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1 Introduction

The Stokes flow in a rectangular cavity with two opposite fixed and two opposite moving walls and a (rotating) cylinder in the centre is studied and an analytical solution is derived. This flow is an extension of the cavity flow with two opposite moving walls, which is considered as a prototype flow for investigation of mixing problems. The basis of the analysis of mixing is the velocity field. Simulation results are presented in the form of streamline patterns and some of these are compared with experimental results.

The motivation for studying this kind of flow is to have some well defined prototype flows available for the investigation of chaotic mixing of viscous fluids in low-Reynolds-number flows in closed regions. These prototype flows should not only be accessible from a theoretical point of view, but also experimentally tractable. Despite real flows in nature and in mixing devices are usually much more complex and, moreover, many fluids show non-Newtonian behaviour, studying more simple cases of Newtonian flows is useful in gaining insight and understanding the basic features of mixing flows. An important, well-known example, is the 2-D flow in a rectangular cavity, driven by the periodic sliding motion of two opposite walls extensively investigated by Ottino and co-workers. The flow as presented here is an extension of this cavity flow.

The basics of understanding the behaviour of such dynamic systems lie in the knowledge of the periodic points of different types (elliptic and hyperbolic) of the system. Estimates of the position of the elliptic points and islands surrounding them can be found with relatively ease experimentally. Exact knowledge of the position of the elliptic points is not possible from experiments. To reveal hyperbolic points experimentally is mostly impossible. Application of numerical or analytical techniques is needed. Although a numerical approach is the only way to deal with real problems, analytical solution for prototype flows are needed to check the numerical codes and, moreover, they are a very powerful tool to study the properties of such dynamic systems.

The theoretical investigation of these problems starts with the analysis of the velocity field. An analytical solution for the cavity flow as used by Leong and Ottino (1989), based on a superposition, was given by Grinchenko, Isaeva and Meleshko (1991). This solutions was used as a basis for an algorithm for the determination of periodic points in this flow.

In this article the derivation of an analytical solution of the velocity field is described. It implies the construction of a general solution for the governing biharmonic equation in terms of a sum of three ordinary Fourier series. Each of these series is "responsible" for fulfilling the boundary conditions only on one part of the boundary and, thus, the sum ("superposition") of these series permits us to construct a solution which potentially is sufficient to satisfy any velocity condition on all the boundaries. Owing to the nonorthogonality of the functions, the coefficients of the Fourier series have to be determined from an infinite system of linear algebraic equations. We use a correct procedure of truncation of that system, based on properties of the well-known solution for Taylor's (1962) flow near the corners. The finite system can be solved by any numerical method and gives us an accurate velocity field within the flow domain. An accurate and low computational cost velocity field is one of the main advantages of the proposed method when intending to study the process of chaotic advection in such a domain.

2 Problem formulation and analytical solution

2.1 Formulation of the problem

Consider a two-dimensional creeping flow of a viscous fluid in a rectangular cavity $|X| \leq W$, $|Y| \leq H$ with a circular cylinder $X^2 + Y^2 \leq R_c^2$ in its centre, see figure 1. The fluid motion is
produced by the horizontal velocities $V_{top}(X)$ and $V_{bot}(X)$ on the top and bottom wall $|Y| = H$ and the circumstantial velocity $U(\theta)$ of the cylinder surface. Consideration are restricted to the specific case of uniform velocities $V_{top}$, $V_{bot}$ and $U$ along the corresponding boundaries. This implies a discontinuity in the velocity field in the corner points $|X| = W$, $|Y| = H$ which, strictly saying, do not belong to the domain of the flow. Taylor showed (1962) that such a discontinuity in the velocity leads to the an infinite value of applied force in order to produce such a jump. More general (physically more correct) boundary conditions with a nonuniform distribution of velocities with and $V_{top}(\pm W/2) = 0$, $V_{bot}(\pm W/2) = 0$ can be considered in a similar way.

For a two-dimensional incompressible flow the stream function $\Psi$ under Stokes approximation has to satisfy the biharmonic equation

$$\nabla^2 \nabla^2 \Psi = 0,$$

(1)

where $\nabla^2$ is the Laplace operator. The components of the velocity field in the Cartesian $(X, Y)$ and polar $(R, \theta)$ coordinates can be expressed by means of stream function $\Psi$ as

$$U_X = \frac{\partial \Psi}{\partial Y}, \quad U_Y = -\frac{\partial \Psi}{\partial X}$$

(2)

and

$$U_R = \frac{1}{R} \frac{\partial \Psi}{\partial \theta}, \quad U_\theta = -\frac{\partial \Psi}{\partial R},$$

(3)

respectively.

The boundary conditions for the equation (1) corresponding to the flow in the cavity are

$$U_X = V_{top}, \quad U_Y = 0, \quad Y = \frac{H}{2},$$

$$U_X = V_{bot}, \quad U_Y = 0, \quad Y = -\frac{H}{2},$$

$$U_X = 0, \quad U_Y = 0, \quad X = \pm \frac{W}{2},$$

$$U_R = 0, \quad U_\theta = U, \quad R = R_0.$$ (4)

Using the relations (2) and (3) it is easy to see that the Stokes flow in the cavity can be described in terms of $\Psi$ and the values of this function and its normal derivative prescribed at the boundaries. This problem is mathematically equivalent to the linear problem of the bending of thin elastic plate with a clamped edges. This remarkable analogy was pointed out first by Rayleigh (1893, 1911), who considered some interesting examples of the creeping flow in a half plane over a sinusoidal boundary and the flow in corners.

Due to the linearity of the boundary problem (1) – (4), the solution for arbitrary velocities $V_{top}$, $V_{bot}$ and $U$ can be expressed as

$$\Psi = \frac{V_{top} - V_{bot}}{2} \Psi_A + \frac{V_{top} + V_{bot}}{2} \Psi_B + U \Psi_C,$$

(5)

where the functions $\Psi_A$, $\Psi_B$ and $\Psi_C$ are the solutions of three basic boundary problems A, B and C with the velocities:

$$A : \quad V_{top} = 1, \quad V_{bot} = -1, \quad U = 0;$$

$$B : \quad V_{top} = 1, \quad V_{bot} = 1, \quad U = 0;$$

$$C : \quad V_{top} = 0, \quad V_{bot} = 0, \quad U = 1.$$ (6)

The importance of such a splitting consists in the fact that the problems A, B and C have to be solved only once. Then, representation (5) gives the solution for any time dependent boundary conditions. Another advantage of this representation is that some symmetry properties of the stream functions can be used. From the schematic picture for the velocities in Fig. 2 it can be
seen that for problems A and C the velocity component $U_X$ is even on $X$ and odd on $Y$, while $U_Y$ is odd on $X$ and even on $Y$. For the problem B the velocity component $U_X$ is even on $X$ and even on $Y$, while $U_Y$ is odd on $X$ and odd on $Y$. Therefore, the following symmetry conditions for stream functions are imposed

$$
\Psi_A(X,Y) = \Psi_A(-X,Y) = \Psi_A(X,-Y), \\
\Psi_B(X,Y) = \Psi_B(-X,Y) = -\Psi_B(X,-Y), \\
\Psi_C(X,Y) = \Psi_C(-X,Y) = \Psi_C(X,-Y).
$$

### 2.2 The analytical solution

The solutions of the problems A, B and C with the conditions (7) can be constructed analytically using the superposition method, originally proposed by Lame (1852). Applying this method to the problem under study, the main idea consists of using the sum of three ordinary Fourier series on the complete systems of trigonometric functions on $X$, $Y$ and $\theta$ coordinates to represent the stream function. All these series satisfy identically the biharmonic equation (1) inside the cavity and have sufficient functional arbitrariness to fulfill the two boundary conditions on each boundary. Because of the interdependency, the expression for a coefficient of a term in one series will depend on all coefficients of the other series and vice versa, and thus, the final solution involves the solution of an infinite system of linear algebraic equations giving the relations between the coefficients.

The solution of all three problems are obtained in a similar manner. Therefore, we will only consider in detail the construction of the solution of problem A. Using the half width of cavity to scale the dimensions, the dimensionless coordinates $x = 2X/H$, $y = 2Y/H$, $r = 2R/H$ and the stream function $\psi = 4\Psi/H^2$ are defined. Then the boundary conditions for $\psi_A$ have the form

$$
\begin{align*}
  &u_y = \frac{\partial \psi_A}{\partial x} = 0, \quad u_x = -\frac{\partial \psi_A}{\partial y} = 0, \quad x = \pm h, \quad |y| \leq 1, \\
  &u_y = \frac{\partial \psi_A}{\partial x} = 0, \quad u_x = -\frac{\partial \psi_A}{\partial y} = \pm 1, \quad y = \pm 1, \quad |x| \leq h, \\
  &u_r = \frac{\partial \psi_A}{\partial r} = 0, \quad u_\theta = \frac{1}{a} \frac{\partial \psi_A}{\partial \theta} = 0, \quad r = a, \quad 0 \leq \theta \leq 2\pi,
\end{align*}
$$

where $h = W/H$ and $a = 2R_0/H$ are the dimensionless length of the cavity and radius of the cylinder, respectively. Note that with the new coordinates, we still keep $V_{top} = 1$, $V_{bot} = -1$, see (6).

### 2.2.1 Representation of the stream function

The main idea of the superposition method consists in the representation of the solution of the biharmonic equation (1) in the form

$$
\psi_A = \psi_{A1} + \psi_{A2} + \psi_{A3}
$$

The solution $\psi_{A1}$ has to be constructed such that it provides the possibility to satisfy any two conditions on $y = \pm 1$. The solution $\psi_{A2}$ has to be sufficient to fulfill any two conditions on $x = \pm h$, while the solution $\psi_{A3}$ has to satisfy any two conditions on $r = a$. The most natural choice is the representation of these solutions in the form of a Fourier series of the complete trigonometric systems on the intervals $|x| \leq h$, $|y| \leq 1$, $0 \leq \theta \leq 2\pi$, respectively.

Using the symmetry of problem A (see Fig. 2), the functions $\psi_{A1}$ and $\psi_{A2}$ are taken in the same form as was done by Grinchenko, Isaeva, Meleshko (1991):

$$
\psi_{A1} = \sum_{m=1}^{\infty} (A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y) \cos \alpha_m x, \quad \alpha_m = \frac{2m - 1}{2h} \pi,
$$
\[ \psi_{A2} = \sum_{l=1}^{\infty} (C_l \cosh \beta_l x + D_l \beta_l x \sinh \beta_l x) \cos \beta_l y, \quad \beta_l = \frac{2l-1}{2} \pi. \] (11)

The solution \( \psi_{A3} \) can be obtained from the general solution of the biharmonic equation (1) in polar coordinates \((r, \theta)\) as originally derived by Michell & Read (1899), (see also Timoshenko & Goodier 1951 and Hellow & Coutanceau 1992):

\[
\psi(r, \theta) = a_0 r^2 + b_0 r^2 \ln(r-1) + c_0 \ln r + d_0 \theta \\
+ (a_1 r + b_1 r^{-1} + b'_1 r \theta + c_1 r^3 + d_1 r \ln r) \cos \theta \\
+ (e_1 r + f_1 r^{-1} + f'_1 r \theta + g_1 r^3 + h_1 r \ln r) \sin \theta \\
+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}) \cos n\theta \\
+ \sum_{n=2}^{\infty} (e_n r^n + f_n r^{-n} + g_n r^{n+2} + h_n r^{-n+2}) \sin n\theta
\] (12)

Taking into account the symmetry properties required, namely:

\[ \psi_{A3}(r, \theta) = \psi_{A3}(r, \pi - \theta) = \psi_{A3}(r, -\theta), \]

and choosing only those terms in (12) that decrease with increasing \( r \) and one term that has a minimal order in \( r \) and is not a function of the angle \( \theta \) (the term with \( \ln r \)) the solution \( \psi_{A3} \) can be presented in the form

\[ \psi_{A3} = E_1 \ln r + \sum_{j=1}^{\infty} \left[ F_j \left( \frac{r}{a} \right)^{-2j} + G_j \left( \frac{r}{a} \right)^{-2j+2} \right] \cos 2j\theta, \] (13)

The contributions of series \( \psi_{A1}, \psi_{A2} \) and \( \psi_{A3} \) to the velocity components \( u_x \) and \( u_y \) can be obtained by using the relations (2) and (3).

From the very beginning the expressions (10), (11), (13) contain the set of arbitrary coefficients \( A_m, B_m, C_l, D_l, E_1, F_j, G_j \) which are sufficient to satisfy any prescribed conditions on all boundaries.

2.2.2 Boundary conditions on the rectangle’s sides

The Cartesian components \( u_x^{(A3)}, u_y^{(A3)} \) of the velocity field from the solution (13) in polar coordinates can be derived as

\[
\begin{align*}
\hat{u}_x^{(A3)}(x, y) &= u_x^{(A3)}(r, \theta) \cos \theta - u_y^{(A3)}(r, \theta) \sin \theta, \\
\hat{u}_y^{(A3)}(x, y) &= u_x^{(A3)}(r, \theta) \sin \theta + u_y^{(A3)}(r, \theta) \cos \theta,
\end{align*}
\] (14)

where \( r = \sqrt{x^2 + y^2} \), \( \theta = \arctan y/x \).

The contributions from \( \psi_{A3} \) to the velocity components on the boundaries \( |x| = h \) and \( |y| = 1 \) may be expanded into Fourier series:

\[
\begin{align*}
\hat{u}_x^{(A3)}(x, 1) &= \sum_{m=1}^{\infty} (-1)^m P_m \cos \alpha_m x, & \hat{u}_y^{(A3)}(x, 1) &= \sum_{m=1}^{\infty} (-1)^m \alpha_m Q_m \sin \alpha_m x, \\
\hat{u}_x^{(A3)}(h, y) &= \sum_{l=1}^{\infty} (-1)^{l+1} \beta_l R_l \sin \beta_l y, & \hat{u}_y^{(A3)}(h, y) &= \sum_{l=1}^{\infty} (-1)^{l+1} S_l \cos \beta_l y.
\end{align*}
\] (15)
Here $P_m$, $Q_m$, $R_l$, $S_l$ can be put in terms of $E_1$, $F_j$, $G_j$ in the following short notations:

$$
\begin{align*}
P_m &= P^{(m)}_{E_1} E_1 + \sum_{j=1}^{\infty} P^{(m)}_{F_j} F_j + \sum_{j=1}^{\infty} P^{(m)}_{G_j} G_j, \\
Q_m &= Q^{(m)}_{E_1} E_1 + \sum_{j=1}^{\infty} Q^{(m)}_{F_j} F_j + \sum_{j=1}^{\infty} Q^{(m)}_{G_j} G_j, \\
R_l &= R^{(l)}_{E_1} E_1 + \sum_{j=1}^{\infty} R^{(l)}_{F_j} F_j + \sum_{j=1}^{\infty} R^{(l)}_{G_j} G_j, \\
S_l &= S^{(l)}_{E_1} E_1 + \sum_{j=1}^{\infty} S^{(l)}_{F_j} F_j + \sum_{j=1}^{\infty} S^{(l)}_{G_j} G_j,
\end{align*}
$$

The coefficients $P^{(m)}_{E_1}, \ldots, S^{(l)}_{G_j}$ are integrals, containing oscillating trigonometric functions – see Appendix A1. These integrals can be computed numerically using standard procedures.

From the requirement that normal components of velocity equals zero at the cavity boundaries, the following relations between coefficients are found using (10), (11), (15), (16):

$$
\begin{align*}
A_m &= -B_m \alpha_m \tanh \alpha_m - (-1)^m \frac{Q_m}{R_l} \\
C_l &= -D_l \beta_l \tanh \beta_l - (-1)^l \frac{R_l}{R_l}.
\end{align*}
$$

Using the known expansions of functions (Prudnikov, Brychkov, Marichev 1981)

$$
\begin{align*}
\cosh \kappa z &= \frac{2}{z_0} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\xi_n}{\xi_n^2 + \kappa^2} \cosh \kappa z_0 \cos \xi_n z, \\
z \sinh \kappa z &= \frac{2}{z_0} \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{\xi_n z_0}{\xi_n^2 + \kappa^2} \sinh \kappa z_0 - \frac{2 \xi_n \kappa}{(\xi_n^2 + \kappa^2)^2} \cosh \kappa z_0 \right] \cos \xi_n z
\end{align*}
$$

where $\xi_n = \frac{2n - 1}{2z_0} \pi z$, $|z| \leq z_0$,

from the boundary conditions on $x = h$ and $y = 1$ for tangential velocities we obtain the following two sets of equations:

$$
\begin{align*}
-X_m \Delta(\alpha_m) + \sum_{l=1}^{\infty} Y_l \frac{4 \alpha_m^2}{(\beta_l^2 + \alpha_m^2)^2} + \sum_{j=1}^{\infty} K_{F_j}^X F_j + \sum_{j=1}^{\infty} K_{G_j}^X G_j + K_{E_1}^X E_1 &= \frac{2}{h}, \\
\quad m = 1, 2, \ldots, \\
Y_l h^2 \Delta(\beta_l h) - \sum_{m=1}^{\infty} X_m \frac{4 \beta_l^2}{(\alpha_m^2 + \beta_l^2)^2} + \sum_{j=1}^{\infty} K_{F_j}^Y F_j + \sum_{j=1}^{\infty} K_{G_j}^Y G_j + K_{E_1}^Y E_1 &= 0, \\
\quad l = 1, 2, \ldots.
\end{align*}
$$

Here, the new unknowns

$$
X_m = (-1)^{m+1} B_m \alpha_m^3 \cosh \alpha_m, \quad Y_l = (-1)^l D_l \beta_l^2 \frac{1}{h} \cosh \beta_l h
$$

and the function

$$
\Delta(z) = \frac{\tanh z}{z} + \frac{1}{\cosh^2 z}
$$

are introduced.

The expressions for the coefficients $K_{F_j}^X, \ldots, K_{E_1}^Y$ are given in Appendix A2.
2.2.3  Boundary conditions on the cylinder’s surface

Two other sets of equations can be obtained from evaluating the normal and tangential velocity components on the cylinder surface \( r = a \).

The contribution from the functions \( \psi_{A1} \) and \( \psi_{A2} \) to velocity components \( u^A_{r} \) and \( u^A_{\theta} \) on the cylinder surface can be derived according to

\[
\begin{align*}
  u^A_{r}(r, \theta) &= u^A_{\infty}(x, y) \cos \theta + u^A_{v}(x, y) \sin \theta, \\
  u^A_{\theta}(r, \theta) &= -u^A_{\infty}(x, y) \sin \theta + u^A_{v}(x, y) \cos \theta,
\end{align*}
\]

with \( x = r \cos \theta \), \( y = r \sin \theta \).

The expansion, similar to the one in the previous subsection, of the velocities \( u_r(a, \theta), u_\theta(a, \theta) \) into Fourier series on \( \sin 2j \theta \) and \( \cos 2j \theta \), respectively, leads to the following sets of linear algebraic equations:

\[
\begin{align*}
  \sum_{m=1}^{\infty} K^r_m X_m + \sum_{l=1}^{\infty} K^r_l Y_l + \sum_{i=1}^{\infty} K^r_i F_i + \sum_{i=1}^{\infty} K^r_i G_i + K^r_{E1} E_1 &= 0, & j &= 1, 2, \ldots, \\
  \sum_{m=1}^{\infty} K^\theta_m X_m + \sum_{l=1}^{\infty} K^\theta_l Y_l + \sum_{i=1}^{\infty} K^\theta_i F_i + \sum_{i=1}^{\infty} K^\theta_i G_i + K^\theta_{E1} E_1 &= 0, & j &= 0, 1, \ldots.
\end{align*}
\]

The expressions for the coefficients \( K^r_m, \ldots, K^\theta_{E1} \) are quite cumbersome (they are presented in Appendix A3), but the way to derive them is, however, straightforward.

2.2.4  Reduction of the infinite system

The infinite sets of the linear algebraic equations (19), (20), (24), (25) form the main system for the coefficients in the representation of the stream function. The general approach to the solution of boundary problems for linear partial equations, based upon their transformation to the solution of an infinite system of linear algebraic equations of the type

\[ z_n = \sum_{k=1}^{\infty} a_k^{(n)} z_k + b_n, \quad n = 1, 2, \ldots \]

was proposed by Fourier in his classical book (Fourier 1822). He adopted a practical approach which consists in regarding first the number of unknowns as finite and equal to \( N \). Thus, all equations which follow the first \( N \) equations are suppressed and, from the sums in the right hand side of (26), all terms that follow the first \( N \) are omitted. Then, for a given number \( N \), the unknowns \( x_n (n = 1, 2, \ldots, N) \) have some fixed values which can be found by any numerical technique. It is obvious that the values of these unknowns will vary as we increase their number and, respectively, the number of equations which ought to determine them. It is required to find the limits towards which the values of the unknowns converge as the number of equations increases. These limits are the true values which satisfy equations (26). Such an approach to the solution of the infinite system is called the method of reduction.

If some preliminary information about the asymptotic behaviour of unknowns yields the conclusion that they tend to zero rapidly enough with the increase of their number, then, the traditional way of solving the system consists of a simple reduction of the finite system by assuming that terms with a number higher than a chosen value may be neglected. The mathematical equivalent problem of an elastic rectangle (with \( X_m \) and \( Y_l \) coefficients only) was extensively examined by many authors. Important properties of the solution of a system of this kind were originally established by Koyalovich (1930). Under some conditions the unknowns exhibit an asymptotic behaviour, which in our case should be

\[ \lim_{n \to \infty} X_n = \lim_{n \to \infty} Y_n = \text{const} \]
Later, the conditions for such an asymptotic behaviour were found to be less rigid (see Grinchenko 1978; Grinchenko, Gomilko, Meleshko 1986; Gomilko 1993) then Koyalovich’s found. It was found that right hand parts of equations like (19) and (20) must decrease with the equation number with at least $O(1/a^m)$ and $O(1/b^2)$, respectively. However, in our case the equations (19) contain a constant $2/h$ (independent of $m$) and it was found that

$$\lim_{m \to \infty} \alpha_m K_{Gj}^X = \lambda_j, \quad \lim_{l \to \infty} \beta_l K_{Gj}^Y = -\lambda_j h,$$

where

$$\lambda_j = 8(2j - 1)a^{2j - 2}(h^2 + 1)^{-j} \frac{\cos(2j \arctan(1/h))}{h}.$$  

All other terms in (19) and (20) have the right decreasing order. Equations (24) and (25) are also supposed to produce no difficulties for system reduction.

The equations (19) and (20) require a more carefull investigation. In order to use the properties established for the system for a cavity without a cylinder, we formally treat all terms without $X_m$ and $Y_l$ in equations (19), (20) as free terms of such a “pure” system. The main task now is to find a possibility to establish the behaviour of the unknowns in the infinite system with the increase of their number. This is important, because these unknowns are, as a matter of fact, the coefficients of the Fourier series. If the asymptotic properties of the unknowns are obtained, then the amount of numerical work decreases and it turns out to be possible to obtain the information about all unknowns of the infinite system after solving a system of finite dimension.

To exclude the terms with an unsuitable decreasing order from the equations (19) and (20), the substitution

$$X_m = x_m + c \alpha_m + \sum_{j=1}^{\infty} g_j G_j, \quad Y_l = y_l + d \beta_l$$

is used, with yet undetermined constants $c$, $d$ and $g_j$. The equations (19), (20) then can be rewritten in the form

$$x_m \Delta(\alpha_m) = \sum_{i=1}^{\infty} y_l \left( \frac{4\alpha_m^2}{(\beta_l^2 + \alpha_m^2)^2} \right) + \Omega^{(x)}_m, \quad m = 1, 2, \ldots,$$

$$y_l h^2 \Delta(\beta_l h) = \sum_{m=1}^{\infty} y_m \left( \frac{4\beta_l^2}{(\alpha_m^2 + \beta_l^2)^2} \right) + \Omega^{(y)}_l, \quad l = 1, 2, \ldots,$$

where

$$\Omega^{(x)}_m = 4d \alpha_m^2 \sum_{i=1}^{\infty} \frac{\beta_l}{(\beta_l^2 + \alpha_m^2)^2} - c \alpha_m \Delta(\alpha_m) + \frac{2}{h} + \sum_{j=1}^{\infty} K_{Gj}^X E_j,$$

$$+ \sum_{j=1}^{\infty} \left( K_{Gj}^X - \Delta(\alpha_m) g_j \right) G_j + K_{E1}^X E_1, \quad m = 1, 2, \ldots$$

$$\Omega^{(y)}_l = 4c \beta_l^2 \sum_{m=1}^{\infty} \frac{\alpha_m}{(\alpha_m^2 + \beta_l^2)^2} - d \beta_l h^2 \Delta(\beta_l h) - \sum_{j=1}^{\infty} K_{Gj}^Y E_j,$$

$$- \sum_{j=1}^{\infty} \left( K_{Gj}^Y + g_j \sum_{m=1}^{\infty} \frac{4\beta_l^2}{(\alpha_m^2 + \beta_l^2)^2} \right) G_j - K_{E1}^Y E_1, \quad l = 1, 2, \ldots$$

Using the values of sums (Prudnikov et al. 1986):

$$\sum_{k=1}^{\infty} \frac{1}{((2k - 1)^2 + a^2)^2} = \frac{\pi}{8a^3} \coth \frac{\pi a}{2} - \frac{\pi^2}{8a^4} \cosh^{-2} \frac{\pi a}{2},$$

9
where \( \psi'(z) = d \ln G(z)/dz^2 \) is well-known three gamma function, and the asymptotic

\[
\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots \quad (z \to \infty, \ |\arg z| < \pi)
\]

it is found that

\[
\Omega_m^{(x)} = \left( \frac{2d}{\pi} - c + \frac{2}{h} \right) + \frac{1}{\alpha_m} \sum_{j=1}^{\infty} (\lambda_j - g_j)G_j + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty,
\]

\[
\Omega_l^{(y)} = \left( \frac{2ch}{\pi} - dh \right) + \frac{h}{\beta_l} \sum_{j=1}^{\infty} (\lambda_j - g_j)G_j + O \left( \frac{1}{\beta_l^2} \right), \quad l \to \infty.
\]

Demanding for \( \Omega_m^{(x)} \) and \( \Omega_l^{(y)} \) to have a decreasing order \( O(1/\alpha_m^2) \) and \( O(1/\beta_l^2) \), respectively, the constants \( c, d \) and \( g_j \) are obtained:

\[
c = \frac{2\pi^2}{h(\pi^2 - 4)}, \quad d = \frac{4\pi}{h(\pi^2 - 4)}, \quad g_j = \lambda_j.
\]

Now the free terms in the system (31) have the necessary decreasing order and it is possible, in principle, to try to solve the system (31) and (24), (25) by means of the method of simple reduction, neglecting all unknowns having a subscript larger than certain values \( M, L \) and \( J \). In that case it is implicitly supposed that the coefficients \( x_m, y_l, F_j, G_j \) have a limiting value of zero when \( m \to \infty, l \to \infty \) and \( j \to \infty \). However, it was shown by Grinchenko, Gomilko, Meleshko (1986) that for systems like (31) the following asymptotic is valid:

\[
\lim_{m \to \infty} x_m = \lim_{l \to \infty} y_l = a_0
\]

Moreover, it is possible to show that for the case of a cavity flow without a cylinder, the value \( a_0 \) can be obtained explicitly without solving the whole system. In our case it can be shown, by repeating such an analytical procedure, that \( a_0 \) does not depend explicitly on the coefficients \( x_m, y_l \), but is expressed in terms of the coefficients \( F_j, G_j, E_1 \) (and implicitly depends on the coefficients \( x_m, y_l \) through the equations (24), (25)). Therefore, such an analytical approach for finding the value \( a_0 \) from the infinite system seems to be not very useful. Here it appears most efficient to form the finite system of equations as follows:

\[
x_m = a_0 \quad (m > M), \quad y_l = a_0 \quad (l > L),
\]

\[
F_j = 0 \quad (j > J), \quad G_j = 0 \quad (j > J),
\]

\[
a_0 = \frac{x_M + y_L}{2}
\]

It is important to notice that, after solving the finite system corresponding to (40), we have essential knowledge about all infinite sets of coefficients. Using the well-known method of improving the convergence of Fourier series, the velocity field in the cavity, including the boundaries, can be expressed in an explicit form (see Appendix A4).

Furthermore, it is important to stress that the construction of the solution for the stream function \( \psi_A \) according to (9), (10), (11), (13) satisfies identically the biharmonic equation (1) for any finite number of arbitrary coefficients. Therefore, the only way to estimate the quality of the solution is to check the accuracy of matching to the boundary conditions, especially near the corner points \( |x| = h, |y| = 1 \). At these points the component \( u_x \) has a discontinuity.
This circumstance is crucial for the method of eigenfunction expansion, developed by Joseph & Sturges (1978) and recently implemented by Shankar (1993) for a rectangular two-dimensional cavity with steady motion of the top wall. After tremendous calculations, taking into account 100 complex eigenfunctions (in fact 400 unknown coefficients) it appears that the accuracy of satisfying the prescribed boundary velocity \( u_x \) on the top wall, within the vicinity of the corner (1% of cavity width), is of the order of its magnitude, despite of the high accuracy on the rest of the boundaries.

High accuracy results for fixed walls and a steady rotating cylinder (corresponding to the problem \( C \) in our notation) were obtained by Hellow & Coutanceau (1992). They used the general solution in polar coordinates (12) for relatively short cavities \( h \leq 3 \), complemented with eigenfunction expansions for \( 3 < h \leq 7 \) with a least square minimization procedure. Using 61 coefficients and 181 minimization points these authors reported about excellent accuracy (up to \( 10^{-7} \)) of satisfying of zero velocities on the wall, whereas the boundary conditions on the cylinder are fulfilled identically.

The accuracy of the superposition method presented in this paper is demonstrated in tables 1–3 where calculated velocities on the boundaries (also in the corner points) and in some test points inside the cavity are given. It is worthwhile to note that, even for a small number of coefficients \( 5 \), a good accuracy is obtained, even close to the corner point . The solution of such small finite system takes only few seconds of computational time on personal computer. It is necessary to emphasize that the values of \( c \) and \( d \) given by (38), which were obtained from the asymptotic analysis of the system (31), are very important near the corner points (see also the expressions for \( \Lambda_{x1}, \Lambda_{y1} \) in Appendix A4). The physical meaning of these values becomes more clear if we put down the expression of the stream function \( \psi_A \) in the polar coordinates \( (\rho, \chi) \), associated with corner point (for example \( x = -h, y = -1 \)). Putting in the expressions for \( \psi_A \) (after it is improved with respect to the convergence, like it was done for velocities) \( x = -h + \rho \cos \chi, y = -1 + \rho \sin \chi \), one obtains

\[
\psi_A(\rho, \chi) = \frac{4\rho}{\pi^2 - 4} \left( \chi \cos \chi + \frac{\pi}{2} \chi \sin \chi - \frac{\pi^2}{4} \sin \chi \right) + O(\rho^2) \tag{41}
\]

The main part of this expression corresponds to the Taylor solution (Taylor 1962, see also Batchelor 1967, Art. 4.8) for the Stokes flow in an infinite quarter plane with a constant tangential velocity of one of the solid walls. The difference between the Taylor’s solution (41) and the results of calculations of the streamfunction \( \psi_A \) is demonstrated in figure 3 the values of \( \psi_A \) and Taylor’s solution (that is the main term in (41)) on the bisector of the angle \( (\chi = \pi/4) \) are plotted versus the distance \( \rho \) from the corner top. It is seen that they correspond, even at relatively large distances, comparable with the cavity size.
angular coordinate $\theta$ of the point $(a, \theta)$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$J$</th>
<th>$\theta$</th>
<th>$u_\theta$</th>
<th>$u_r$</th>
<th>$u_\theta$</th>
<th>$u_r$</th>
</tr>
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<td>$-9.0 \times 10^{-9}$</td>
<td>$-9.8 \times 10^{-9}$</td>
<td>$-5.4 \times 10^{-9}$</td>
<td>$8.4 \times 10^{-16}$</td>
</tr>
<tr>
<td>10'</td>
<td>6'</td>
<td>6'</td>
<td>$\pi/4$</td>
<td>$-1.1 \times 10^{-8}$</td>
<td>$-9.8 \times 10^{-9}$</td>
<td>$-5.5 \times 10^{-9}$</td>
<td>$6.1 \times 10^{-15}$</td>
</tr>
<tr>
<td>5'</td>
<td>3'</td>
<td>3'</td>
<td>$\pi/2$</td>
<td>$-3.9 \times 10^{-6}$</td>
<td>$-7.6 \times 10^{-6}$</td>
<td>$9.1 \times 10^{-8}$</td>
<td>$3.5 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td></td>
<td>$3.2 \times 10^{-5}$</td>
<td>$1.0 \times 10^{-4}$</td>
<td>$3.8 \times 10^{-4}$</td>
<td>$3.5 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

3 Results

When the reduced system of linear algebraic equations for problem $A$ is solved, and the values of the coefficients in the representation of the stream function are obtained for the given geometrical parameters $a$ and $h$, it is possible to calculate the velocity field using the expressions given in Appendix A4. The same holds for the stationary boundary problems $B$ and $C$ (6). Next, we can examine the flow regimes for different geometrical parameters and combinations of the boundary velocities. The most convenient and widely exploited way to reveal the structure of the flow is by means of streamline patterns. Streamline patterns can be plotted as contour maps of the stream function itself without computation of the velocity field. We will follow this technique. The main part of the results presented in this article is obtained for cylinder radius $a = 0.3$ and aspect ratio of the cavity $h = 1.67$. The choice of the value of aspect ratio is motivated by the wide availability of experimental results for a rectangular cavity without the cylinder with such an aspect ratio (see, for example, Leong & Ottino 1989). Flow visualization experiments were carried out in a cavity similar to the one used by Ottino and co-workers which

coordinates of point $(x, y)$ inside the cavity

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$J$</th>
<th>$(1, 0)$</th>
<th>$u_x$</th>
<th>$u_y$</th>
<th>$(1, 0.8)$</th>
<th>$u_x$</th>
<th>$u_y$</th>
<th>$(0, 0.8)$</th>
<th>$u_x$</th>
<th>$u_y$</th>
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<td>0.47687</td>
<td>$-0.08039$</td>
<td>0.73941</td>
<td>$-6.2 \times 10^{-18}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10'</td>
<td>6'</td>
<td>6'</td>
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<td>$-0.40181$</td>
<td>0.47687</td>
<td>$-0.08039$</td>
<td>0.73941</td>
<td>$-6.2 \times 10^{-18}$</td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>$-0.08036$</td>
<td>0.73939</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3</td>
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<td>$-6.2 \times 10^{-18}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Dependence of the calculated values of $u_x$ and $u_y$ for a few test points inside the cavity on the number of unknowns
was extended with a cylinder in the centre.

First, the flow in the cavity generated by moving the top and bottom wall ("top" and "bottom" refer to the figure 1) in opposite directions with the same velocity and a fixed cylinder, is considered. This corresponds to the stationary boundary problem A (6). In this case the structure of the flow strongly depends on the geometrical parameters of the cavity, most of all on the aspect ratio \( h \). In relatively a short cavity, for example a square one (figures 4a,b), two vortex zones, attached to the cylinder, are observed. The total height of these vortexes and the cylinder, which can be treated as the "effective size" of an obstacle (Grinchenko, Galaktionov 1993), only slightly depends on the radius of the cylinder. As the aspect ratio of the cavity increases the vortex zones rapidly become more "flat" and disappear. Then, within a wide range of the \( h \), the flow has a very simple structure without any vortexes (see figure 4c and, for the experimental results, figure 4d). The flow near the corner points exhibits remarkable similarities with the above mentioned Taylor (1962) solution. If we continue to elongate the cavity, a new pair of vortexes arises on the left and right hand side of the cylinder. Figure 4e shows the streamline pattern for \( h = 2.5 \) and \( a = 0.3 \). This case will be discussed later in connection with the analogy to a pure shear flow.

Secondly, the flow generated by moving the top and bottom walls of the cavity in the same direction with equal velocities is considered. This corresponds to the stationary boundary problem B (6). The flow becomes separated into 2 symmetrical domains (see figure 5). In both domains pairs of vortexes are seen, separated by a saddle-type stagnation point. This structure is also found in the visualization experiment as is also shown in figure 5b. The flow structure weakly depends on the geometrical parameters. The most strong change occurs in the case of relatively short cavity (for example, \( h = 1 \)): with the decreasing of the cylinder radius both the vortex pairs in the upper and bottom halves of the cavity merge, forming single vortexes.

The flow generated by a rotating cylinder and fixed cavity walls corresponds to the stationary boundary problem C (6). This flow was already carefully investigated, both numerically and experimentally, by Hellou & Coutanceau (1992). The most remarkable feature of this kind of flow regime are the corner eddies. As it was shown by Moffatt (1961), this is a universal feature of the Stokes flows: infinite series of the vortexes can be found under some conditions in the vicinity of the sharp corner. In figure 6 the streamline pattern is presented for the geometrical parameters \( h = 1.67 \) and \( a = 0.3 \). The result of the visualization experiment is shown in figure 6b. In case of a 90° coner angle the ratio of intensities of the adjacent vortexes in the series predicted by Moffatt (1961) is at least about \( 2 \times 10^6 \). So even the primary vortexes are usually weak in comparison with the main flow. The secondary vortexes were revealed experimentally by Taneda (1980) but for a more sharp angle. In our case, retaining a significant number of coefficients in the representation of the streamfunction, it is possible to reveal the secondary vortexes in case of square cavity. However, such a precision is not necessary in the majority of cases and, moreover, the possibility of revealing experimentally the secondary vortexes for this geometry is doubtful. The evolution of the primary vortexes with the increasing of the aspect ratio of the cavity (merging of vortexes and formation of a typical cellular flow structure) was already shown by Hellou & Coutanceau (1992).

The next flow regime considered is the combination of opposite moving walls and a rotating cylinder and the analogy of this flow with an unbounded shear flow. Robertson & Acrivos (1970) and later Jeffrey & Sherwood (1980) investigated the case of a rotating cylinder, immersed in an unbounded shear flow and centered on the stagnation line of undisturbed flow. These authors used for their analysis the exact Stokes solution for an infinite domain, originally proposed by Bretherton (1962). With a relatively long cavity and top and bottom walls moving with the same, opposite speed, we can mimic this flow in the central zone of the cavity. In our case we have a solution, valid for a particular flow in a finite cavity. So, it is possible to see the influence of the outer walls.
To simulate this quasi-shear flow, the velocities of the top and bottom wall are set equal to $V_{\text{top}} = 1$ and $V_{\text{bot}} = -1$ (the intensity of shear flow $k = \frac{dv_y}{dy} = 1$). The geometrical parameters of the cavity were chosen to be $a = 0.3$ and $h = 2.5$. Using the calculated velocity profiles in the vicinity of the cylinder it was verified that, for the given parameters, the cavity flow characteristics near the cylinder are very close to that of the unbounded shear flow. Five qualitatively different flow regimes, analogous to that described by Jeffrey & Sherwood (1980), may be generated varying the cylinder rotation speed $U$. Following them, to systematize this for our particular problem, the existence, position and type of stagnation points as well as separation streamlines were examined. The coordinates of stagnation points, located on the vertical and horizontal axes of symmetry and the coordinates where separation streamlines cross these axes are plotted in figure 7 as functions of the cylinder rotation speed $U$.

When the cylinder is counterrotating with respect to shear flow ($U > 0$), the flow domain is subdivided into 4 zones. The streamline pattern for $U = 0.25$ is presented in figure 8a. In the vicinity of the cylinder the fluid is rotating in the same direction as the cylinder does, whereas fluid in outer zone is forced to move in opposite direction. This leads to the arising of saddle-type stagnation points on the vertical axis of symmetry and of large vortices on the left- and right-hand sides of the cylinder. In case of unbounded shear flow the vortexes transform into zones of blocked flow (see Jeffrey & Sherwood 1980) extending to infinity.

If the cylinder rotation speed is slowed down, no qualitative changes are observed until the cylinder is stopped – see figure 7. When the cylinder is fixed, the zone where fluid is rotating against the shear flow, disappears. The flow structure becomes more simple with only two vortex zones attached to right and left side of the cylinder (see figure 8b).

When the cylinder is slowly co-rotating with respect to the shear flow, the flow domain again becomes subdivided into 4 zones. There are saddle points on the horizontal axis of symmetry and the separation streamline that crosses these stagnation points, separates two vortices and a zone around the cylinder from the main flow. In figure 8c the streamline pattern is presented for $U = -0.075$.

In an unbounded shear flow, according to Jeffrey & Sherwood (1980), the next qualitative change occurs when the speed of cylinder rotation achieves a critical value, for which the cylinder is freely rotating and the stagnation points disappear. In the classification of flow regimes suggested by Jeffrey & Sherwood the case of a freely rotating cylinder is of special importance. "Freely rotating" means that total torque on the cylinder applied by the fluid is zero. Following the theoretical predictions of Robertson & Acrivos (1970), we obtain that for the cavity flow for a freely rotating cylinder the tangential velocity on its surface should be $U = -0.15 (U = -\frac{1}{4}ka)$. Also in our case, a straightforward computation of the torque for the given geometrical parameters of the cavity leads to the same value for $U$ for which the torque is zero. The angular velocity $\Omega_{\text{free}}$ (in fact, $\Omega_{\text{free}} = U_{\text{free}}/a$, so in case of unbounded shear flow, according to Robertson & Acrivos, $\Omega_{\text{free}} = U_{\text{free}}/a = -0.5$) was computed for different geometrical parameters of the cavity. The results are briefly summarized in the table 4. In case of a cavity flow, the presence of the outer walls stimulates the collapse of the vortex structures and the disappearance of stagnation saddle points at a lower speed of cylinder rotation.

<table>
<thead>
<tr>
<th>Cylinder radius $a$</th>
<th>Aspect ratio of the cavity $h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>-0.3905</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.3903</td>
</tr>
</tbody>
</table>

Table 4: Dependence of the angular velocity $\Omega_{\text{free}}$ of the freely rotating cylinder on the aspect ratio $h$ of the cavity and the radius of the cylinder $a$. In all cases $V_{\text{top}} = 1$, $V_{\text{bot}} = -1$. 

To simulate this quasi-shear flow, the velocities of the top and bottom wall are set equal to $V_{\text{top}} = 1$ and $V_{\text{bot}} = -1$ (the intensity of shear flow $k = \frac{dv_y}{dy} = 1$). The geometrical parameters of the cavity were chosen to be $a = 0.3$ and $h = 2.5$. Using the calculated velocity profiles in the vicinity of the cylinder it was verified that, for the given parameters, the cavity flow characteristics near the cylinder are very close to that of the unbounded shear flow. Five qualitatively different flow regimes, analogous to that described by Jeffrey & Sherwood (1980), may be generated varying the cylinder rotation speed $U$. Following them, to systematize this for our particular problem, the existence, position and type of stagnation points as well as separation streamlines were examined. The coordinates of stagnation points, located on the vertical and horizontal axes of symmetry and the coordinates where separation streamlines cross these axes are plotted in figure 7 as functions of the cylinder rotation speed $U$.

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the given geometry of cavity the critical value is about $U_{\text{crit}} = -0.12$ (see also figure 7). The streamline pattern for this critical regime is presented in figure 8d.

When the cylinder is co-rotating with higher speed the flow structure becomes very simple; no stagnation points or vortices are observed (see figure 8e). According to Robertson & Acrivos (1970) in unbounded shear flow all streamlines become closed in a Stokes approximation.

From the results presented one can see that all regimes that are typical for a shear flow around an obstacle (in our case, a cylinder) are reproducable in a relatively small domain with a simple shape, at least if we are interested in the flow details in the vicinity of the obstacle.

A variety of sophisticated flow structures can be generated in the cavity by specifying different combinations of the geometrical parameters $a$ and $h$ and boundary velocities $V_{\text{top}}, V_{\text{bot}}$ and $U$. It is not the objective of the current article to try to demonstrate all types of the flow structures. Nevertheless, there is a practical interesting case that deserves to be mentioned. The study of chaotic mixing in a rectangular cavity was carried out by J.M.Ottino and co-workers (see, for example, Leong & Ottino 1989), Carey & Shen (1995) and other authors for the periodic flow driven by the successive motion of the two opposite walls. So, the flow pattern in the rectangular cavity without the obstacle with one moving wall is extensively examined.

The presence of a cylinder can strongly influence the structure of the flow generated by one moving wall. Two illustrative cases, a fixed and a counter rotating cylinder respectively, are presented. For the fixed cylinder (figure 9a), unlike for the cavity without the cylinder, a vortex zone arises on the side of the moving wall, attached to the cylinder. It is separated from the rest of the cavity by the separation streamline, attached to the cylinder. This zone contains two vortices, joined by a saddle-type stagnation point located on the vertical axis of symmetry. All these details of the flow structure are also found in the visualization experiments (figure 9b).

For the counter rotating cylinder the vortex zone with the pair of vortices is separated from the cylinder (figure 10a). An additional stagnation point arises in the bottom half of the cavity and a "dog snout" streamline pattern is formed. The corresponding experimentally obtained streamline pattern is given in figure 10b. The presence of the isolated vortex zone in the main flow can be of significant importance for the usage of this kind of flow for studying mixing phenomena. However, this is beyond the scope of the present article.

4 Conclusions

One of the most important items of the presented results is to demonstrate the usefulness and reliability of the analytical solution as given here and, moreover, of such an analitycal technique at all. Despite the effort spent to obtain the necessary analitycal expressions, it provides us with high-accuracy formulas for the stream function and velocity field, which in the given case are reliable even in the vicinity of corner points. The computation time, necessary to achieve an accuracy that is good enough for most applications (like the analysis of the flow structure) is relatively small.

The solutions obtained give the possibility to analyse the flow structure in a cavity with a cylinder. The quality of the solution is well demonstrated by the fact that it exhibits such local peculiarities as corner eddies, originally predicted by Moffatt (1964a, 1964b). A variety of flow regimes with complicated velocity structure, containing vortices, stagnation points, separation lines and cellular structure, is revealed in this relatively simple system. The remarkable feature is the strong dependence of the flow structure on the geometrical parameters.

The computed streamline patterns were verified by comparison with flow visualization experiments. The calculated results also provide the possibility to compare with some well-known, important studies on the shear flow around a cylinder.

The flow in a rectangular cavity is used as a prototype flow for studying fluid mixing phenomena in Stokes flows. The presented system, a cavity with a cylinder, is usefull to investigate
if the presence of an obstacle can stimulate the mixing (is stretching and folding of material) due to existence of the vortex zones, created by the obstacle. Adding of the cylinder rotation might be a simple way to remove islands (dead zones where no mixing takes place) in mixing patterns. Just as for the cavity without a cylinder, the system is analytically and experimentally tractable with relative ease.

A Appendixes

A.1 Expressions for the coefficients in formulas (16)

In the formulas presented below, the functions

\[ \xi_j(z) = 2j \arctan \frac{1}{z}, \quad \eta_j(z) = 2j \arctan \frac{z}{h}, \tag{42} \]

are introduced to simplify things. Of course, \( \xi_j(h) = \eta_j(1) \).

\[
P_{E1}^{(m)} = \frac{(-1)^m}{h} \int_{-h}^{h} \frac{\cos \alpha_m x}{x^2 + 1} \, dx = \frac{2}{\alpha_m h(h^2 + 1)} + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty, \tag{43}
\]

\[
P_{Fj}^{(m)} = \frac{(-1)^m 4ja^{2j}}{h} \int_{-h}^{h} \frac{x \sin \xi_j(x) + \cos \xi_j(x)}{(x^2 + 1)^{j+1}} \cos \alpha_m x \, dx = \frac{2j a^{2j}}{\alpha_m h(h^2 + 1)^{j+1}} \left[ \sin \xi_j(h) + \cos \xi_j(h) \right] + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty, \tag{44}
\]

\[
P_{Gj}^{(m)} = \frac{(-1)^m 2a^{2j-2}}{h} \int_{-h}^{h} \frac{j x \sin \xi_j(x) + (j - 1) \cos \xi_j(x)}{(x^2 + 1)^j} \cos \alpha_m x \, dx = \frac{2j^{2j-2}}{\alpha_m h(h^2 + 1)^j} \left[ \sin \xi_j(h) + (j - 1) \cos \xi_j(h) \right] + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty, \tag{45}
\]

\[
Q_{E1}^{(m)} = \frac{(-1)^{m+1}}{\alpha_m h} \int_{-h}^{h} \frac{x}{x^2 + 1} \sin \alpha_m x \, dx = \frac{2(1 - h^2)}{\alpha_m^3 h(h^2 + 1)^2} + O \left( \frac{1}{\alpha_m^4} \right), \quad m \to \infty, \tag{46}
\]

\[
Q_{Fj}^{(m)} = \frac{(-1)^{m+1} 4ja^{2j}}{\alpha_m h} \int_{-h}^{h} \frac{\sin \xi_j(x) - x \cos \xi_j(x)}{(x^2 + 1)^{j+1}} \sin \alpha_m x \, dx = \frac{2j a^{2j}}{\alpha_m^3 h(h^2 + 1)^{j+2}} \left[ 2j^2 h^2 - 3h^2 j + h^2 - 2j^2 - j + 1 \cos \xi_j(h) \right] + O \left( \frac{1}{\alpha_m^4} \right), \quad m \to \infty, \tag{47}
\]

\[
Q_{Gj}^{(m)} = \frac{(-1)^{m+1} 2a^{2j-2}}{\alpha_m h} \int_{-h}^{h} \frac{j x \sin \xi_j(x) - (j - 1) \cos \xi_j(x)}{(x^2 + 1)^j} \sin \alpha_m x \, dx = \frac{2j^{2j-2}}{\alpha_m^3 h(h^2 + 1)^j} \left[ 2j^2 h^2 - 2j h^2 - 2j^2 - j + 1 \cos \xi_j(h) \right] + O \left( \frac{1}{\alpha_m^4} \right), \quad m \to \infty, \tag{48}
\]

\[
R_{E1}^{(l)} = \frac{(-1)^{l+1}}{\beta_l} \int_{-1}^{1} \frac{\sin \beta_l y}{h^2 + y^2} \, dy = \frac{2(h^2 - 1)}{\beta_l^3 (h^2 + 1)^2} + O \left( \frac{1}{\beta_l^4} \right), \quad l \to \infty, \tag{49}
\]

\[
R_{Fj}^{(l)} = \frac{(-1)^l 4ja^{2j}}{\beta_l} \int_{-1}^{1} \frac{h \sin \eta_j(y) + y \cos \eta_j(y)}{(h^2 + y^2)^{j+1}} \sin \beta_l y \, dy = \frac{2j a^{2j}}{\beta_l^3 (h^2 + 1)^{j+2}} \left[ 2j^2 h^2 + h^2 - 2j^2 - j + 1 \cos \eta_j(1) \right] + O \left( \frac{1}{\beta_l^4} \right), \quad l \to \infty, \tag{50}
\]
In these equations the coefficients with unknowns $P_j$, $G_j$, $E_1$ have a similar structure, for example

\begin{align}
R_{ij} &= \frac{(-1)^i a_{2j}^{2j-2}}{\beta_i} \int_{-1}^{1} \frac{jh \sin \eta_j(y) + (j - 1)y \cos \eta_j(y) \sin \beta_i y}{(h^2 + y^2)^{j+1}} dy = \\
&= - \frac{4a_{2j}^{2j-2}}{\beta_i^2} [2j^2 h^2 + jh^2 - h^2 - 2j^2 + 3j - 1] \cos \eta_j(1) \\
&\quad - (4jh - 2jh) \sin \eta_j(1) + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty, \\
S_{E1}^{(l)} &= (1)^l h \int_{-1}^{1} \frac{\cos \beta_i y}{h^2 + y^2} dy = - \frac{2h}{\beta_i(h^2 + 1)^{j+1}} + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty, \\
S_{Fj}^{(l)} &= (1)^i a_{2j}^{2j} \int_{-1}^{1} \frac{y \sin \eta_j(y) - h \cos \eta_j(y)}{(h^2 + y^2)^{j+1}} \cos \beta_i y \; dy = \\
&= - \frac{4j a_{2j}^{2j}}{\beta_i(h^2 + 1)^{j+1}} [\sin \eta_j(1) - h \cos \eta_j(1)] + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty, \\
S_{Gj}^{(l)} &= (1)^i a_{2j}^{2j-2} \int_{-1}^{1} \frac{y \sin \eta_j(y) - (j - 1)y \cos \eta_j(y)}{(h^2 + y^2)^{j}} \cos \beta_i y \; dy = \\
&= - \frac{4j a_{2j}^{2j-2}}{\beta_i(h^2 + 1)^{j}} [j \sin \eta_j(1) - (j - 1)h \cos \eta_j(1)] + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty.
\end{align}

A.2 Expressions for the coefficients in formulas (19) and (20)

In these equations the coefficients with unknowns $P_j$, $G_j$, $E_1$ have a similar structure, for example

\begin{align}
K_{Fj}^{X_m} &= \frac{2}{h} \alpha_m \sum_{i=1}^{\infty} R_{ij}^{(m)} \frac{\beta_i}{\beta_i^2 + \alpha_m} + \alpha_m P_{Fj}^{(m)} - \alpha_m Q_{Fj}^{(m)} \tanh \alpha_m, \\
K_{Fj}^{Y_m} &= \frac{2\beta_i}{h} \alpha_m \sum_{i=1}^{\infty} Q_{Fj}^{(m)} \frac{\beta_i}{\alpha_m^2 + \beta_i^2} + \beta_i S_{Fj}^{(l)} - \beta_i R_{Fj}^{(l)} \tanh \beta_i h.
\end{align}

Infinite sums (denoted as $\tilde{R}_{Fj}^{(m)}$, $\tilde{Q}_{Fj}^{(m)}$, etc.) in these expressions make their usage inconvenient. Nevertheless, using the expressions for the values $R_{ij}^{(l)}$, $R_{ij}^{(l)}$, $R_{ij}^{(l)}$, $Q_{Fj}^{(m)}$, $Q_{Fj}^{(m)}$ and $Q_{Fj}^{(m)}$ in the form of the integrals (46)-(51) and changing the order of summing and integration, we obtain the following formulas for these sums:

\begin{align}
\tilde{R}_{Fj}^{(m)} &= \frac{\alpha_m}{h} \frac{2 a_{2j}^{2j}}{h} \int_{-1}^{1} \frac{h \sin \eta_j(y) + y \cos \eta_j(y) \sinh \alpha_m y}{(h^2 + y^2)^{j+1}} \cosh \alpha_m \; dy = \\
&= - \frac{4j a_{2j}^{2j}}{h(h^2 + 1)^{j+1}} [h \sin \eta_j(1) + \cos \eta_j(1)] + \\
&\quad + \frac{4j a_{2j}^{2j}}{\alpha_m(h^2 + 1)^{j+1}} [(2jh^2 + h^2 - 2j^2 - 1) \cos \eta_j(1) + \\
&\quad - (4jh + 2h) \sin \eta_j(1)] + O \left( \frac{1}{\alpha_m} \right), \quad m \to \infty, \\
\tilde{R}_{Gj}^{(m)} &= \frac{\alpha_m}{h} \frac{2 a_{2j}^{2j-2}}{h} \int_{-1}^{1} \frac{jh \sin \eta_j(y) + (j - 1)y \cos \eta_j(y) \sinh \alpha_m y}{(h^2 + y^2)^{j}} \cosh \alpha_m \; dy = \\
&= - \frac{4j a_{2j}^{2j-2}}{h(h^2 + 1)^{j}} [jh \sin \eta_j(1) + (j - 1) \cos \eta_j(1)] + \\
&\quad + \frac{4j a_{2j}^{2j-2}}{\alpha_m(h^2 + 1)^{j+1}} [(2j^2 h^2 + jh^2 - h^2 - 2j^2 + 3j - 1) \cos \eta_j(1) + \\
&\quad - (4jh^2 + 2jh) \sin \eta_j(1)] + O \left( \frac{1}{\alpha_m} \right), \quad m \to \infty, \\
\tilde{R}_{E1}^{(m)} &= \frac{\alpha_m}{h} \int_{-1}^{1} \frac{y \sinh \alpha_m y}{h \cosh \alpha_m} \; dy = \frac{2}{h(h^2 + 1)^{j}} - \\
&\quad - \frac{2(h^2 - 1)}{\alpha_m(h^2 + 1)^{j+1}} + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty.
\end{align}
\[
\tilde{Q}^{(i)}_{F;j} = \beta_i 2 j a 2 j \int_{-h}^{h} \frac{\sin \xi_j(x)}{(x^2 + 1)^{j+1}} \frac{x \cos \xi_j(x)}{\cosh \beta_i h} \, dx = \\
\frac{4 j a 2 j}{(h^2 + 1)^{j+1}} [\sin \xi_j(h) - h \cos \xi_j(h)] - \\
- \beta_i (h^2 + 1)^{j+2} [(2 j h^2 + h^2 - 2 j - 1) \cos \xi_j(h) + \\
-(4 j h + 2 h) \sin \xi_j(h)] + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty,
\]

(60)

Then the following exact expressions with specific asymptotic behavior are obtained:

\[
A.3
\]

The equations (24) represent the condition that the j-th Fourier component of the velocity \( u_r(a, \theta) \), expanded into the series on the functions \( \sin 2 j \theta \), should be equal to zero. Likewise, the equations (25) represent the conditions that the Fourier components of the tangential velocity on the cylinder surface \( u_\theta(a, \theta) \), expanded into the series on \( \cos 2 j \theta \), should be equal to zero. Note, that the presence of only such functions in the expansions of \( u_r(a, \theta) \) and \( u_\theta(a, \theta) \) is caused by the symmetry conditions

\[
A(T, \theta) = \lambda A(T, -\theta) = \lambda A(T, \pi - \theta).
\]

Expressions for the coefficients in equations (24) and (25) are expressed by the following formulas:

\[
K_{F;j}^{X_m} = \tilde{R}_{F;j}^{(m)} + \alpha_m P_{F;j}^{(m)} - \alpha_m^2 Q_{F;j}^{(m)} \tanh \alpha_m = O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty,
\]

(63)

\[
K_{G;j}^{X_m} = \tilde{R}_{G;j}^{(m)} + \alpha_m P_{G;j}^{(m)} - \alpha_m^2 Q_{G;j}^{(m)} \tanh \alpha_m = \lambda_j \frac{1}{\alpha_m} + O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty,
\]

\[
K_{E1}^{X_m} = \tilde{R}_{E1}^{(m)} + \alpha_m P_{E1}^{(m)} - \alpha_m^2 Q_{E1}^{(m)} \tanh \alpha_m = O \left( \frac{1}{\alpha_m^2} \right), \quad m \to \infty,
\]

(64)

\[
K_{F;j}^{Y_j} = \tilde{Q}_{F;j}^{(i)} + \beta_i S_{F;j}^{(i)} - \beta_i^2 R_{F;j}^{(i)} \tanh \beta_i h = O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty,
\]

\[
K_{G;j}^{Y_j} = \tilde{Q}_{G;j}^{(i)} + \beta_i S_{G;j}^{(i)} - \beta_i^2 R_{G;j}^{(i)} \tanh \beta_i h = -\lambda_j \frac{1}{\beta_i} + O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty,
\]

\[
K_{E1}^{Y_j} = \tilde{Q}_{E1}^{(i)} + \beta_i S_{E1}^{(i)} - \beta_i^2 R_{E1}^{(i)} \tanh \beta_i h = O \left( \frac{1}{\beta_i^2} \right), \quad l \to \infty,
\]

The equations (24) represent the condition that the j-th Fourier component of the velocity \( u_r(a, \theta) \), expanded into the series on the functions \( \sin 2 j \theta \), should be equal to zero. Likewise, the equations (25) represent the conditions that the Fourier components of the tangential velocity on the cylinder surface \( u_\theta(a, \theta) \), expanded into the series on \( \cos 2 j \theta \), should be equal to zero. Note, that the presence of only such functions in the expansions of \( u_r(a, \theta) \) and \( u_\theta(a, \theta) \) is caused by the symmetry conditions

\[
\psi_A(r, \theta) = \psi_A(r, -\theta) = \psi_A(r, \pi - \theta).
\]

Coefficients of the equation (24) are expressed by the following formulas:

\[
K_{X;m}^{r;j} = \frac{(-1)^m}{\alpha_m \cosh \alpha_m} \frac{2}{\pi} \int_0^{\pi} \sin \theta \sin(\alpha_m a \cos \theta) \{ \tanh \alpha_m \cosh(\alpha_m a \sin \theta) - \\
- a \sin \theta \sinh(\alpha_m a \sin \theta) \} - \cos \theta \cos(\alpha_m a \cos \theta) \{ \sinh(\alpha_m a \sin \theta) \times \\
x (\alpha_m^{-1} - \tanh \alpha_m) + a \sin \theta \cosh(\alpha_m a \sin \theta) \} \sin 2 j \theta \, d\theta
\]

(65)
\[
K^{\gamma j}_{Y_1} = \frac{(-1)^j h}{\beta_1 \cosh \beta_1 \pi} \int_0^\pi \left[ \cos \theta \sin (\beta_1 a \sin \theta) \right] h \tanh \beta_1 h \cosh (\beta_1 a \cos \theta) - \\
- a \cos \theta \sinh (\beta_1 a \cos \theta) \big] - \sin \theta \cos (\beta_1 a \sin \theta) \big] \left[ \sinh (\beta_1 a \cos \theta) x \\
\times (\beta_1^{-1} - h \tanh \beta_1 h) + a \cos \theta \cosh (\beta_1 a \cos \theta) \big] \right] \sin 2j \theta \ d\theta
\] (66)

\[
K^{\gamma j}_{F_1} = -\delta_{ij} \frac{2 j}{a} + \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{F_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{F_1},
\]

\[
K^{\gamma j}_{G_1} = -\delta_{ij} \frac{2 j}{a} + \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{G_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{G_1},
\]

\[
K^{\gamma j}_{E_1} = \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{E_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{E_1},
\] (67)

where \( \delta_{ij} \) is the Kronecker delta:

\[
\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j
\end{cases}
\] (68)

and the values \( \Omega^{\gamma j}_{Qm} \) and \( \Omega^{\gamma j}_{R\ell} \) are expressed as follows:

\[
\Omega^{\gamma j}_{Qm} = \frac{(-1)^{m+1} a m}{\cosh \alpha_m \pi} \int_0^\pi \left[ \sin \theta \sin (\alpha_m a \cos \theta) \cosh (\alpha_m a \sin \theta) + \\
\cos \theta \cos (\alpha_m a \cos \theta) \sinh (\alpha_m a \sin \theta) \right] \sin 2j \theta \ d\theta
\] (69)

\[
\Omega^{\gamma j}_{R\ell} = \frac{(-1)^{j+1} \beta_1 \beta_1}{\cosh \beta_1 \pi} \int_0^\pi \left[ \cos \theta \sin (\beta_1 a \sin \theta) \cosh (\beta_1 a \cos \theta) + \\
\sin \theta \cos (\beta_1 a \sin \theta) \sinh (\beta_1 a \cos \theta) \right] \sin 2j \theta \ d\theta
\] (70)

Coefficients in the equations (25) have a similar form and they are expressed by the following formulas:

\[
K^{\gamma j}_{X_m} = \frac{(-1)^m}{\alpha_m \cosh \alpha_m \pi} \int_0^\pi \left[ \cos \theta \sin (\alpha_m a \cos \theta) \right] \tanh \alpha_m \cosh (\alpha_m a \sin \theta) - \\
- a \sin \theta \sinh (\alpha_m a \sin \theta) \big] + \sin \theta \cos (\alpha_m a \cos \theta) \big] \left[ \sinh (\alpha_m a \sin \theta) x \\
\times (\alpha_m^{-1} - \tanh \alpha_m) + a \sin \theta \cosh (\alpha_m a \sin \theta) \big] \right] \cos 2j \theta \ d\theta
\] (71)

\[
K^{\gamma j}_{Y_1} = \frac{(-1)^{j+1} h}{\beta_1 \cosh \beta_1 \pi} \int_0^\pi \left[ \cos \theta \sin (\beta_1 a \sin \theta) \right] h \tanh \beta_1 h \cosh (\beta_1 a \cos \theta) - \\
- a \cos \theta \sinh (\beta_1 a \cos \theta) \big] + \cos \theta \cos (\beta_1 a \sin \theta) \big] \left[ \sinh (\beta_1 a \cos \theta) x \\
\times (\beta_1^{-1} - h \tanh \beta_1 h) + a \cos \theta \cosh (\beta_1 a \cos \theta) \big] \right] \cos 2j \theta \ d\theta
\] (72)

\[
K^{\gamma j}_{F_1} = \delta_{ij} \frac{2 j}{a} + \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{F_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{F_1},
\]

\[
K^{\gamma j}_{G_1} = \delta_{ij} \frac{2 (j - 1)}{a} + \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{G_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{G_1},
\]

\[
K^{\gamma j}_{E_1} = -\delta_{ij} \frac{2 j}{a} + \sum_{m=1}^\infty \Omega^{\gamma j}_{Qm} Q^{(m)}_{E_1} + \sum_{l=1}^\infty \Omega^{\gamma j}_{R\ell} R^{(l)}_{E_1},
\] (73)
where the values $\Omega_{Qm}^{ij}$ and $\Omega_{Rm}^{ij}$ are defined as:

$$\Omega_{Qm}^{ij} = \frac{(-1)^{m+1} \alpha_m}{\cosh \alpha_m} 2 \int_0^\pi \left[ \cos \theta \sin(\alpha_m \alpha \cos \theta) \cosh(\alpha_m \alpha \sin \theta) - 
- \sin \theta \cos(\alpha_m \alpha \cos \theta) \sinh(\alpha_m \alpha \sin \theta) \right] \cos \theta \, d\theta$$

$$\Omega_{Rm}^{ij} = \frac{(-1)^{m+1} \beta_m}{\cosh \beta_m} 2 \int_0^\pi \left[ \sin \theta \sin(\beta_m \alpha \sin \theta) \cosh(\beta_m \alpha \cos \theta) - 
- \cos \theta \cos(\beta_m \alpha \sin \theta) \sinh(\beta_m \alpha \cos \theta) \right] \cos \theta \, d\theta$$

(74)

(75)

A.4 Expressions for the velocity components in stationary problem A

The expression for $u_x$ in problem A has the form:

$$u_x = \sum_{m=1}^M (-1)^m \cos \alpha_m \alpha \frac{X_m}{\alpha_m} \left[ \frac{\sinh \alpha_m y}{\cosh \alpha_m} - \frac{\cosh \alpha_m y}{\cosh \alpha_m} - \frac{1}{\alpha_m} \sinh \alpha_m y \right]$$

$$-a_m Q_m \sinh \alpha_m y + c\gamma(y) \left( 1 - |y| \right) e^{-\alpha_m(1-|y|)}$$

$$-a_x \gamma(y) \frac{1}{\alpha_m} e^{-\alpha_m(1-|y|)}$$

$$+ \sum_{l=1}^L (-1)^l \sin \beta_l y \left[ \frac{Y_l}{\beta_l} \left( h \tanh \beta_l h - x \frac{\sin \beta_l x}{\cosh \beta_l h} + \beta_l R_l \frac{\cosh \beta_l x}{\cosh \beta_l h} \right) 
- h \delta(h - |x|) e^{-\beta_l (h - |x|)} \right]$$

$$- \sum_{j=1}^J \sin(2j \theta(x, y)) 2x \left[ F_j \lambda_j a^{2j} (x^2 + y^2)^{-j-1} + G_j \lambda_j a^{2j-2} (x^2 + y^2)^{-j} \right]$$

$$- \sum_{j=1}^J \cos(2j \theta(x, y)) 2y \left[ F_j \lambda_j a^{2j} (x^2 + y^2)^{-j-1} + G_j (j-1) a^{2j-2} (x^2 + y^2)^{-j} \right]$$

$$+ E_j \frac{y}{x^2 + y^2} + \Lambda x_1 + \Lambda x_2,$$

where

$$a_x = a_0 + \sum_{j=1}^J \lambda_j G_j, \quad a_y = a_0.$$

Here $\theta(x, y)$ is the polar angle of the point $(x, y)$ which can be calculated from

$$\theta(x, y) = \begin{cases} 
\arctan(y/x), & x > 0 \\
\pi/2, & x = 0, y > 0 \\
-\pi/2, & x = 0, y < 0 \\
\pi + \arctan(y/x), & x < 0
\end{cases}$$

and $\gamma$ is the signum function:

$$\gamma(y) = \begin{cases} 
1, & y > 0 \\
0, & y = 0 \\
-1, & y < 0
\end{cases}$$
The term $A_{x1}$ is added to improve the convergence of series with $c$ and $d$, the term $A_{x2}$ for series with $a_x$ and $a_y$ because it is possible that $a_x \neq 0$ even if $a_0 = 0$.

$$A_{x1} = \begin{cases} 
-c\gamma(y)(1-|y|)\frac{\cos \frac{\pi y}{2h}}{\cosh \frac{\pi(1-|y|)}{2h}} + c\gamma(y)\frac{h}{\pi} \arctan \frac{\pi(1-|y|)}{2h} \sinh \frac{\pi(h-|x|)}{2} - h d(h-|x|) \frac{\gamma(y) \pi(1-|y|)}{2} \cos \pi(1-|y|) - 1, & |x| < h, |y| < 1 \\
-c\gamma(y)(1-|y|)\frac{\cos \frac{\pi y}{2h}}{\cosh \frac{\pi(1-|y|)}{2h}} + c\gamma(y)\frac{h}{\pi} \arctan \frac{\pi(1-|y|)}{2h} \sinh \frac{\pi(h-|x|)}{2} - h d(h-|x|) \frac{\gamma(y) \pi(1-|y|)}{2} \cos \pi(1-|y|) - 1, & |x| = h, |y| < 1 \\
0, & |x| = h, |y| = 1.
\end{cases}$$

$$A_{x2} = \begin{cases} 
-a_x \gamma(x)(1-|y|) \frac{h}{2\pi} \arctan \frac{\pi(1-|y|)}{2h} - a_y \frac{h}{2\pi} \ln \frac{\pi(1-|y|)}{2h} + \frac{\gamma(y) \pi(1-|y|)}{2} \cos \pi(1-|y|) - 1, & |x| < h, |y| < 1 \\
-a_x \gamma(x)(1-|y|) \frac{h}{2\pi} \arctan \frac{\pi(1-|y|)}{2h} - a_y \frac{h}{2\pi} \ln \frac{\pi(1-|y|)}{2h} + \frac{\gamma(y) \pi(1-|y|)}{2} \cos \pi(1-|y|) - 1, & |x| = h, |y| = 1 \\
0, & |x| = h.
\end{cases}$$

The expression for $u_y$ in problem $A$ is

$$u_y = \sum_{m=1}^{M} (-1)^m \sin \alpha_m x \left[ \frac{X_m}{\alpha_m} \left( \tanh \alpha_m \cosh \beta_m y + \sinh \beta_m y \right) - a_x (1 - |y|) e^{-\alpha_m (1-|y|)} \right]$$

$$+ \sum_{l=1}^{L} (-1)^l \cos \beta_l y \left[ \frac{Y_l}{\beta_l} \left( h \tanh \beta_l h \sinh \beta_l x + \cosh \beta_l h \right) - x \cosh \beta_l h - \frac{1}{\beta_l} \sinh \beta_l x \right]$$

$$+ a_y \gamma(x) (h-|x|) \frac{1}{\beta_l} e^{-\beta_l (h-|x|)}$$

$$\sum_{j=1}^{J} \sin(2j \theta(x, y)) 2y \left[ F_j j a^{2j} (x^2 + y^2)^{j-1} + G_j j a^{2j-2} (x^2 + y^2)^{j-1} \right]$$

$$+ \sum_{j=1}^{J} \cos(2j \theta(x, y)) 2y \left[ F_j j a^{2j} (x^2 + y^2)^{j-1} + G_j j a^{2j-2} (x^2 + y^2)^{j-1} \right]$$

$$- E_1 \frac{x}{x^2 + y^2} + A_{y1} + A_{y2}.$$
\[ a_x \text{ and } a_y. \]

\[ \Lambda_{y1} = \begin{cases} 
-\epsilon(1 - |y|) \frac{\sin \frac{\pi y}{2h} \sinh \frac{\pi (1 - |x|)}{2h}}{\cosh \frac{\pi (1 - |x|)}{h}} + \cos \frac{\pi y}{2h} \frac{\pi (h - |x|)}{2h} \\
- d h \gamma(x) (h - |x|) \frac{\cosh \frac{\pi y}{2h} \cosh \frac{\pi (h - |x|)}{2h}}{\cosh \pi (h - |x|) + \cos \pi y} \\
+ d h \frac{1}{\pi} \gamma(x) \arctan \frac{\pi (h - |x|)}{\sinh \frac{\pi (h - |x|)}{2h}} \\
- c \gamma(x) (1 - |y|) \frac{\sinh \frac{\pi (1 - |x|)}{2h}}{\cosh \frac{\pi (1 - |x|)}{h}} + \frac{d}{2} h \gamma(x), \\
0, \end{cases} \]

\[ \Lambda_{y2} = \begin{cases} 
- a_x (1 - |y|) \frac{h}{2\pi} \ln \frac{\cosh \frac{\pi (1 - |x|)}{2h} + \sin \frac{\pi x}{2h}}{\cosh \frac{\pi (1 - |x|)}{2h} - \sin \frac{\pi x}{2h}} \\
- a_y \gamma(x) (h - |x|) \frac{h}{\pi} \arctan \frac{\cosh \frac{\pi x}{2h} \cosh \frac{\pi (h - |x|)}{2h}}{\sinh \frac{\pi (h - |x|)}{2h}} \\
- a_x (1 - |y|) \frac{h}{2\pi} \ln \frac{\cosh \frac{\pi (1 - |x|)}{2h} + \gamma(x)}{\cosh \frac{\pi (1 - |x|)}{2h} - \gamma(x)}, \\
0, \end{cases} \]

\[ |x| < h, |y| < 1 \]

\[ |x| = h, |y| < 1 \]

\[ |y| = 1, \]
Figure captions

**Figure 1.** A schematic presentation of the flow region under study. The cavity length is $W$, its width is $H$ and the cylinder radius is $R_0$. After scaling the cavity the dimensionless half-width become 1, half length is $h$, and cylinder radius is $a$.  

**Figure 2.** Representation of the solution of boundary problem with arbitrary velocities $V_{\text{top}}(t)$, $V_{\text{bot}}(t)$ and $U(t)$ as linear combination of the solutions of three stationary boundary problems $A$, $B$ and $C$. Here $V_A(t) = \frac{1}{2}(V_{\text{top}} - V_{\text{bot}})$, $V_B(t) = \frac{1}{2}(V_{\text{top}} + V_{\text{bot}})$, $V_C(t) = U(t)$. To show the symmetry properties of the velocity field, described by the problems $A$, $B$ and $C$, velocity components are depicted at arbitrarily chosen points $(x, y)$, $(x, -y)$ and $(-x, y)$. 

**Figure 3.** Dependence of the stream function value at the bisector of the corner on the distance $\rho$ from the corner top as described by the solution of the boundary problem $A$ ($V_{\text{top}} = 1$, $V_{\text{bot}} = -1$, $U=0$) for the geometrical parameters $h = 1.67$ and $a = 0.3$. The dashed line represents the Taylor solution (1962). 

**Figure 4.** Streamline patterns for the cavity flow, generated by the motion of top and bottom walls in opposite directions with $V_{\text{top}} = 1$ and $V_{\text{bot}} = -1$ and a fixed cylinder ($U=0$) corresponding to the stationary boundary problem $A$. The different sets of geometrical parameters $a$ and $h$ are: a) $h = 1$, $a = 0.3$; b) $h = 1$, $a = 0.1$; c) $h = 1.67$, $a = 0.3$; d) Streamline pattern from flow visualization; e) $h = 2.5$, $a = 0.3$. 

**Figure 5.** a) Streamline pattern for the cavity flow, generated by the motion of top and bottom walls in the same direction ($V_{\text{top}} = 1$, $V_{\text{bot}} = 1$) and a fixed cylinder ($U=0$) corresponding to the stationary boundary problem $B$. The aspect ratio of the cavity is $h = 1.67$ and the cylinder radius is $a = 0.3$. b) Streamline pattern from flow visualization. 

**Figure 6.** a) Streamline pattern for the cavity flow, generated by the rotating cylinder ($U = 1$) with fixed walls ($V_{\text{top}} = 0$, $V_{\text{bot}} = 0$) corresponding to stationary boundary problem $C$. The geometrical parameters are $h = 1.67$, $a = 0.3$. b) Streamline pattern from flow visualization. 

**Figure 7.** Coordinates of stagnation points and the position of the separation streamlines on the vertical (upper plot) and horizontal axes of symmetry versus the cylinder rotation speed $U$ in a quasi-shear flow ($V_{\text{top}} = 1$, $V_{\text{bot}} = -1$), generated in the cavity with geometrical parameters $h = 2.5$ and $a = 0.3$. Solid line represent the vortex center, dashed line - the saddle point, dash-dot line - the position of the separation streamline on the axis. Because the flow pattern is symmetrical, only half of the axes are represented on the plots. 

**Figure 8.** Streamline pattern for different flow regimes caused by the cylinder rotation in the quasi-shear flow ($V_{\text{top}} = 1$, $V_{\text{bot}} = -1$), generated in a relatively long cavity ($h = 2.5$, $a = 0.3$). The cylinder is respectively: a) counterrotating, $U = 0.25$; b) fixed, $U = 0$; c) slowly corotating, $U = -0.075$; d) rotating with the critical velocity $U = -0.12$; e) fast corotating. 

**Figure 9.** a) The structure of the flow generated by the moving top wall while the bottom wall and the cylinder are fixed in a cavity with geometrical parameters $h = 1.67$ and $a = 0.3$: b) Streamline pattern from flow visualization.

**Figure 10.** a) The structure of the flow generated by the simultaneous motion of the top wall ($V_{\text{top}} = 1$) and the cylinder ($U = 0.9$). The bottom wall is fixed. The geometrical parameters of the cavity are $h = 1.67$ and $a = 0.3$. b) Streamline pattern from flow visualization.
Figure 1.

Figure 2.
Figure 3.
Figure 7.
Figure 9a.

Figure 9b.
Figure 10a.

Figure 10b.
References


