationary flow. Suppose the body has a large radius R.

The equation for continuity is:
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
The dimension of u will be of the order \(U_0\), which is the velocity of the main stream. The gradient \(\partial u/\partial x\) is therefore of the order \(U_0/L\). If the gradient \(\partial v/\partial y\) is related to the boundary layer thickness \(\delta\) then \(v = \delta/L\) \(U_0\).

Using these assumptions and the mass conservation law, the production term from (A3.12) reduces to \(\mu_t(\partial u/\partial y)^2\). In the diffusion term we have:
\[
\frac{\partial}{\partial x}\left( \frac{\partial k}{\partial x} \right) = \frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2} - \frac{k}{L^2} + \frac{k}{\delta^2}
\]
since \(\delta << L\) this reduces to \(\partial k/\partial y\). The k equation can now be written as:
\[
\frac{\partial k}{\partial t} + \frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial y} = \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) - c_\rho \sigma_k^{3/2}
\]
Writing \(\mu + \frac{\mu_t}{\sigma_k} = \mu(1 + 1 + \frac{1}{\sigma_k}) = \mu(1 + R)\)

\[
= \mu_k \frac{\partial}{\partial y} \left( \frac{\partial k}{\partial y} + \frac{k}{\sigma_k} \frac{\partial u}{\partial y} \right)
\]

with \(R\) the turbulent Reynolds number: \(R = gk^{1/2}/\nu\) / \(\nu = \mu_t/\mu^\star\)

Hence if \(R \gg 1\) the laminar viscosity effects in the boundary layer are negligible, and the k-equation reduces to:
\[
\frac{\partial k}{\partial t} + \frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial y} = \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial}{\partial y} \left( \frac{\partial k}{\partial y} + \frac{k}{\sigma_k} \frac{\partial u}{\partial y} \right) - c_\rho \sigma_k^{3/2}
\]
(A3.13)
Appendix 4: Transport equation for \( \varepsilon \)

First equation (A3.3) is differentiated with respect to \( x_k \):

\[
\frac{\partial}{\partial t} \left( \frac{\partial \rho^*}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \left( \rho^* \frac{\partial u_i^*}{\partial x_k} \right) - \frac{\partial}{\partial x_j} \left( \rho \frac{\partial u_i^*}{\partial x_j} \right) = - \frac{1}{\rho} \frac{\partial^2 p^*}{\partial x_i^2} + \nu \frac{\partial^3 u_i^*}{\partial x_i^3} + \nu \frac{\partial^2 u_i^*}{\partial x_i^2} + \frac{\partial}{\partial x_i} \left( \frac{\partial u_i^*}{\partial x_i} \right)^2
\]

Multiply (A4.1) by \( 2 \nu \frac{\partial u_i^*}{\partial x_i} \) and take the mean:

\[
\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \left( \frac{u_i^* u_j^*}{2} \right) - u_i^* \frac{\partial}{\partial x_j} \left( \frac{u_i^* + u_j^*}{2} \right) - \frac{\partial u_i^*}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i^* + u_j^*}{2} \right)
\]

\[
= \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \left( \frac{u_i^* u_j^*}{2} \right) - \frac{\partial u_i^*}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i^* + u_j^*}{2} \right)
\]

\[
= \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \left( \frac{u_i^* u_j^*}{2} \right) - \frac{\partial u_i^*}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i^* + u_j^*}{2} \right)
\]

\[
= 2 \nu \frac{\partial u_i^*}{\partial x_i} \left( \frac{\partial u_i^*}{\partial x_i} \right)^2
\]

Define \( \varepsilon^* = \frac{\mu}{C_D} \frac{\partial u_i^*}{\partial x_i} \) (note that \( \varepsilon = \varepsilon^* \))

\[
\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \left( \frac{u_i^* u_j^*}{2} \right) - \frac{\partial u_i^*}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i^* + u_j^*}{2} \right)
\]

The last term of (A4.2) can be written as:

\[
\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \left( \frac{u_i^* u_j^*}{2} \right) = \frac{\partial u_i^*}{\partial x_k} \frac{\partial u_i^*}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i^*}{\partial x_k} \left( \frac{\partial u_i^*}{\partial x_k} \right)^2
\]

and with the definition of \( \varepsilon^* \):

\[
2 \nu \frac{u_i^*}{\partial x_j} \left( \frac{\partial u_i^*}{\partial x_k} \right)^2 = 2 \nu \frac{u_i^* C_D}{\partial x_j} \frac{\partial \varepsilon^*}{\partial x_j} = 2 u_j \frac{\partial \varepsilon^*}{\partial x_j}
\]
Substituting the equations above leads to:

\[
\frac{\partial}{\partial t} (C_D \epsilon) + 2 \nu \sum_j \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} + 2 \nu u \frac{\partial}{\partial x_j} \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_i} \right)^2 \right) \]

\[
- \frac{2}{\eta} \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_i} \right) + 2 \nu^2 \frac{\partial^2}{\partial x_j^2} \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_i} \right)^2 \right) - 2 \nu^2 \left( \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right)^2 \]

\[
- 2 \nu \frac{\partial^2 u_i}{\partial x_i \partial x_j} - c_D \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j} - 2 \nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j} \]

(4.6)

Rearrangement gives:

\[
C_D \frac{\partial \epsilon}{\partial t} + C_D u_j \frac{\partial \epsilon}{\partial x_j} = -2 \nu \sum_j \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right) \]

(1) (2) (3)

\[
-2 \nu \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_i} \right) + \nu c_D \frac{\partial^2 \epsilon}{\partial x_j^2} - 2 \nu^2 \left( \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right)^2 \]

(4) (5) (6)

\[
-2 \nu \frac{\partial^2 u_i}{\partial x_i \partial x_j} - c_D \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j} - 2 \nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j} \]

(A4.7)

(7) (8) (9)
Eq. (A4.7) gives the exact transport equation for $\varepsilon$. In its present form it contains several unknown correlations. As with the $k$-model some assumptions are made in order to proceed.

1) The first term on the righthand side is a production term. For dimensional consistency it is treated similar to the production term of the $k$-equation, thus (Taylor et al., 1981):

$$-2 \nu \frac{\partial u_i}{\partial x_k} \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_k}{\partial x_i} \right) = -\frac{\partial u_i}{\partial x_k} \left(-C_4 C_2 \frac{e^{u_i u_k}}{C_2} + C_6 \frac{\delta_{ik}}{k} \right)$$

Using the Reynolds stress hypothesis:

$$= C_1 C_2 \frac{e^{u_i}}{k} \left( \nu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) -2/3 k \frac{\delta_{ik}}{k} \right) =$$

$$C_1 C_2 \frac{e^{u_i}}{k} \left( \nu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) -2/3 k \frac{\delta_{ik}}{k} \right)$$

Using the continuity equation:

$$-c_0 \frac{u_j}{\partial x_j} \frac{\partial e^{'j}}{\partial x_j} = -c_0 \frac{\partial (u_j e^{'j})}{\partial x_j}$$

This term is interpreted as a diffusion term. Writing the expression according to the general diffusion law (see 2.14)

$$-\frac{\partial (u_j e^{'j})}{\partial x_j} = 1/\rho \frac{\partial}{\partial x_j} \left( \mu \frac{\partial e^{'j}}{\partial x_j} \right)$$

It can be shown that (Taylor et al. 1981) terms (4) and (7) are negligible with respect to the remaining terms of (A4.7)

3) For dimensional consistency the terms (6) and (9) are approximated by:
Substituting from (A4.8)-(A4.10) in (A4.7) yields:
\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial \varepsilon}{\partial x_j} \right) = C_1 \varrho \frac{\varepsilon}{\mu} \frac{\partial^2 \varepsilon}{\partial x_k^2} + C_2 \varrho \frac{\varepsilon^2}{\mu} + \frac{\partial}{\partial x_j} \left( \mu + \mu_s \right) \frac{\partial \varepsilon}{\partial x_j}
\]

Using \(\nu_t = \frac{\varrho}{\mu} k^{1/2}\):
\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial \varepsilon}{\partial x_j} \right) = C_1 \varrho \frac{\varepsilon}{\mu} \frac{\partial^2 \varepsilon}{\partial x_k^2} + C_2 \varrho \frac{\varepsilon^2}{\mu} + \frac{\partial}{\partial x_j} \left( \mu + \mu_s \right) \frac{\partial \varepsilon}{\partial x_j}
\]

In the derivation of the transport equation for "\(\varepsilon\)" \(C_1, C_2, \sigma_\varepsilon, \sigma_k, \sigma_\mu\) are assumed to be constants.

For boundary layers with high Reynolds numbers this equation can be treated similar to the \(k\)-equation (see appendix 3). This results in:
\[
\frac{d\varepsilon}{dt} = \varrho C_1 \frac{\varepsilon}{\mu} (\frac{\partial u_j}{\partial y})^2 - C_2 \varrho \frac{\varepsilon^2}{\mu} + \frac{\partial}{\partial y} (\mu + \mu_s) \frac{\partial \varepsilon}{\partial y}
\]

The general form of the \(z\)-equation for boundary layers at high Reynolds numbers may be written as (Lauder & Spalding, 1972):
\[
\frac{dz}{dt} = z(\frac{C_1 \mu_k}{\mu} (\frac{\partial u}{\partial y})^2 - C_2 \frac{\varrho^2}{\mu_k}) + \frac{\partial}{\partial y} (\mu_k \frac{\partial z}{\partial y}) + S_z
\]

where \(S_z\) represents source terms.

The \(\varepsilon\)-equation can easily be derived from the general form by substituting:
\[
z = \varepsilon = k^{3/2} \quad \text{and using} \quad \mu_k = \frac{\varrho}{\mu} 1 k^{1/2}\]
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